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Partial Differential Equations

Reiterated homogenization for elliptic operators

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Abstract

In this Note, using the periodic unfolding method (see D. Cioranescu et al., C. R. Acad. Sci. Paris, Ser. I 335 (1) (2002) 99–104), we study reiterated homogenization for equations of the form $-\operatorname{div}(a_\epsilon(x, Du_\epsilon)) = f$, where a_ϵ is Carathéodory and satisfies some monotone and growth conditions. We show that if we assume that $\mathcal{T}'_{\delta(\epsilon)}(\mathcal{T}_\epsilon(a_\epsilon))(x, y, z, \xi)$ converges, for almost all $(x, y, z) \in \Omega \times Y \times Z$, to a Carathéodory operator, then the sequences u_ϵ and Du_ϵ converge in a certain sense to the solution $(u_0, \hat{u}, \tilde{u})$ of a limit variational problem, as $\epsilon \to 0$. In particular this contains the case $a_\epsilon(x, \xi) = a(x, \frac{x}{\epsilon}, \frac{\{x/\epsilon\}_Y}{\delta(\epsilon)}, \xi)$, where a is periodic in the second and third arguments, and continuous in each argument. ***To cite this article: N. Meunier, J. Van Schaftingen, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Homogénéisation réitérée pour des opérateurs elliptiques. Dans cette note, on étudie, en utilisant la méthode d'éclatement périodique (voir D. Cioranescu et al., C. R. Acad. Sci. Paris, Ser. I 335 (1) (2002) 99–104), l'homogénéisation réitérée pour des équations de la forme $-\operatorname{div}(a_\epsilon(x, Du_\epsilon)) = f$, où a_ϵ est de Carathéodory et satisfait des conditions de monotonie et de croissance. On montre que si l'on suppose la convergence de $\mathcal{T}'_{\delta(\epsilon)}(\mathcal{T}_\epsilon(a_\epsilon))(x, y, z, \xi)$, pour presque tout $(x, y, z) \in \Omega \times Y \times Z$, vers un opérateur de Carathéodory, alors les suites u_ϵ et Du_ϵ convergent dans un certain sens vers la solution $(u_0, \hat{u}, \tilde{u})$ d'un problème variationnel limite, quand $\epsilon \to 0$. Ce résultat s'applique en particulier au cas $a_\epsilon(x, \xi) = a(x, \frac{x}{\epsilon}, \frac{\{x/\epsilon\}_Y}{\delta(\epsilon)}, \xi)$, où a est periodique par rapport aux deuxième et troisième variables, et continue par rapport à chaque variable. ***Pour citer cet article : N. Meunier, J. Van Schaftingen, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

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La notion d'homogénéisation réitérée a été introduite dans Bensoussan, Lions et Papanicolaou [2] et Sanchez-Palencia [8] pour les opérateurs linéaires et périodiques. Le cas non linéaire convexe a été traité en utilisant la Γ-convergence [3]. Le cas des opérateurs monotones périodiques non linéaires a été étudié par Lions, Lukkassen, Persson et Wall [6] en utilisant une méthode d'énergie et une convergence multi-échelle, voir [1] pour cette théorie. Des applications de ces résultats pour des matériaux non linéaires peuvent être trouvées dans [3]. La généralisation de l'homogénéisation au cas linéaire non periodique elliptique a été faite par Tartar [9] et Murat et Tartar [7] en utilisant la H-convergence.

Dans cette Note, nous étudions l'homogénéisation réitérée pour des opérateurs non-linéaires monotones elliptiques en utilisant la méthode d'éclatement périodique introduite par Cioranescu, Damlamian et Griso [4].

Nous considérons la classe d'équations aux dérivées partielles de la forme :

$$\begin{cases} -\operatorname{div}\big(a_\epsilon(x, Du_\epsilon)\big) = f \quad \text{sur } \Omega, \\ u_\epsilon \in W_0^{1,p}(\Omega), \end{cases} \tag{1}$$

où Ω est un ouvert borné lipschitzien de $\mathbf{R}^N$, $1 < p < \infty$, $p^{-1} + q^{-1} = 1$ et $f \in W^{-1,q}(\Omega)$. On suppose que $a_\epsilon(x, \xi)$ est de Carathéodory et satisfait des conditions de monotonie (3) et de croissance (4).

Le Théorème 3.1 assure que si pour presque tout $(x, y) \in \Omega \times Y$, la suite $(\mathcal{T}_\epsilon(a_\epsilon)(x, y, \xi))_\epsilon$ converge simplement vers un opérateur de Carathéodory $a_{\text{hom}}(x, y, \xi)$, alors la suite $(u_\epsilon)_\epsilon$ converge faiblement dans $W_0^{1,p}(\Omega)$ vers u_0 où $(u_0, \hat{u})$ est l'unique solution du problème variationnel (7) et $\mathcal{T}_\epsilon(u)(x, y) = u(\epsilon[\frac{x}{\epsilon}]_Y + \epsilon y)$. On obtient en outre la convergence forte dans $L^p(\Omega \times Y)$ de la suite $(\mathcal{T}_\epsilon(Du_\epsilon))_\epsilon$ vers $D_x u_0 + D_y \hat{u}$ ainsi que la convergence forte des correcteurs dans le Théorème 3.2.

Remarque 1. Le Théorème 3.1 contient en particulier le cas où $a_\epsilon(x, \xi) = a(x, \frac{x}{\epsilon}, \xi)$, $a(x, y, \xi)$ étant une fonction Y-périodique dans la seconde variable et continue par rapport à chacune des variables.

Dans le Théorème 4.1 nous étudions l'homogénéisation réitérée. Nous introduisons l'opérateur d'éclatement réitéré, voir [4], pour toute fonction u, étendue par 0 en dehors de Ω, $u \in L^p(\Omega)$: $\mathcal{T}'_{\delta(\epsilon)}(\mathcal{T}_\epsilon(u))(x, y, z) = u(\epsilon[\frac{x}{\epsilon}]_Y + \epsilon\delta(\epsilon)[\frac{y}{\delta(\epsilon)}]_Z + \epsilon\delta(\epsilon)z)$, où $\lim_{\epsilon\to 0} \delta(\epsilon) = 0$.

Si la suite $(\mathcal{T}_{\delta(\epsilon)}(\mathcal{T}_\epsilon(a_\epsilon))(x, y, z, \xi))_\epsilon$ converge vers un opérateur de Carathéodory $a_{\text{homrei}}(x, y, z, \xi)$, pour presque tout $(x, y, z) \in \Omega \times Y \times Z$, alors la suite $(u_\epsilon)_\epsilon$ converge faiblement dans $W_0^{1,p}(\Omega)$ vers u_0 où $(u_0, \hat{u}, \tilde{u})$ est l'unique solution du problème variationnel (9). On a aussi la convergence forte dans $L^p(\Omega \times Y \times Z; \mathbf{R}^N)$ de la suite $(\mathcal{T}_{\delta(\epsilon)}(\mathcal{T}_\epsilon(Du_\epsilon)))_\epsilon$ vers $D_x u_0 + D_y \hat{u} + D_z \tilde{u}$. La convergence forte des correcteurs est assurée par le Théorème 4.2.

Remarque 2. Le Théorème 4.1 contient le cas où $a_\epsilon(x, \xi) = a(x, \frac{x}{\epsilon}, \frac{\{x/\epsilon\}_Y}{\delta(\epsilon)}, \xi)$, $a(x, y, z, \xi)$ étant une fonction Y-périodique dans la seconde variable, Z-périodique dans la troisième variable et continue par rapport à chacune des variables. De plus, si $Y = Z$ et $\frac{1}{\delta(\epsilon)} \in \mathbf{N}^*$ alors on a $a_\epsilon(x, \xi) = a(x, \frac{x}{\epsilon}, \frac{\{x/\epsilon\}_Y}{\delta(\epsilon)}, \xi) = a(x, \frac{x}{\epsilon}, \frac{x}{\epsilon\delta(\epsilon)}, \xi)$. Cela généralise ainsi les résultats obtenus dans [6] où il était nécessaire de supposer une hypothèse plus forte sur la fonction $a(x, \cdot, z, \xi)$.

Remarque 3. On peut généraliser le Théorème 4.1 au cas n fois réitéré, si on suppose que pour presque tout $(x, y_1, \ldots, y_n) \in \Omega \times Y_1 \times \cdots \times Y_n$, la suite $((\mathcal{T}'_{\delta_n(\epsilon)} \circ \cdots \circ \mathcal{T}'_{\delta_1(\epsilon)} \circ \mathcal{T}_\epsilon) a_\epsilon(x, y_1, \ldots, y_n, \xi))_\epsilon$ converge vers un opérateur de Carathéodory $a_{\text{homrei}}(x, y_1, \ldots, y_n, \xi)$.

Remarque 4. Le Théorème 4.1 contient en particulier le cas où $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, Ω_1 et Ω_2 étant des ouverts disjoints de frontière lipschitzienne et $a_\epsilon(x, \xi)$ est tel que $a_\epsilon(x, \xi) = a^1(x, \xi)$ si $x \in \Omega_1$, $a_\epsilon(x, \xi) = a^2(x, \frac{x}{\epsilon}, \frac{\{x/\epsilon\}_Y}{\delta(\epsilon)}, \xi)$ si

$x \in \Omega_2$, où a^1 et a^2 sont continues par rapport à chacune des variables et satisfont (3) et (4). Cela recouvre des situations plus générales que ce qui a été envisagé dans [6].

Remarque 5. Comme dans le cas linéaire, [4,5], le Théorème 4.1 peut être étendu à l'étude de domaines perforés.

1. Introduction

The reiterated homogenization was first introduced in Bensoussan, Lions and Papanicolaou [2] and in Sanchez-Palencia [8] for linear and periodic operators. The nonlinear case was studied for elliptic and convex problems in [3]. The non-linear case for periodic monotone operators was obtained in [6], using a method of energy and multiscales convergence, see Allaire and Briane [1] for this method. Applications of these results for nonlinear materials can be found in [3]. The non periodic case in homogenization was first studied in the linear case by Tartar [9] and Murat and Tartar [7], using the H-convergence theory.

In this Note, we study reiterated homogenization for monotone operators by using the periodic unfolding method introduced in Cioranescu, Damlamian and Griso [4].

We consider equations of the form:

$$\begin{cases} -\operatorname{div}\big(a_\epsilon(x, Du_\epsilon)\big) = f \quad \text{sur } \Omega, \\ u_\epsilon \in W_0^{1,p}(\Omega), \end{cases} \tag{2}$$

where Ω is a Lipschitz open bounded set of $\mathbf{R}^N$, $1 < p < \infty$, $p^{-1} + q^{-1} = 1$ and $f \in W^{-1,q}(\Omega)$.

2. Unfolding reiterated operator

Let Y and Z be two reference cells (sets having the paving property with respect to basis, defining the periods, $(b_1, \dots, b_N)$ and $(c_1, \dots, c_N)$, respectively) associated with the scales ϵ and $\epsilon\delta(\epsilon)$, with $\lim_{\epsilon\to 0} \delta(\epsilon) = 0$. For $z \in \mathbf{R}^N$, we denote $[z]_Y$ the unique integer combination $\sum_{j=1}^N k_j b_j$ of the periods such that $z - [z]_Y$ belongs to Y, and we set $\{z\}_Y = z - [z]_Y$. Then, for each $x \in \mathbf{R}^N$, we immediately see that $x = \epsilon([\frac{x}{\epsilon}]_Y + \{\frac{x}{\epsilon}\}_Y)$.

We define similarly for all $y \in \mathbf{R}^N$, $[y]_Z$ and $\{y\}_Z$.

For $u \in L^p(\Omega)$, $p \in [1, \infty]$, extended by zero outside of Ω, we define the unfolding operator $\mathcal{T}_\epsilon : L^p(\Omega) \to L^p(\Omega \times Y)$ by

$$\mathcal{T}_\epsilon(u)(x, y) = u\left(\epsilon\left[\frac{x}{\epsilon}\right]_Y + \epsilon y\right), \quad \text{for } x \in \Omega \text{ and } y \in Y.$$

Now we apply to $\tilde{u} \in L^p(\Omega \times Y)$, $p \in [1, \infty]$, extended by zero outside of $\Omega \times Y$, a similar unfolding operation for the variable y, x being seen merely as a parameter. Adding a new variable $z \in Z$, we obtain the unfolding operator $\mathcal{T}'_{\delta(\epsilon)} : L^p(\Omega \times Y) \to L^p(\Omega \times Y \times Z)$ given by

$$\mathcal{T}'_{\delta(\epsilon)}(\tilde{u})(x, y, z) = \tilde{u}\left(x, \delta(\epsilon)\left[\frac{y}{\delta(\epsilon)}\right]_Z + \delta(\epsilon) z\right), \quad \text{for } (x, y, z) \in (\Omega \times Y \times Z).$$

Therefore, for $u \in L^p(\Omega)$, we can define the reiterated unfolding operator by

$$\mathcal{T}'_{\delta(\epsilon)}\big(\mathcal{T}_\epsilon(u)\big)(x, y, z) = u\left(\epsilon\left[\frac{x}{\epsilon}\right]_Y + \epsilon\delta(\epsilon)\left[\frac{y}{\delta(\epsilon)}\right]_Z + \epsilon\delta(\epsilon) z\right), \quad \text{for } (x, y, z) \in (\Omega \times Y \times Z).$$

We immediately see that for every $x \in \Omega$, we have $\mathcal{T}'_{\delta(\epsilon)}(\mathcal{T}_\epsilon(u))(x, \{x/\epsilon\}_Y, \{\frac{\{x/\epsilon\}_Y}{\delta(\epsilon)}\}_Z) = u(x)$.

3. Homogenization results

Theorem 3.1. *Let $1 < p < \infty$, $p^{-1} + q^{-1} = 1$ and $a_\epsilon : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$, with $a_\epsilon(\cdot, \xi)$ measurable for all $\xi \in \mathbf{R}^N$ and $a_\epsilon(x, \cdot)$ continuous for almost all $x \in \Omega$, be such that*

- *there exists $c > 0$ such that for all $x \in \Omega$, $\xi_1, \xi_2 \in \mathbf{R}^N$,*
$$|\xi_1 - \xi_2|^p \leqslant c\big(a_\epsilon(x, \xi_1) - a_\epsilon(x, \xi_2) | \xi_1 - \xi_2\big), \tag{3}$$
- *there exists $C > 0$ such that for all $x \in \Omega$ and $\xi \in \mathbf{R}^N$,*
$$|a_\epsilon(x, \xi)| \leqslant C\big(1 + |\xi|^{p-1}\big). \tag{4}$$

Furthermore, we assume that for almost every $(x, y) \in \Omega \times Y$,
$$\mathcal{T}_\epsilon(a_\epsilon)(x, y, \xi) \to a_{\mathrm{hom}}(x, y, \xi), \quad \text{as } \epsilon \text{ goes to zero,} \tag{5}$$
where $a_{\mathrm{hom}}(x, y, \xi)$ is Carathéodory. Let $f_\epsilon \in W^{-1,q}(\Omega)$ be such that $f_\epsilon \to f$ strongly in $W^{-1,q}(\Omega)$.

Let $u_\epsilon \in W_0^{1,p}(\Omega)$ be the solution of the following problem
$$\begin{cases} \int_\Omega \big(a_\epsilon(x, Du_\epsilon) | D\varphi\big)\,\mathrm{d}x = \int_\Omega f_\epsilon \varphi\,\mathrm{d}x, \\ \forall \varphi \in W^{1,p}(\Omega), \end{cases} \tag{6}$$
then we have,
$$u_\epsilon \rightharpoonup u_0 \quad \text{weakly in } W_0^{1,p}(\Omega),$$
where u_0 is the first term of the unique solution $(u_0, \hat{u})$ of the following variational problem:
$$\begin{cases} u_0 \in W_0^{1,p}(\Omega),\ \hat{u} \in L^p\big(\Omega; W_{\mathrm{per}}^{1,p}(Y)\big),\ \text{with } \int_Y \hat{u}(x, y)\,\mathrm{d}y = 0, \\ \forall \Psi \in W_0^{1,p}(\Omega), \forall \Phi \in L^p\big(\Omega; W_{\mathrm{per}}^{1,p}(Y)\big), \\ \frac{1}{|Y|} \int_{\Omega \times Y} \big(a_{\mathrm{hom}}(x, y, D_x u_0 + D_y \hat{u}) | D_x \Psi(x) + D_y \Phi(x, y)\big)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f \Psi\,\mathrm{d}x. \end{cases} \tag{7}$$

Moreover, the following strong convergence holds
$$\mathcal{T}_\epsilon(Du_\epsilon) \to D_x u_0 + D_y \hat{u} \quad \text{in } L^p(\Omega \times Y \times Z; \mathbf{R}^N), \text{ when } \epsilon \text{ goes to zero.}$$

The proof of Theorem 3.1 is very simple with the unfolding method. We proceed in four steps, we shall give it in details in a forthcoming publication. First, we prove the weak convergences of the unfolding sequences of $(\mathcal{T}_\epsilon(u_\epsilon))_\epsilon$ and $(\mathcal{T}_\epsilon(Du_\epsilon))_\epsilon$, then we establish an equation satisfied by the limit. In the third step, we show that the unfolding sequence of the gradient $(\mathcal{T}_\epsilon(Du_\epsilon))_\epsilon$ converges strongly and at the end we proceed to the identification of the homogenized equation.

Moreover, if we introduce the averaging operator $\mathcal{U}_\epsilon : L^p(\Omega \times Y) \to L^p(\Omega \times Y \times Z)$ defined by $\mathcal{U}_\epsilon(v) = \frac{1}{|Y|} \int_Y v(\epsilon[\frac{x}{\epsilon}]_Y + \epsilon y, \{\frac{x}{\epsilon}\}_Y)\,\mathrm{d}y$, we obtain the following result for correctors:

Theorem 3.2. *We have the following strong convergences*:
$$D_x u_\epsilon - D_x u_0 - \mathcal{U}_\epsilon(D_y \hat{u}) \to 0 \quad \text{in } L^p(\Omega).$$

Remark 1. Condition (3) used in Theorem 3.1 implies the following condition, which was used in [6]: there exists $c > 0$ such that for all $x \in \Omega$, $\xi_1, \xi_2 \in \mathbf{R}^N$,
$$\big(a_\epsilon(x, \xi_1) - a_\epsilon(x, \xi_2) | \xi_1 - \xi_2\big) \geqslant c\big(1 + |\xi_1| + |\xi_2|\big)^{p-\alpha} |\xi_1 - \xi_2|^\alpha,$$
with $\max\{p, 2\} \leqslant \alpha < +\infty$.

Remark 2. In particular, Theorem 3.1 applies to the case where $a_\epsilon(x,\xi) = a(x, \frac{x}{\epsilon}, \xi)$, $a(x,y,\xi)$ being Y-periodic in the second argument and continuous with respect to every argument.

4. Reiterated homogenization

We can now give the reiterated result.

Theorem 4.1. *Let us assume that* $p, q, a_\epsilon, f_\epsilon, u_\epsilon$ *are as in Theorem* 3.1 *and that for almost every* $(x,y,z) \in \Omega \times Y \times Z$,

$$\mathcal{T}'_{\delta(\epsilon)}\big(\mathcal{T}_\epsilon(a_\epsilon)\big)(x,y,z,\xi) \to a_{\mathrm{homrei}}(x,y,z,\xi), \tag{8}$$

as ϵ *goes to zero, with* $\lim_{\epsilon\to 0}\delta(\epsilon) = 0$ *and* $a_{\mathrm{homrei}}(x,y,z,\xi)$ *is Carathéodory. We have*

$$u_\epsilon \rightharpoonup u_0 \quad \textit{weakly in } W_0^{1,p}(\Omega), \textit{ as } \epsilon \to 0,$$

where u_0 *is the first term of the unique solution* $(u_0, \hat{u}, \tilde{u})$ *of the variational problem*

$$\begin{cases} u_0 \in W_0^{1,p}(\Omega),\ \hat{u} \in L^p\big(\Omega; W^{1,p}_{\mathrm{per}}(Y)\big),\ \textit{with } \int_Y \hat{u}(x,y)\,\mathrm{d}y = 0, \\ \tilde{u} \in L^p\big(\Omega\times Y; W^{1,p}_{\mathrm{per}}(Z)\big),\ \textit{with } \int_Z \tilde{u}(x,y,z)\,\mathrm{d}z = 0, \\ \forall\Psi \in W_0^{1,p}(\Omega), \forall\Phi \in L^p\big(\Omega; W^{1,p}_{\mathrm{per}}(Y)\big), \forall\Theta \in L^p\big(\Omega\times Y; W^{1,p}_{\mathrm{per}}(Z)\big) \\ \frac{1}{|Y||Z|}\int_{\Omega\times Y\times Z}\big(a_{\mathrm{homrei}}(x,y,z,D_x u_0 + D_y\hat{u} + D_z\tilde{u})| \\ \quad D_x\Psi(x) + D_y\Phi(x,y) + D_z\Theta(x,y,z)\big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_\Omega f\Psi\,\mathrm{d}x. \end{cases} \tag{9}$$

Moreover, the following strong convergence holds, when ϵ *goes to zero,*

$$\mathcal{T}'_{\delta(\epsilon)}\big(\mathcal{T}_\epsilon(Du_\epsilon)\big) \to D_x u_0 + D_y\hat{u} + D_z\tilde{u} \quad \textit{in } L^p(\Omega\times Y\times Z; \mathbf{R}^N).$$

Furthermore, if we define the averaging reiterated operator $\mathcal{U}'_{\epsilon,\delta(\epsilon)}$ for all $w \in L^p(\Omega\times Y\times Z)$:

$$\mathcal{U}'_{\epsilon,\delta(\epsilon)}(w) = \frac{1}{|Y||Z|}\int_{Y\times Z} w\left(\epsilon\left[\frac{x}{\epsilon}\right]_Y + \epsilon y, \delta(\epsilon)\left[\frac{\{x/\epsilon\}_Y}{\delta(\epsilon)}\right]_Z + \delta(\epsilon)z, \left\{\frac{\{x/\epsilon\}_Y}{\delta(\epsilon)}\right\}_Z\right)\mathrm{d}y\,\mathrm{d}z,$$

we deduce the following result for the correctors:

Theorem 4.2. *We have the following strong convergences*:

$$D_x u_\epsilon - D_x u_0 - \mathcal{U}_\epsilon(D_y\hat{u}) - \mathcal{U}'_{\epsilon,\delta(\epsilon)}(D_z\tilde{u}) \to 0 \quad \textit{in } L^p(\Omega).$$

Remark 3. In particular, Theorem 4.1 applies to the case where $a_\epsilon(x,\xi) = a(x, \frac{x}{\epsilon}, \frac{\{x/\epsilon\}_Y}{\delta(\epsilon)}, \xi)$, $a(x,y,z,\xi)$ being Y-periodic in the second argument, Z-periodic in the third and continuous with respect to every argument. Furthermore, if $Y = Z$ and $\frac{1}{\delta(\epsilon)}$ is an integer, then $a_\epsilon(x,\xi) = a(x, \frac{x}{\epsilon}, \frac{\{x/\epsilon\}_Y}{\delta(\epsilon)}, \xi)$ also equals $a(x, \frac{x}{\epsilon}, \frac{x}{\epsilon\delta(\epsilon)}, \xi)$. This generalizes the results of [6] in which it was necessary to assume a stronger condition on $a(x,\cdot,z,\xi)$.

Remark 4. Theorem 4.1 can easily be generalized to the case of n times reiteration if we assume that for almost all $(x, y_1, \ldots, y_n) \in \Omega \times Y_1 \times \cdots \times Y_n$, the sequence $((\mathcal{T}'_{\delta_n(\epsilon)} \circ \cdots \circ \mathcal{T}'_{\delta_1(\epsilon)} \circ \mathcal{T}_\epsilon)a_\epsilon(x, y_1, \ldots, y_n, \xi))_\epsilon$ converges to a Carathéodory operator $a_{\mathrm{homrei}}(x, y_1, \ldots, y_n, \xi)$.

Remark 5. As a particular case, Theorem 4.1 applies to the following situation: $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, with Ω_1, Ω_2 disjoint Lipschitzian open sets and $a_\epsilon(x, \xi)$ is such that $a_\epsilon(x, \xi) = a^1(x, \xi)$ if $x \in \Omega_1$ and $a_\epsilon(x, \xi) = a^2(x, \{\frac{x}{\epsilon}\}_Y, \{\frac{\{x/\epsilon\}_Y}{\delta(\epsilon)}\}_Z, \xi)$ if $x \in \Omega_2$, where a^1 and a^2 are continuous with respect to every argument and satisfy (3), (4). This is more general than what was treated in [6].

Remark 6. As in the linear case, see [4,5], Theorem 4.1 can be generalized to perforated domains.

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References

[1] G. Allaire, M. Briane, Multiscale convergence and reiterated homogenization, Proc. Roy. Soc. Edinburgh Sect. A 126 (1996) 297–342.
[2] A. Bensoussan, J.L. Lions, G.C. Papanicolaou, Asymptotic Analysis for Periodic Structure, North-Holland, Amsterdam, 1978.
[3] A. Braides, D. Lukkassen, Reiterated homogenization of integral functionals, Math. Models Methods Appl. Sci. 10 (1) (2000) 1–25.
[4] D. Cioranescu, A. Damlamian, G. Griso, Periodic unfolding and homogenization, C. R. Acad. Sci. Paris, Ser. I 335 (1) (2002) 99–104.
[5] A. Damlamian, P. Donato, H^0-convergence and iterated homogenization, Asymptotic Anal. 39 (2004) 45–60.
[6] J.L. Lions, D. Lukkassen, L.E. Persson, P. Wall, Reiterated homogenization of nonlinear monotone operators, Chinese Ann. Math. Ser. B 22 (2001) 1–12.
[7] F. Murat, L. Tartar, H-convergence, in: Topics in the Mathematical Modelling of Composite Materials, 1997, pp. 21–43.
[8] E. Sanchez-Palencia, Non-Homogeneous Media and Vibration Theory, Springer-Verlag, Berlin, 1980, p. 1–12.
[9] L. Tartar, Cours Peccot, Collège de France, 1977.