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Algebra

When is $A + XB[[X]]$ Noetherian?

Sana Hizem, Ali Benhissi

Department of Mathematics, Faculty of Sciences, 5000 Monastir, Tunisia

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Abstract

Let $A \subseteq B$ be an extension of commutative rings with identity, X an analytic indeterminate over B, and $R := A + XB[[X]]$, the subring of the formal power series ring $B[[X]]$, consisting of the series with constant terms in A. In this Note we study when the ring R is Noetherian. We prove that R is Noetherian if and only if A is Noetherian and B is a finitely generated A-module. ***To cite this article: S. Hizem, A. Benhissi, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Quand $A + XB[[X]]$ est-il noethérien ? Soient $A \subseteq B$ une extension d'anneaux commutatifs unitaires, X une indeterminée sur B, et $R := A + XB[[X]]$, le sous-anneau de l'anneau des séries formelles $B[[X]]$, formé par les séries dont le terme constant est dans A. Nous donnons une condition nécessaire et suffisante pour que l'anneau R soit noethérien. Nous démontrons que R est noethérien si et seulement si A est noethérien et B est un A module de type fini. ***Pour citer cet article : S. Hizem, A. Benhissi, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Let $A \subseteq B$ be an extension of commutative rings with identity, X an analytic indeterminate over B, and $R := A + XB[[X]]$, the subring of the formal power series ring $B[[X]]$, consisting of the series with constant terms in A. This construction has been studied by many authors and has proven to be useful in constructing interesting examples and counterexamples. See for instance [1–3].

Lemma 1. *R with the $XB[[X]]$-adic topology is the completion of $A + XB[X]$ with the $XB[X]$-adic topology.*

Proof. Since $I = XB[[X]]$ is an ideal of R and $\bigcap_{n\in\mathbb{N}} I^n = (0)$, R is a Hausdorff space with the I-adic topology. Since $XB[[X]] \cap (A + XB[X]) = XB[X]$, the $XB[[X]]$-adic topology on R induces the $XB[X]$-adic topology on $A + XB[X]$.

E-mail address: hizems@yahoo.fr (S. Hizem).

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Let $f=\sum_{i=0}^{+\infty}a_iX^i\in R$ and for any integer $k\geqslant 0$, $g_k=\sum_{i=0}^{k}a_iX^i\in A+XB[X]$. Then $f-g_k=\sum_{i=k+1}^{+\infty}a_iX^i\in X^{k+1}B[[X]]$, so $f=\lim_{k\to+\infty}g_k$. Conversely, let $(g_k)_k$ be a Cauchy sequence of $A+XB[X]$ for the $XB[X]$-adic topology and $g=g_0+(g_1-g_0)+(g_2-g_1)+\cdots\in A+XB[[X]]$. Since for any $l\in\mathbb{N}$, there exists $k_0\in\mathbb{N}$ such that for any $k\geqslant k_0$, $g_{k+1}-g_k\in(XB[[X]])^l$, then $g-g_k\in(XB[[X]])^l$ and $g=\lim_{k\to+\infty}g_k$. □

Lemma 2. *An ideal of R containing $XB[[X]]$ is of type $I+XB[[X]]$ for some ideal I of A.*

Proof. Let J an ideal of R containing $XB[[X]]$ and $I=\{f(0),\ f\in J\}$. Then I is an ideal of A and $J\subseteq I+XB[[X]]$. Conversely, let $a\in I$. Then there exists $f=\sum_{i=0}^{+\infty}a_iX^i\in J$ such that $a=a_0=f-\sum_{i=1}^{+\infty}a_iX^i\in J$, since $\sum_{i=1}^{+\infty}a_iX^i\in XB[[X]]\subseteq J$. So $I\subseteq J$ and then $I+XB[[X]]\subseteq J$. □

Lemma 3. *Assume that B is a finitely generated A-module and p a finitely generated ideal of A. Then the ideal $P=p+XB[[X]]$ of $A+XB[[X]]$ is finitely generated.*

Proof. Let $b_1,\dots,b_s$ generators of the A-module B and $a_1,\dots,a_n$ in A such that $p=a_1A+\cdots+a_nA$. Then $B[[X]]=b_1A[[X]]+\cdots+b_sA[[X]]$, and $p+XB[[X]]=a_1A+\cdots+a_nA+b_1XA[[X]]+\cdots+b_sXA[[X]]$. Since $A[[X]]\subseteq R$ we have $p+XB[[X]]=(a_1,\dots,a_n,b_1X,\dots,b_sX)_R$. □

Theorem 4. *Let $A\subseteq B$ be commutative rings with identity, then the ring $R=A+XB[[X]]$ is Noetherian if and only if A is Noetherian and B is a finitely generated A-module.*

Proof. If R is Noetherian, then so is the ring $R/XB[[X]]\simeq A$. On the other hand, the ideal $XB[[X]]$ of R is finitely generated. Let $f_1,\dots,f_n$ in $B[[X]]$ such that $XB[[X]]=Xf_1R+\cdots+Xf_nR$, then $B[[X]]=f_1R+\cdots+f_nR$ and $B=f_1(0)A+\cdots+f_n(0)A$. So B is a finitely generated A-module. Conversely, let A be a Noetherian ring and B a finitely generated A-module. If R is not Noetherian, then (by Zorn's lemma) there exists an ideal P of R, maximal among the ideals of R which are not finitely generated. In fact, let Σ be the set of all ideals which are not finitely generated in R. Order Σ by inclusion; Σ is not empty, since R is not Noetherian. Let (a_α) be a chain of ideals in Σ. Let $a=\bigcup_\alpha a_\alpha$. Then a is an ideal of R and a is not finitely generated. Hence by Zorn's lemma Σ has a maximal element P. By Lemmas 2 and 3, $XB[[X]]\nsubseteq P$. Since B is a finitely generated A-module, then we can choose $b_1,\dots,b_s\in B$ such that $B=b_1A+\cdots+b_sA$, so $XB[[X]]=b_1XA[[X]]+\cdots+b_sXA[[X]]$. Since $XB[[X]]\nsubseteq P$, then there exists i_0, $1\leqslant i_0\leqslant s$, such that $b_{i0}X\notin P$. We can suppose that $b_1X\notin P$. Hence $P\subset P+Xb_1R$. Therefore, $P+b_1XR$ is finitely generated. So, there exists $J=\langle f_1,\dots,f_n\rangle\subseteq P$ a finitely generated ideal of R such that $P+b_1XR=J+b_1XR$. We claim that $P=J+P\cap b_1XR$. Indeed, $J\subseteq P$ and $P\cap b_1XR\subseteq P$, so $J+P\cap b_1XR\subseteq P$. Conversely, let $f\in P$, then $f\in J+b_1XR$, so there exists $g\in J$, and $h\in R$ such that $f=g+b_1Xh$, so $b_1Xh=f-g\in P$ and $b_1Xh\in P\cap b_1XR$. Moreover, $P=J+(P:b_1XR)b_1XR$ with $P:b_1XR=\{f\in R;\ fb_1XR\subseteq P\}$. Indeed, $(P:b_1XR)b_1XR\subseteq P$, so $J+(P:b_1XR)b_1XR\subseteq P$. Conversely, it suffices to prove that $P\cap b_1XR\subseteq(P:b_1XR)b_1XR$. Let $f\in P\cap b_1XR$, then there exists $g\in R$ such that $f=b_1Xg$, so $b_1X\in b_1XR$ and $g\in(P:b_1XR)$, which implies that $f\in(P:b_1XR)b_1XR$. Moreover, $P\subseteq P:b_1XR$. If $P\subset P:b_1XR$ then, by maximality of P, $P:b_1XR$ is finitely generated and so $P=J+(P:b_1XR)b_1XR$ is also finitely generated which is impossible. So we have the equality $P=P:b_1XR$ which implies $P=J+b_1XP$. We deduce then that $P=J$. In fact, for $g\in P$, we construct by induction on $k\in\mathbb{N}^*$, a sequence $(g_k)_{k\in\mathbb{N}^*}$ of elements of J such that for any $k\in\mathbb{N}^*$, $g_k=\sum_{i=1}^{n}s_{k,i}f_i$ with for any $1\leqslant i\leqslant n$, $s_{k,i}\in(b_1X)^{k-1}R$ and $g-g_1-\cdots-g_k\in(b_1X)^kP$. For $k=1$, we have $g\in P=J+b_1XP$, so there exists $g_1\in J$ such that $g-g_1\in(b_1X)P$. Let $s_{1,i}\in R$ for $1\leqslant i\leqslant n$, be such that $g_1=\sum_{i=1}^{n}s_{1,i}f_i$. Assume by induction on k, that $g_1,\dots,g_k$ are constructed. Since $g-g_1-\cdots-g_k\in(b_1X)^kP=(b_1X)^k[J+b_1XP]$, then there exists $g'\in J$ such that $g-g_1-\cdots-g_k-(b_1X)^kg'\in(b_1X)^{k+1}P$. Let $s_i\in R$, for $1\leqslant i\leqslant n$ be such that $g'=\sum_{i=1}^{n}s_if_i$ and take $g_{k+1}=(b_1X)^kg'=\sum_{i=1}^{n}X^kb_1^ks_if_i$ and $s_{k+1,i}=b_1^kX^ks_i\in(b_1X)^kR$. Then $g-g_1-\cdots-g_k-g_{k+1}\in(b_1X)^{k+1}P$.

Since $\sum_{j=1}^{k+1} s_{j,i} - \sum_{j=1}^{k} s_{j,i} = s_{k+1,i} \in (b_1X)^k R \subseteq (XB[[X]])^k$, then, for any $1 \leqslant i \leqslant n$, the sequence $(\sum_{j=1}^{k} s_{j,i})_k$ is a Cauchy one for the $XB[[X]]$-adic topology. Let, by Lemma 1, for $1 \leqslant i \leqslant n$, $s_i = \lim_k \sum_{j=1}^{k} s_{j,i} \in R$ and $g' = \sum_{i=1}^{n} s_i f_i$. Then $g = g'$. Indeed, for $m \in \mathbb{N}$ and $k \geqslant m$, $g - g' = (g - \sum_{j=1}^{k} g_j) + (\sum_{j=1}^{k} g_j - g')$, with $g - \sum_{j=1}^{k} g_j \in (b_1X)^k P \subseteq (XB[[X]])^k \subseteq (XB[[X]])^m$ and $\sum_{j=1}^{k} g_j - g' = \sum_{j=1}^{k} g_j - \sum_{i=1}^{n} s_i f_i = \sum_{i=1}^{n} f_i(\sum_{j=1}^{k} s_{j,i} - s_i) \in (XB[[X]])^m$ for any $k \geqslant k_0$, for some integer $k_0 \geqslant m$. So $g - g' \in (XB[[X]])^m$, for any $m \in \mathbb{N}$, which implies that $g - g' \in \bigcap_{m\in\mathbb{N}} (XB[[X]])^m = (0)$, so $g = g' \in J$. And thus $P = J$, which is impossible. □

Example 1. Let $K \subseteq L$ be an extension of fields, then $K + XL[[X]]$ is Noetherian if and only if $[L : K]$ is finite. For example $\mathbb{R} + X\mathbb{C}[[X]]$ is Noetherian but $\mathbb{Q} + X\mathbb{R}[[X]]$ is not.

Remark 1. Let $A \subset B$ be an extension of integral domains, then the domain $R = A + XB[[X]]$ is never principal (in fact it can never be an UFD). To see this, observe that the element X is irreducible but not prime. In fact, let $f = \sum_{i=0}^{+\infty} a_i X^i$ and $g = \sum_{i=0}^{+\infty} b_i X^i \in R$ such that $fg = X$. Then $a_0b_0 = 0$, for example $a_0 = 0$ and $a_0b_1 + a_1b_0 = 1$, therefore $a_1b_0 = 1$, which implies that g is a unit in R. On the other hand, the ideal $X \cdot (A + XB[[X]]) = AX + X^2B[[X]]$ is not prime, because if $b \in B \backslash A$, then $X \cdot (A + XB[[X]])$ contains $(bX)^2$ but not bX.

References

[1] D.F. Anderson, D.N. El Abidine, The $A + XB[X]$ and $A + XB[[X]]$ constructions from *GCD*-domains, J. Pure Appl. Algebra 159 (2001) 15–24.

[2] D.E. Dobbs, M. Khalis, On the prime spectrum, Krull dimension and catenarity of integral domains of the form $A + XB[[X]]$, J. Pure Appl. Algebra 159 (2001) 57–73.

[3] T. Dumitrescu, S.O.I. Al-Salihi, N. Radu, T. Shah, Some factorization properties of composite domains $A + XB[X]$ and $A + XB[[X]]$, Commun. Algebra 28 (3) (2000) 1125–1139.