Calculus of Variations

Homogenization of a Ginzburg–Landau functional

Leonid Berlyand [a], Doina Cioranescu [b], Dmitry Golovaty [c]

[a] *Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA*
[b] *Université Pierre et Marie Curie (Paris VI), laboratoire d'analyse numérique, 4, place Jussieu, 75252 Paris cedex 05, France*
[c] *Department of Theoretical and Applied Mathematics, The University of Akron, Akron, OH 44325, USA*

Abstract

We consider a nonlinear homogenization problem for a Ginzburg–Landau functional with a (positive or negative) surface energy term describing a nematic liquid crystal with inclusions. Assuming that sizes and distances between inclusions are of the same order ε, we obtain a limiting functional as $\varepsilon \to 0$. We generalize the method of mesocharacteristics to show that a corresponding homogenized problem for arbitrary, periodic or non-periodic geometries is described by an anisotropic Ginzburg–Landau functional. We give computational formulas for material characteristics of an effective medium. ***To cite this article: L. Berlyand et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).***
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Résumé

Homogénéisation d'une fonctionnelle de Ginzburg–Landau. Nous considérons un problème non linéaire d'homogénéisation pour une fonctionnelle de Ginzburg–Landau avec un terme correspondant à l'energie de surface (positive ou négative) décrivant un milieu cristallin liquide avec des inclusions. On suppose que la distance ε entre les inclusions est comparable à leur taille. En appliquant la méthode des mesocharactéristiques nous donnons la fonctionnelle limite lorsque $\varepsilon \to 0$ et prouvons que le problème homogénéisé pour des géometries arbitraires (périodiques ou non), est décrit par une fonctionnelle de Ginzburg–Landau anisotrope. Nous donnons des formules pour calculer les caractéristiques effectives des matériaux ainsi obtenus. ***Pour citer cet article : L. Berlyand et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).***
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Version française abrégée

Nous considérons un modèle mathématique d'une classe de cristaux liquides nématiques avec des inclusions d'un volume total donné. L'énergie d'un tel milieu est décrite par une fonctionnelle de Ginzburg–Landau (2) avec un terme (positif ou négatif) de surface. La densité de l'énergie surfacique, la taille des hétérogénéités ainsi que la distance moyenne entre les inclusions, sont décrites par un petit paramètre ε. Ainsi, les deux énergies-volumique et surfacique-sont du même ordre de grandeur.

La présence d'un terme de surface dans le problème variationnel implique que le minimiseur $\mathbf{u}_\varepsilon$ de (2), solution de l'équation non linéaire de Ginzburg–Landau (3), vérifie une condition de Robin sur la surface des inclusions. Notre but est de décrire le comportement asymptotique de $\mathbf{u}_\varepsilon$ lorsque $\varepsilon \to 0$. Le Théorème 1 donne le problème limite dans un cadre géométrique assez général. Le domaine Ω_ε de (2) est obtenu en retirant d'un domaine fixe Ω un ensemble de boules $\{B_{\varepsilon i}\}_{i=1}^{N_\varepsilon}$ de centre $\mathbf{x}_{\varepsilon i}$ et de rayon $r_{\varepsilon i}$, vérifiant les conditions (1). La preuve du théorème est basée sur la méthode des mésocaractéristiques de [9] et [3]. Cela revient à considèrer, à la place de (2), le problème de minimisation locale (8), où K_h^α, l'"échantillon", est un cube de côté h et de centre $\mathbf{x}_\alpha \in \Omega$. La mésocaractéristique $T(\varepsilon, h, \mathbf{x}_\alpha, \mathbf{A}, \ell)$ s'écrit sous la forme (10), en termes des minimiseurs $\mathbf{w}_{0j}^{\varepsilon,h,\alpha}$, $\mathbf{w}_{ik}^{\varepsilon,h,\alpha}$ de (7) pour des choix particuliers de $\mathbf{A}$ et ℓ. En utilisant les estimations (13), on établit ensuite des convergences pour (10) lorsque $\varepsilon \to 0$. Dans le cas périodique, les coefficients homogénéisés sont donnés par les formules explicites (16).

1. Introduction

We consider a mathematical model for a class of nematic liquid crystal (NLC) composites with a high volume fraction of small inclusions. A NLC is described by a Ginzburg–Landau functional with a (positive or negative) surface energy term. We assume that the surface energy density and the sizes of inclusions are controlled by a small parameter ε. The intensity of the surface energy density is chosen to model weak anchoring conditions on an inclusion-nematic host interface. In this case, both the surface and the bulk energy terms provide comparable contributions to the overall energy of the composite. To model composites with a high total volume fraction of inclusions, we let the distances between inclusions be of order ε. The case of a geometry characterized by a small volume fraction of inclusions was considered in [2,4]. It was shown that the presence of inclusions is accounted for by an explicitly computed effective potential.

The presence of the surface energy term in the variational formulation of our problem implies that the minimizer is subject to Robin boundary conditions on surfaces of inclusions. For linear scalar problems (Laplace operator) the homogenization problem for perforated domains with a Robin boundary condition on the boundaries of holes was studied in [7]. In [3] the case of large holes, where the homogenized operator becomes anisotropic was considered by using the method of mesocharacteristics. It is this method that we employ here.

The results of this Note are proved under rather general conditions: the inclusions remain sufficiently far apart, the domain is not required to have periodic geometry, and the surface energy term can be negative. The main consequence of the lack of non-negativity is that there is no a priori lower bound on the energy. This bound has to be established independently.

We show that we can account for inclusions via the anisotropy of the homogenized operator as well as an effective potential. The only contribution of the surface energy to the homogenized energy is through this potential which accounts for misalignment of molecules on surfaces of inclusions. Physically, when the anchoring is weak, there are no significant bulk elastic distortions that can be attributed to surface effects. The results we present here are detailed in [5].

2. Statement of the problem

Let $\Omega \in \mathbb{R}^3$ be an open bounded set with a smooth boundary and let $\{B_{\varepsilon i}\}_{i=1}^{N_\varepsilon}$ be a set of balls $B_{\varepsilon i} = B(\mathbf{x}_{\varepsilon i}, r_{\varepsilon i})$, such that

$$2\varepsilon C_1 \leqslant \min\left\{\min_j \text{dist}(\mathbf{x}_{\varepsilon i}, \mathbf{x}_{\varepsilon j}), \text{dist}(\mathbf{x}_{\varepsilon i}, \partial\Omega)\right\} \leqslant 2\varepsilon C_2; \qquad \varepsilon C_3 \leqslant r_{\varepsilon i} \leqslant \varepsilon C_4, \tag{1}$$

for every $1 \leqslant i \leqslant N_\varepsilon$ uniformly in ε, where $C_3 < C_4 < C_1 < C_2$.

Set $\Omega_\varepsilon = \Omega \setminus \bigcup_i B_{\varepsilon i}$, $S_{\varepsilon i} = \partial B_{\varepsilon i}$, $S_\varepsilon = \bigcup_i S_{\varepsilon i}$ and consider the functional

$$E_\varepsilon[\mathbf{u}] = \int_{\Omega_\varepsilon} \left\{\frac{1}{2}|\nabla\mathbf{u}|^2 + \frac{1}{4\delta^2}\big(|\mathbf{u}|^2 - 1\big)^2\right\} d\mathbf{x} + \frac{\varepsilon q}{2}\int_{S_\varepsilon} (\mathbf{u}, \mathbf{n})^2 d\sigma. \tag{2}$$

Here the bulk energy density corresponds to the equal elastic constants case in the Ericksen's model for nematic liquid crystals with variable degree of orientation in the absence of flow [8,6]. The boundary term is the Rapini–Papoular phenomenological surface free energy [11].

Let $\mathbf{u}_\varepsilon$ be a minimizer of $E_\varepsilon[\mathbf{u}]$ over the class $H^1_{\mathbf{U}}(\Omega_\varepsilon) := \{\mathbf{u} \in H^1(\Omega_\varepsilon) | (\mathbf{u} - \mathbf{U})|_{\partial\Omega} = \mathbf{0}\}$, where the function $\mathbf{U} \in C^2(\Omega): \mathbb{R}^3 \to \mathbb{R}^3$. The minimizer of (2) satisfies the equation

$$\begin{cases} -\Delta\mathbf{u}_\varepsilon + \dfrac{1}{\delta^2}\big(|\mathbf{u}_\varepsilon|^2 - 1\big)\mathbf{u}_\varepsilon = \mathbf{0} & \text{in } \Omega_\varepsilon, \\ \nabla\mathbf{u}_\varepsilon\mathbf{n} + q_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{n})\mathbf{n} = \mathbf{0} & \text{on } S_\varepsilon, \end{cases} \tag{3}$$

where $\mathbf{n}$ is the unit vector, normal to $\bigcup_i S_{\varepsilon i}$. There exists at least one global minimizer of (3) in $H^1_{\mathbf{U}}(\Omega_\varepsilon)$ that is also a solution of (2) (see, for instance [10]).

Our goal is to find the partial differential equation satisfied by an asymptotic limit of the minimizers of $E_\varepsilon[\mathbf{u}]$ in order to obtain a homogenized description of the problem (2)–(3) as $\varepsilon \to 0$. The main difficulty results from the fact that q can be negative, and there are no immediate bounds on the $H^1(\Omega_\varepsilon)$-norm of $\mathbf{u}_\varepsilon$ that would normally follow from (3).

3. Compactness

We begin by proving compactness of a sequence of minimizers $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ of (2) as $\varepsilon \to 0$, when $q < 0$. Throughout the Note, C will denote a constant independent of ε.

By using a result from [4], we first show the following estimate:

$$E_\varepsilon[\mathbf{u}_\varepsilon] \geqslant \left(\frac{1}{2} - C\varepsilon^2\right)\int_{\Omega_\varepsilon} |\nabla\mathbf{u}_\varepsilon|^2 d\mathbf{x} + \frac{1}{4\delta^2|\Omega_\varepsilon|}\left[\int_{\Omega_\varepsilon} |\mathbf{u}_\varepsilon|^2 d\mathbf{x}\right]^2 - \left(C + \frac{1}{2\delta^2}\right)\int_{\Omega_\varepsilon} |\mathbf{u}_\varepsilon|^2 d\mathbf{x}. \tag{4}$$

Then, for ε small enough, (4) and Hölder's inequality imply that

$$\int_{\Omega_\varepsilon} |\nabla\mathbf{u}_\varepsilon|^2 d\mathbf{x} + \int_{\Omega_\varepsilon} |\mathbf{u}_\varepsilon|^2 d\mathbf{x} \leqslant C, \quad \text{uniformly in } \varepsilon. \tag{5}$$

Due to assumptions (1), the domains Ω_ε are strongly connected (cf. [9]). Consequently, there exists an extension operator $\mathcal{P} \in \mathcal{L}(H^1(\Omega_\varepsilon); H^1(\Omega))$ satisfying $\|\mathcal{P}\mathbf{v}\|_{H^1(\Omega)} \leqslant C\|\mathbf{v}\|_{H^1(\Omega_\varepsilon)}$. Note that a sufficient condition for this was proposed in [1]: it holds as long as there is a "security layer" around each inclusion, with a thickness of order ε as $\varepsilon \to 0$. From (5) it follows that, up to a subsequence, there exists $\mathbf{u}_0 \in H^1(\Omega)$ such that

$$\mathcal{P}\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u}_0 \quad \text{weakly in } H^1(\Omega), \quad \text{hence} \quad \mathcal{P}\mathbf{u}_\varepsilon \to \mathbf{u}_0 \quad \text{strongly in } L^2(\Omega). \tag{6}$$

By the trace theorem, $(\mathbf{u}_0 - \mathbf{U})|_{\partial\Omega} = \mathbf{0}$. The extension of this result to the case $q > 0$ is trivial.

4. Main result

The method we present here is based on the ideas introduced in [9] and [3]. First, we define a mesocharacteristic that describes the homogenized solution. Set

$$T(\varepsilon, h, \mathbf{x}_\alpha, \mathbf{A}, \ell) = \min_{\mathbf{w} \in H^1(K_h^\alpha \cap \Omega_\varepsilon)} \left[\int_{K_h^\alpha \cap \Omega_\varepsilon} \left\{ |\nabla \mathbf{w}|^2 + h^{-2-\theta} |\mathbf{w} - \ell - \mathbf{A}(\mathbf{x} - \mathbf{x}_\alpha)|^2 \right\} d\mathbf{x} + \varepsilon q \int_{\partial \Omega_{\varepsilon h}^\alpha} (\mathbf{w}, \mathbf{n})^2 \, d\sigma \right]. \quad (7)$$

Here $\theta > 0$, both the vector $\ell \in \mathbb{R}^3$ and the matrix $\mathbf{A} \in M^{3 \times 3}$ are arbitrary, $K_h^\alpha = K_h(\mathbf{x}_\alpha)$ is a 3-dimensional cube with a side of length h and centered at a given $\mathbf{x}_\alpha \in \Omega$. Depending on the sign of the parameter q, we distinguish between the following two choices for $\Omega_{\varepsilon h}^\alpha$: when $q > 0$, $\Omega_{\varepsilon h}^\alpha = \Omega_\varepsilon \cap K_h^\alpha$; when $q < 0$, $\Omega_{\varepsilon h}^\alpha = \bigcup_\mu B_{\varepsilon \mu}$, where the union is taken over all balls $B_{\varepsilon \mu}$ having centers $\mathbf{x}_{\varepsilon \mu} \in K_h^\alpha$, such that $\mathrm{dist}\{\mathbf{x}_{\varepsilon \mu}, \partial K_h^\alpha\} \geqslant \varepsilon C_1$. Note that, when $q < 0$, we *disregard* contributions from surfaces of the inclusions that lie too close to ∂K_h^α. The distinction between the definitions of $\Omega_{\varepsilon h}^\alpha$ for different choices of q is necessary because the boundary term in the definition of T can become very large when q is negative.

Instead of considering the minimizers of the energy over the entire domain Ω_ε, we consider a local minimization problem for the functional

$$\frac{1}{h^3} \left\{ \int_{K_h^\alpha \cap \Omega_\varepsilon} \left\{ |\nabla \mathbf{w}_\varepsilon|^2 + h^{-2-\theta} |\mathbf{w}_\varepsilon - \ell - \mathbf{A}(\mathbf{x} - \mathbf{x}_\alpha)|^2 \right\} d\mathbf{x} + \varepsilon q \int_{K_h^\alpha \cap S_\varepsilon} (\mathbf{w}_\varepsilon, \mathbf{n})^2 \, d\sigma \right\}, \quad (8)$$

where the penalty term $h^{-5-\theta} |\mathbf{w}_\varepsilon - \ell - \mathbf{A}(\mathbf{x} - \mathbf{x}_\alpha)|^2$ enforces the closeness of minimizers $\mathbf{w}_\varepsilon$ to the linear function $\ell + \mathbf{A}(\mathbf{x} - \mathbf{x}_\alpha)$ in $L^2(K_h^\alpha \cap \Omega_\varepsilon)$ when h is small. If we can show that, for some $\theta > 0$, the penalty term is of order $o(1)$ when $\varepsilon \to 0$, then the asymptotic limit of the local functional (8) should yield an effective density in terms of $\mathbf{A}$ and ℓ when $\varepsilon, h \to 0$.

Denote by $\mathbf{w}_{0i}^{\varepsilon, h, \alpha}$ a minimizer of (7) when $\mathbf{A} = \mathbf{0}$ and $\ell = \mathbf{e}_i$. Then $\mathbf{w}_{0i}^{\varepsilon, h, \alpha}$ satisfies

$$\begin{cases} -\Delta \mathbf{w}_{0i}^{\varepsilon, h, \alpha} + h^{-2-\theta} \mathbf{w}_{0i}^{\varepsilon, h, \alpha} = h^{-2-\theta} \mathbf{e}_i & \text{in } K_h^\alpha \cap \Omega_\varepsilon, \\ \dfrac{\partial}{\partial \mathbf{n}} \mathbf{w}_{0i}^{\varepsilon, h, \alpha} + \varepsilon q (\mathbf{w}_{0i}^{\varepsilon, h, \alpha} \cdot \mathbf{n}) \mathbf{n} = \mathbf{0} & \text{on } \partial \Omega_{\varepsilon h}^\alpha, \\ \dfrac{\partial}{\partial \mathbf{n}} \mathbf{w}_{0i}^{\varepsilon, h, \alpha} = \mathbf{0} & \text{on } \partial \{K_h^\alpha \cap \Omega_\varepsilon\} \setminus \partial \Omega_{\varepsilon h}^\alpha. \end{cases} \quad (9)$$

If $\mathbf{w}_{ik}^{\varepsilon, h, \alpha}$ is a minimizer of (7) when $\mathbf{A} = (\mathbf{e}_{ik})_{1 \leqslant i, j \leqslant 3}$ and $\ell = \mathbf{0}$, then it will satisfy a system similar to (9). Due to the linearity, the minimizer of (7) corresponding to an arbitrary matrix $\mathbf{A} = a_{ik} \mathbf{e}_{ik}$ and a vector $\ell = \ell_j \mathbf{e}_j$, is given by $\mathbf{w}_{\mathbf{A}, \ell}^{\varepsilon, h, \alpha} = \ell_j \mathbf{w}_{0j}^{\varepsilon, h, \alpha} + a_{ik} \mathbf{w}_{ik}^{\varepsilon, h, \alpha}$, where the summation over repeated indices is assumed. Then we deduce from (7) that

$$T(\varepsilon, h, \mathbf{x}_\alpha, a_{ik} \mathbf{e}_{ik}, \ell_j \mathbf{e}_j) = t_{ikmn}^{\varepsilon, h}(\mathbf{x}_\alpha) a_{ik} a_{mn} + c_{ikj}^{\varepsilon, h}(\mathbf{x}_\alpha) a_{ik} \ell_j + r_{jp}^{\varepsilon, h}(\mathbf{x}_\alpha) \ell_j \ell_p, \quad (10)$$

where, for instance (for every $i, j, k, m, n, p = 1, 2, 3$),

$$t_{ikmn}^{\varepsilon, h}(\mathbf{x}_\alpha) = \int_{K_h^\alpha \cap \Omega_\varepsilon} \left(\nabla \mathbf{w}_{ik}^{\varepsilon, h, \alpha}, \nabla \mathbf{w}_{mn}^{\varepsilon, h, \alpha} \right) dx + \varepsilon q \int_{\partial \Omega_{\varepsilon h}^\alpha} (\mathbf{w}_{ik}^{\varepsilon, h, \alpha}, \mathbf{n})(\mathbf{w}_{mn}^{\varepsilon, h, \alpha}, \mathbf{n}) \, d\sigma$$

$$+ h^{-2-\theta} \int_{K_h^\alpha \cap \Omega_\varepsilon} \left(\mathbf{w}_{ik}^{\varepsilon, h, \alpha} - \mathbf{e}_{ik}(\mathbf{x} - \mathbf{x}_\alpha), \mathbf{w}_{mn}^{\varepsilon, h, \alpha} - \mathbf{e}_{mn}(\mathbf{x} - \mathbf{x}_\alpha) \right) dx. \quad (11)$$

The expressions for $c_{ikj}^{\varepsilon, h}(\mathbf{x}_\alpha)$ and $r_{jp}^{\varepsilon, h}(\mathbf{x}_\alpha)$ are of the same integral type.

Assume that for every $\mathbf{x}_\alpha \in \Omega$ and $i, j, k, m, n, p = 1, 2, 3$, there exist s, t_{ikmn}, and r_{jp} such that

$$\text{(i)} \quad \lim_{h \to 0} \lim_{\varepsilon \to 0} \frac{|K_h^\alpha \cap \Omega_\varepsilon|}{h^3} = s(\mathbf{x}_\alpha); \qquad \text{(ii)} \quad \lim_{h \to 0} \overline{\lim_{\varepsilon \to 0}} \frac{r_{jp}^{\varepsilon, h}(\mathbf{x}_\alpha)}{h^3} = \lim_{h \to 0} \underline{\lim_{\varepsilon \to 0}} \frac{r_{jp}^{\varepsilon, h}(\mathbf{x}_\alpha)}{h^3} = r_{jp}(\mathbf{x}_\alpha),$$

$$\text{(iii)} \quad \lim_{h \to 0} \overline{\lim_{\varepsilon \to 0}} \frac{t_{ikmn}^{\varepsilon, h}(\mathbf{x}_\alpha)}{h^3} = \lim_{h \to 0} \underline{\lim_{\varepsilon \to 0}} \frac{t_{ikmn}^{\varepsilon, h}(\mathbf{x}_\alpha)}{h^3} = t_{ikmn}(\mathbf{x}_\alpha), \quad (12)$$

uniformly in h for some $\theta \in (0, 2)$, where the functions s, t_{ikmn}, and r_{jp} are continuous at $\mathbf{x}_\alpha \in \Omega$, and the tensor $\mathcal{T} = (t_{ikmn})_{1 \leqslant i,k,m,n \leqslant 3}$ is positive definite in Ω. The function s in (12), characterizes the porosity of the material. Note that (12i) implies the following convergence:

$$\chi_{\Omega_\varepsilon} \rightharpoonup s \quad \text{weakly} \star \text{ in } L^\infty(\Omega). \tag{13}$$

The following result describes the problem satisfied by the limit of minimizers of $E_\varepsilon[\mathbf{u}]$ as $\varepsilon \to 0$.

Theorem 4.1. *Under assumptions* (12)*, the sequence of minimizers* $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ *of the functionals* $E_\varepsilon[\mathbf{u}_\varepsilon]$ *converges in the sense of* (6) *to the solution* $\mathbf{u}_0$ *of the problem*

$$\begin{cases} -\dfrac{\partial}{\partial x_k}\left(t_{ikmn}(\mathbf{x})\dfrac{\partial u_{0m}}{\partial x_n}\right) + r_{ik}(\mathbf{x})u_{0k} + \dfrac{s(\mathbf{x})}{\delta^2}\big(|\mathbf{u}_0|^2 - 1\big)u_{0i} = 0 & \text{in } \Omega, \\ \mathbf{u}_0 - \mathbf{U} = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{14}$$

for $i = 1, 2, 3$*, where* t_{ikmn}*,* r_{jp}*, and* s *are defined in* (12)*. Moreover,* (12) *hold for all* $\theta > 0$*.*

Corollary 4.2. *The functional* F *defined by*

$$F[\mathbf{v}] := \int_\Omega \left\{ \sum_{ikmn} t_{ikmn}\frac{\partial v_i}{\partial x_k}\frac{\partial v_m}{\partial x_n} + \sum_{jp} r_{jp}v_j v_p + \frac{s(x)}{4\delta^2}\big(|\mathbf{v}|^2 - 1\big)^2 \right\} \mathrm{d}\mathbf{x}, \quad \mathbf{v} \in H^1(\Omega), \tag{15}$$

is a $\Gamma(H^1(\Omega))$*-limit of the sequence* E_ε*.*

Remark 1. The definitions of 'homogenized coefficients' imply that neither the limit problem (14) nor $\mathbf{u}_0$ depend on the choice of the extension operator $\mathcal{P}$.

Remark 2. The effective contributions from the bulk and the surface energy terms described by the tensor $\mathcal{T}$ and the functions r_{ik} ($i, k = 1, 2, 3$) respectively, are completely decoupled and there are no elastic interactions between inclusions under our assumptions on their distribution. Thus the anchoring effects (surface misalignment) are accounted for in the potential terms $r_{ik}u_{0k}$.

Note that (14) does not contain a contribution from a limit of $c_{ikj}^{h,\varepsilon}$. Indeed, we have

Lemma 4.3. *Under conditions* (12)*, one has* $\lim_{h,\varepsilon \to 0} 1/h^3 c_{ikj}^{h,\varepsilon} = 0$*.*

5. Sketch of the proof

The proofs of Theorem 4.1 and Lemma 4.3 are based on a set of sharp estimates of the H^1-norms of functions $\mathbf{w}_{0i}^{\varepsilon,h_\varepsilon,\alpha}$ and $\mathbf{w}_{ik}^{\varepsilon,h_\varepsilon,\alpha}$-minimizers of (7) when K_h^α is replaced by $K_{h_\varepsilon}^\alpha$, where $h_\varepsilon = h + 4C_1\varepsilon$ and C_1 is the constant introduced in (1). When ε and h are small, one has

$$\int_{K_{h_\varepsilon}^\alpha \cap \Omega_\varepsilon} |\nabla\mathbf{w}_{0i}^{\varepsilon,h_\varepsilon,\alpha}|^2 \,\mathrm{d}\mathbf{x} \leqslant Ch^3; \qquad h^{-2-\theta}\int_{K_{h_\varepsilon}^\alpha \cap \Omega_\varepsilon} |\mathbf{w}_{0i}^{\varepsilon,h_\varepsilon,\alpha} - \mathbf{e}_i|^2 \,\mathrm{d}\mathbf{x} \leqslant Ch^3,$$

$$\int_{K_{h_\varepsilon}^\alpha \cap \Omega_\varepsilon} |\nabla\mathbf{w}_{ik}^{\varepsilon,h_\varepsilon,\alpha}|^2 \,\mathrm{d}\mathbf{x} \leqslant Ch^3; \qquad h^{-2-\theta}\int_{K_{h_\varepsilon}^\alpha \cap \Omega_\varepsilon} \big|\mathbf{w}_{ik}^{\varepsilon,h_\varepsilon,\alpha} - \mathbf{e}_{ik}(\mathbf{x} - \mathbf{x}_\alpha)\big|^2 \,\mathrm{d}\mathbf{x} \leqslant Ch^3. \tag{16}$$

The proof of Theorem 4.1 consists of two steps. We first obtain an upper bound of the energy and show that $\overline{\lim}_{\varepsilon\to 0}E_\varepsilon[\mathbf{u}_\varepsilon] \leqslant F[\mathbf{z}]$ for every $\mathbf{z} \in C^2(\Omega)$. The main idea is to approach an arbitrary $\mathbf{z} \in C^2(\Omega)$ by a sequence of functions $\mathbf{Z}^{\varepsilon,h}$ written locally in terms of $\mathbf{w}_{ik}^{\varepsilon,h_\varepsilon,\alpha}$ and $\mathbf{w}_{0i}^{\varepsilon,h_\varepsilon,\alpha}$. Then each term of $E_\varepsilon[\mathbf{Z}^{\varepsilon,h}]$ is estimated by using (10), (12) and (16). In the second step, we prove that $\underline{\lim}_{\varepsilon\to 0}E_\varepsilon[\mathbf{u}_\varepsilon] \geqslant F[\mathbf{u}_0]$. The fact that the assumptions of Theorem 4.1 are independent of θ is proved by following the arguments of [9].

6. The periodic case

Theorem 4.1 was proved under a very general assumption that the limits (12) exist. This assumption holds for a wide array of composite materials. In particular, we show that these limits exist for periodic geometries, and can be explicitly computed.

Suppose that the centers $\mathbf{x}_{\varepsilon i}$ of the spheres $\{B_{\varepsilon i}\}_{i=1}^{N_\varepsilon}$ are at the nodes of a periodic lattice with a period $\gamma_n \varepsilon$ in the direction $\mathbf{e}_n$, where $n = 1, 2, 3$. Assume also that $r_{\varepsilon i} = r\varepsilon$ for all $i = 1, \ldots, N_\varepsilon$ where $2r < \min_n \gamma_n$. Then we can choose a cell $G_{\varepsilon i}$ in Ω by removing the ball $B_{\varepsilon i}$ of the radius $r\varepsilon$ from the parallelepiped with sides $\gamma_1 \varepsilon, \gamma_2 \varepsilon$ $\gamma_3 \varepsilon$ and centered at the point $\mathbf{x}_{\varepsilon i}$. By fixing the origin at $\mathbf{x}_{\varepsilon i}$ and rescaling the cell by the factor $r\varepsilon$, we obtain the parallelepiped $\Gamma = \{\mathbf{x} \in \Omega : |\mathbf{x}_n| < \gamma_n/(2r), \ n = 1, 2, 3\}$ with the ball $B = B(\mathbf{0}, 1)$ removed from it. Let $\Pi = \Gamma \setminus B$. and for every $i, k = 1, 2, 3$, consider the following 'cell' problem:

$$\begin{cases} \Delta \mathbf{u}_{ik} = 0 & \text{in } \Pi, \\ \mathbf{u}_{ik} = \pm \dfrac{\gamma_k}{2r} \mathbf{e}_i & \text{on } \partial \Gamma \cap \left\{ x_k = \pm \dfrac{\gamma_k}{2r} \right\}, \\ \dfrac{\partial \mathbf{u}_{ik}}{\partial \mathbf{n}} = 0 & \text{on } \partial \Gamma \setminus \left\{ x_k = \pm \dfrac{\gamma_k}{2r} \right\}. \end{cases} \tag{17}$$

Set $A_{ikmn} = \int_\Pi (\nabla \mathbf{u}_{ik}, \nabla \mathbf{u}_{mn}) \, d\mathbf{x}$, where $i, k, m, n = 1, 2, 3$. It is easily seen from the symmetry of (17) that $A_{ikmn} = 0$, whenever $i \neq k$ or $m \neq n$. Then we prove that $t_{ikmn} = \frac{r^3}{\gamma_1 \gamma_2 \gamma_3} A_{ikmn}$; $r_{jp} = \frac{4\pi q r^2}{3\gamma_1 \gamma_2 \gamma_3} \delta_{jp}$; $s = 1 - \frac{4\pi r^3}{3\gamma_1 \gamma_2 \gamma_3}$. Hence, the homogenized problem corresponds to the anisotropic Ginzburg–Landau functional $F[\mathbf{u}] = \int_\Omega \{ \frac{3v_\Gamma}{4\pi} \sum_{ikmn} A_{ikmn} \frac{\partial u_i}{\partial x_k} \frac{\partial u_m}{\partial x_n} + \frac{q v_\Gamma}{r} |\mathbf{u}|^2 + \frac{1-s}{4\delta^2} (|\mathbf{u}|^2 - 1)^2 \} \, d\mathbf{x}$, where $v_\Gamma = (4\pi r^3)/(3\gamma_1 \gamma_2 \gamma_3)$, is the volume fraction of the inclusions in the cell Γ.

Acknowledgements

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