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Functional Analysis

On the invariance of weighted Sobolev spaces under Fourier transform

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Abstract

We prove that the image of a class of weighted Sobolev spaces under Fourier transform can be characterized by means of an algebraic elementary calculus on multi-indices. A consequence of this result is the invariance of a family of weighted spaces under Fourier transform. ***To cite this article: T.Z. Boulmezaoud, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Invariance d'espaces de Sobolev avec poids par transformation de Fourier. On montre que l'image par transformation de Fourier d'une classe d'espaces de Sobolev avec poids peut être caractérisée par un calcul élementaire sur les multi-indices. Une conséquence directe de ce résultat est l'invariance de certains espaces à poids par transformation de Fourier. ***Pour citer cet article : T.Z. Boulmezaoud, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

L'une des propriétés remarquables de la transformation de Fourier est qu'elle permute la décroissance à l'infini et la régularité. A titre d'exemple, rappelons que l'espace de Sobolev H^s est transformé en L^2_s, l'espace de toutes les fonctions mesurables dont le carré est intégrable par rapport à la mesure de densité $(1+|\xi|^2)^s$. Il est donc tout à fait naturel de s'interroger sur l'image par cette transformation des espaces de Sobolev avec poids, qui combinent une description de la régularité et du comportement à l'infini des fonctions. Les espaces à poids s'avèrent en effet un cadre très adéquat pour résoudre les équations aux dérivées partielles dans des domaines non bornés (voir [1] et références incluses). Dans le cas où le domaine non borné est l'espace tout entier, la transformation de Fourier devient un outil puissant, ce qui rend souvent nécessaire la connaissance de l'image des espaces utilisés. Ici on veut

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caractériser essentiellement l'image par cette transformation des espaces de fonctions définies sur $\mathbb{R}^n$, localement intégrable et satisfaisant des conditions de la forme

$$\left(1+|x|^2\right)^{b_\mu/2} x^\lambda D^\mu u \in L^p(\mathbb{R}^n) \quad \text{pour tous } |\mu| \leqslant m \text{ et } |\lambda| = a_\mu,$$

où les exposants a_μ et b_μ, $|\mu| \leqslant m$, sont des réels positifs ou nuls dépendant uniquement de l'ordre de dérivation $|\mu|$ (espaces isotropes). On s'intéresse dans un premier temps au cas où $a_\mu = 0$ et $b_\mu = r$ pour tout $|\mu| \leqslant m$, r étant un réel positif. Un tel espace est noté $\mathscr{V}_r^{m,p}(\mathbb{R}^n)$. Après avoir étendu cette définition à des valeurs réelles de m, on établit le résultat suivant :

Théorème 0.1. *Soient r, s et p trois réels tels que $s \geqslant 0$, $r \geqslant 0$ et $1 < p \leqslant 2$. Alors, on a l'injection continue*

$$\mathscr{F}\left(\mathscr{V}_s^{r,p}(\mathbb{R}^n)\right) \hookrightarrow \mathscr{V}_r^{s,p^*}(\mathbb{R}^n), \tag{1}$$

où $1/p + 1/p^* = 1$. De plus, cette injection devient une identité si $p = 2$. En particulier, pour tout $s \geqslant 0$, l'espace $\mathscr{V}_s^{s,2}(\mathbb{R}^n)$ est invariant par transformation de Fourier.

Afin de généraliser ce résultat on introduit un calcul de multi-indices. L'objectif étant de montrer que l'image d'un espace à poids du type ci-dessus, avec des exposants a_μ, b_μ naturels mais quelconques, peut être obtenue par un calcul algébrique élémentaire sur les multi-indices. L'idée de départ consiste à associer à tout triplet (m, β, ω), où $m \geqslant 0$ est un entier, β et ω sont deux $(m+1)$-uplets, c'est-à-dire des applications de $\{0, \ldots, m\}$ dans $\mathbb{N}$, l'espace de Sobolev avec poids noté $\mathscr{W}^p(m, \beta, \omega)$ de toutes les distributions u définies sur $\mathbb{R}^n$ et satisfaisant les conditions $(1+|x|^2)^{w(|\mu|)-\beta(|\mu|)} x^\lambda D^\mu u \in L^p(\mathbb{R}^n)$, pour tous λ et μ tels que $|\mu| \leqslant m$ et $|\lambda| = \beta(|\mu|)$. On dit que ce triplet (m, β, ω) est presque $\mathscr{F}$-cohérent s'il satisfait les conditions (a)–(c) de la Définition 2.1. Si de plus $\beta(m) \leqslant \omega(m)$ alors on dit qu'il est $\mathscr{F}$-cohérent. On appelle conjugué d'un triplet presque $\mathscr{F}$-cohérent (m, β, ω), et on note $(m, \beta, \omega)_*$, le triplet $(m_*, \omega^-, \min(\omega^+, \beta^\oplus))$ où $m_* = \max_{0 \leqslant i \leqslant m} \omega(i)$ tandis que ω^- et ω^+ sont définies par (5). C'est aussi un triplet presque $\mathscr{F}$-cohérent. Il est $\mathscr{F}$-cohérent si (m, β, ω) est $\mathscr{F}$-cohérent. On montre le théorème suivant :

Théorème 0.2. *Soit (m, β, ω) un triplet presque $\mathscr{F}$-cohérent et p un réel tel que $1 < p \leqslant 2$. Alors*

$$\mathscr{F}\left(\mathscr{W}^p(m, \beta, \omega)\right) \hookrightarrow \mathscr{W}^{p^*}(m_*, \beta_*, \omega_*),$$

où $(m_*, \beta_*, \omega_*) = (m, \beta, \omega)_*$. Si $p = 2$, alors l'injection précédente devient une identité.

Ce résultat n'est pas valable pour $p > 2$ (le cas $L^p(\mathbb{R}^n)$ étant un contre-exemple). Toutefois, en utilisant l'inégalité de Hölder on obtient le Corollaire 3.3 pour $p > 2$. L'une des autres conséquences du Théorème 0.2 est l'invariance par transformation de Fourier d'une catégorie d'espaces de Sobolev avec poids. Il sont caractérisés de la manière suivante :

Théorème 0.3. *Soit $m \in \mathbb{N}$ et soit ω un $(m+1)$-uplet tel que $\omega(i) \leqslant m$, $\omega(\omega(i)) \geqslant i$, $\omega(i+1) \leqslant \omega(i) + 1$ pour $0 \leqslant i \leqslant m$. Alors l'espace $\mathscr{W}^2(m, \omega^-, \omega)$ est invariant par transformation de Fourier.*

1. Introduction: the first main result

The aim of this Note is to characterize the image of some weighted Sobolev spaces under Fourier transform. The motivation for this characterization stands in studying functional properties of weighted Sobolev spaces which are often used for solving partial differential equations in unbounded domains. The first result concerns a particular family of weighted spaces which we prove to be preserved by Fourier transform. The second result is an extension to a more general class of weighted Sobolev spaces: we prove that the image of these spaces can be obtained by means of a simple algebraic calculus. This calculus is briefly exposed in Section 2.1.

A straightforward consequence of these results stands in the invariance of weighted Sobolev spaces under Fourier transform.

Throughout this Note, $n \geqslant 1$ is an integer and $x = (x_1, \dots, x_n)$ is a typical point in $\mathbb{R}^n$. Let $\alpha = (\alpha_0, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ be two n-tuples of nonnegative integers. We write $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$, $\langle x\rangle = \sqrt{1+|x|^2}$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and

$$\min(\alpha,\beta) = \bigl(\min(\alpha_1,\beta_1), \dots, \min(\alpha_n,\beta_n)\bigr), \qquad \max(\alpha,\beta) = \bigl(\max(\alpha_1,\beta_1), \dots, \max(\alpha_n,\beta_n)\bigr).$$

The notation $a_u \lesssim b_u$, a_u and b_u being two quantities depending on a function u, means that there exists a constant c not depending on u such that $a \leqslant cb$. For each real $p > 1$, we denote by p^* the dual exponent of p defined by $1/p + 1/p^* = 1$. The Fourier transform of any complex valued Lebesgue integrable function $f : \mathbb{R} \to \mathbb{C}$ is

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)\,\mathrm{d}x,$$

where $\xi \in \mathbb{R}^n$. Let $\mathscr{S}(\mathbb{R}^n)$ refer to the usual Schwartz space of functions of class $\mathscr{C}^\infty$ with rapid decrease at infinite and let $\mathscr{S}'(\mathbb{R}^n)$ its dual space, i.e. the space of tempered distributions. The Fourier transform is an invertible mapping from $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}(\mathbb{R}^n)$, and by duality, from $\mathscr{S}'(\mathbb{R}^n)$ into $\mathscr{S}'(\mathbb{R}^n)$. For any Banach space Y, such that $Y \subset \mathscr{S}'(\mathbb{R}^n)$, we set $\mathscr{F}(Y) = \{\mathscr{F}(u);\ u \in Y\}$ its image by Fourier transform: it is a Banach space when equipped with norm $\|v\|_{\mathscr{F}(Y)} = \|\mathscr{F}^{-1}v\|_Y$.

Given an integer $m \geqslant 0$ and two real numbers s and p, $1 < p < +\infty$, we set

$$\mathscr{V}_s^{m,p}(\mathbb{R}^n) = \bigl\{u \in \mathscr{D}'(\mathbb{R}^n) \mid \langle x\rangle^s D^\lambda u \in L^p(\mathbb{R}^n)\ \forall\lambda \in \mathbb{N}^n,\ 0 \leqslant |\lambda| \leqslant m\bigr\}.$$

This space is equipped with the norm $\|u\|_{\mathscr{V}_s^{m,p}(\mathbb{R}^n)} = \sum_{|\alpha|\leqslant m}(\|\langle x\rangle^s D^\alpha u\|_{L^p(\mathbb{R}^n)}^p)^{1/p}$.

The definition of $\mathscr{V}_s^{m,p}(\mathbb{R}^n)$ is extended to real values of m as follows: if $r = m + \sigma$, with $m \in \mathbb{N}$ and $0 < \sigma < 1$, then $\mathscr{V}_s^{r,p}(\mathbb{R}^n)$ is the space of all the functions $u \in \mathscr{V}_s^{m,p}(\mathbb{R}^n)$ such that $\forall|\alpha| = m$, $\forall|\beta| \leqslant [s]$, $x^\beta D^\alpha u \in W^{\sigma,p}(\mathbb{R}^n)$. Here $W^{\sigma,p}(\mathbb{R}^n)$ is the usual Sobolev space of order σ.

We state the following:

Theorem 1.1. *Let r, s and p be three real numbers with $s \geqslant 0$, $r \geqslant 0$ and $1 < p \leqslant 2$. Then, the following imbedding holds algebraically and topologically*

$$\mathscr{F}\bigl(\mathscr{V}_s^{r,p}(\mathbb{R}^n)\bigr) \hookrightarrow \mathscr{V}_r^{s,p^*}(\mathbb{R}^n). \tag{2}$$

Moreover, if $p = 2$ then $\mathscr{F}(\mathscr{V}_s^{r,2}(\mathbb{R}^n)) = \mathscr{V}_r^{s,2}(\mathbb{R}^n)$. In particular, for each real $s \geqslant 0$, the space $\mathscr{V}_s^{s,2}(\mathbb{R}^n)$ is invariant under Fourier transform.

A straightforward consequence of this theorem is that the operator $-\Delta + I$ is an isomorphism from $\mathscr{V}_s^{r+2,p}(\mathbb{R}^n)$ into $\mathscr{V}_s^{r,p}(\mathbb{R}^n)$ for all real exponents s, $r \geqslant 0$.

The proof of Theorem 1.1 rests on the use of *spaces of Bessel potentials* (called also Lizorkin spaces). The space of Bessel potential of order $s \geqslant 0$, s being a real, is composed of all tempered distributions v such that $\mathscr{F}^{-1}\langle\xi\rangle^s\mathscr{F}v \in L^p(\mathbb{R}^n)$. This space is denoted by $\mathscr{L}^{s,p}(\mathbb{R}^n)$ and is equipped with the norm $\|v\|_{\mathscr{L}^{s,p}(\mathbb{R}^n)} = \|\mathscr{F}^{-1}\langle\xi\rangle^s\mathscr{F}v\|_{L^p(\mathbb{R}^n)}$. The reader can refer to [4], [2] or [6] for more details about these spaces. Notice that Lizorkin spaces are intimately linked to usual Sobolev spaces. Actually, if $s \geqslant 0$ is a real then following imbeddings hold (see, e.g., [2])

$$\mathscr{L}^{s,p}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n) \quad \text{if } 2 \leqslant p < +\infty, \qquad W^{s,p}(\mathbb{R}^n) \hookrightarrow \mathscr{L}^{s,p}(\mathbb{R}^n) \quad \text{if } 1 < p \leqslant 2. \tag{3}$$

Furthermore, these imbeddings become identities if $p = 2$ and s is a nonnegative real, or if $1 < p < +\infty$ and s is a positive integer. Using some properties of Lizorkin spaces, one can prove the three following lemmas from which Theorem 1.1 is deduced.

Lemma 1.2. *Let $u \in \mathscr{V}_s^{r,p}(\mathbb{R}^n)$. Then, $x^\alpha u \in W^{r,p}(\mathbb{R}^n)$, for each α such that $|\alpha| \leqslant [s]$. Furthermore,*

$$\sum_{|\alpha|\leqslant[s]} \|x^\alpha u\|_{W^{r,p}(\mathbb{R}^n)} \lesssim \|u\|_{\mathscr{V}_s^{r,p}(\mathbb{R}^n)}.$$

Lemma 1.3. *Let $u \in \mathscr{V}_s^{r,p}(\mathbb{R}^n)$. Then, $\hat{u} \in \mathscr{V}_r^{[s],p}(\mathbb{R}^n)$.*

Lemma 1.4. *Let $s \geqslant 0$ and $s \neq \mathbb{N}$. Then, for any function u in $\mathscr{V}_s^{m,p}(\mathbb{R}^n)$, its Fourier transform $\hat{u}$ satisfies; $\forall|\alpha| = [s]$, $\forall|\lambda| \leqslant m$, $\xi^\lambda D^\alpha \hat{u} \in W^{s-[s],p^*}(\mathbb{R}^n)$. Moreover,*

$$\sum_{|\lambda|\leqslant m,\ |\alpha|=[s]} \|\xi^\lambda D^\alpha \hat{u}\|_{W^{s-[s],p^*}(\mathbb{R}^n)} \lesssim \|u\|_{\mathscr{V}_s^{m,p}(\mathbb{R}^n)}.$$

2. Generalization

Theorem 1.1 asserts that the Fourier transform maps the weighted space $\mathscr{V}_r^{s,p}(\mathbb{R}^n)$ into a subspace of $\mathscr{V}_s^{r,p^*}(\mathbb{R}^n)$ if $1 < p < 2$ and exactly into $\mathscr{V}_s^{r,2}(\mathbb{R}^n)$ if $p = 2$. Our objective here is to extend this result to a larger class of weighted Sobolev spaces. Namely, we shall consider spaces of all generalized functions satisfying formally

$$x^\lambda \langle x\rangle^{b_\mu} D^\mu u \in L^p,$$

for each $|\mu| \leqslant m$, $|\lambda| = a_\mu$, m being a positive integer, and a_μ, b_μ are positive exponents (these exponents describe respectively the behavior at infinity and in the neighborhood of the origin). The role played by the origin is not a fancy but appears naturally when one tries to translate the decay conditions at infinity of a function u in terms of its Fourier transform.

The generalization of Theorem 1.1 requires a special algebraic calculus on multi-indices. This calculus is developed in the next section.

2.1. The multi-indices calculus

In what follows, for each integer $m \geqslant 0$ we set $\mathcal{S}_m = \{0, 1, \ldots, m\}$. An $(m+1)$-tuple $\omega = (\alpha_0, \ldots, \alpha_m)$ will be identified here to a mapping, still denoted by ω, from $\mathcal{S}_m$ into $\mathbb{N}$, and such that $\omega(i) = \alpha_i$, $0 \leqslant i \leqslant m$. The set of all these mappings will be denoted by $\mathscr{M}_m$. If $\omega \in \mathscr{M}_m$, we write $|\omega|_\infty = \max_{0\leqslant i\leqslant m} \omega(i)$. Given an integer $s \in \mathbb{Z}$, τ_s stands for the translation operator defined from $\mathbb{Z}$ into $\mathbb{Z}$ by $\tau_s(i) = i + s$. In the sequel, a $(m+1)$-uplet $\omega \in \mathscr{M}_m$ is said *concave* iff

$$j \leqslant k \leqslant s \Longrightarrow \omega(k) \geqslant \min\bigl(\omega(s),\ \omega(j)\bigr). \tag{4}$$

Definition 2.1. Let $m \geqslant 0$ be an integer and let β, $\omega \in \mathscr{M}_m$. Then, the triplet (m, β, ω) is said *nearly $\mathscr{F}$-coherent* if

(a) $\beta(0) = 0$.
(b) $\beta(i - s) \leqslant \max(s, \beta(i)) - s$ for any s, i such that $0 \leqslant s \leqslant i \leqslant m$.
(c) $\omega(i + 1) \leqslant \omega(i) + 1$, for each $0 \leqslant i \leqslant m$.

If in addition $\beta(m) \leqslant \omega(m)$, then the triplet (m, β, ω) is said $\mathscr{F}$-coherent.

Conditions (a) and (b) imply that there exists an integer $i_0 \leqslant m$ such that $\beta(i) = 0$ iff $i \leqslant i_0$. This integer i_0 is unique. We denote it by $\#\beta$. On the other hand, one can prove that if (m, β, ω) is an $\mathscr{F}$-coherent triplet, then necessarily $\beta(i) \leqslant \omega(i)$ for each $i \leqslant m$.

Now, given a triplet (m, β, ω), with $m \geqslant 0$ and β, $\omega \in \mathscr{M}_m$, we set

$$\omega^-(j) = \min\{i \in \mathcal{S}_m;\ \omega(i) \geqslant j\}, \qquad \omega^+(j) = \max\{i \in \mathcal{S}_m;\ \omega(i) \geqslant j\}, \tag{5}$$

for each integer $j \in \{0, \ldots, |\omega|_\infty\}$. Hence, $\omega^-,\ \omega^+ \in \mathscr{M}_{m_*}$, with $m_* = |\omega|_\infty$. Similarly, if $\beta \in \mathscr{M}_m$ is such that $\beta(0) = 0$, then we set for each integer $j \in \mathbb{N}$

$$\beta^\oplus(j) = \max\{i \in \mathcal{S}_m;\ \beta(i) \leqslant j\}.$$

Definition 2.2. The conjugate of a nearly $\mathscr{F}$-coherent triplet (m, β, ω), denoted by $(m, \beta, \omega)_*$, is the triplet $(|\omega|_\infty, \omega^-, \min(\beta^\oplus, \omega^+))$.

Let (m_*, β_*, ω_*) be the conjugate of a nearly $\mathscr{F}$-coherent triplet (m, β, ω). The following properties hold:

(a) (m_*, β_*, ω_*) is also nearly $\mathscr{F}$-coherent,
(b) $\#\beta_* = \omega(0)$ and $\omega_*(0) = \#\beta$,
(c) ω_* is concave,
(d) $|\omega_*|_\infty \leqslant m$,
(e) $|\omega_*|_\infty = m$ if and only if (m, β, ω) $\mathscr{F}$-coherent.
Moreover, if (m, β, ω) is $\mathscr{F}$-coherent and if $(m_{**}, \beta_{**}, \omega_{**}) = (m_*, \beta_*, \omega_*)_*$ denotes its biconjugate then
(f) (m_*, β_*, ω_*) is $\mathscr{F}$-coherent,
(g) $m_{**} = m$, $\beta_{**} = \beta$ and $\omega_{**} \geqslant \omega$,
(h) $(m_{**}, \beta_{**}, \omega_{**}) = (m, \beta, \omega)$ if and only if ω is concave.

3. The second main result

Now, we are in position to give a more general version of Theorem 1.1. Let $m \geqslant 0$ be an integer and p, $1 < p < +\infty$, a real. To each triplet (m, β, ω), $\beta,\ \omega \in \mathscr{M}_m$, we associate the weighted space $\mathscr{W}^p(m, \beta, \omega)$ of all distributions u whose generalized derivatives of order less or equal to m satisfy

$$\forall |\mu| \leqslant m,\ \forall |\lambda| = \beta(|\mu|), \quad \langle x\rangle^{\omega(|\mu|)-\beta(|\mu|)} x^\lambda D^\mu u \in L^p(\mathbb{R}^n).$$

This space is equipped with the norm

$$\|u\|_{\mathscr{W}^p(m,\beta,\omega)} = \sum_{|\mu| \leqslant m} \sum_{|\lambda| = \beta(|\mu|)} \|\langle x\rangle^{\omega(|\mu|)-\beta(|\mu|)} x^\lambda \partial^\mu u\|_{L^p(\mathbb{R}^n)}. \tag{6}$$

Theorem 3.1. *Let (m, β, ω) be an $\mathscr{F}$-coherent triplet and let p be a real with $1 < p \leqslant 2$. Then*

$$\mathscr{F}\big(\mathscr{W}^p(m, \beta, \omega)\big) \hookrightarrow \mathscr{W}^{p^*}\big((m, \beta, \omega)_*\big).$$

Moreover, if $p = 2$, and if (m, β, ω) is $\mathscr{F}$-coherent, then the identity $\mathscr{F}(\mathscr{W}^2(m, \beta, \omega)) = \mathscr{W}^2((m, \beta, \omega)_*)$ holds topologically and algebraically.

Corollary 3.2. *Let (m, β, ω) be a nearly $\mathscr{F}$-coherent triplet. Then $\mathscr{W}^2(m, \beta, \omega) \hookrightarrow \mathscr{W}^2((m, \beta, \omega)_{**})$. The last imbedding becomes an identity if (m, β, ω) is $\mathscr{F}$-coherent.*

Corollary 3.3. *Suppose that $p > 2$ and let $a,\ b \in \mathbb{N}$ such that $a < n(1 - 2/p) < b$. Let (m, β, ω) a nearly $\mathscr{F}$-coherent triplet with $\omega(i) \geqslant b$ for each $i \leqslant m$. Then,*

$$\mathscr{F}\big(\mathscr{W}^p(m, \beta, \omega)\big) \hookrightarrow \mathscr{W}^p\big(|\omega|_\infty - b, \omega^- \circ \tau_b, \min(\beta^+ \circ \tau_a, \omega^+ \circ \tau_b)\big).$$

Corollary 3.4. *Suppose that $p < 2$ and let $a,\ b \in \mathbb{N}$, $a < n(2/p - 1) < b \leqslant m$. Let (m, β, ω) be a nearly $\mathscr{F}$-coherent triplets with $\beta(b) = 0$ and $\omega^+(|\omega|_\infty) \geqslant b$. Then,*

$$\mathscr{F}\big(\mathscr{W}^p(m,\beta,\omega)\big)\hookrightarrow\mathscr{W}^p\big(m_*,\max(\tau_{-a}\circ\beta_*,0),\tau_{-b}\circ\omega_*\big)$$

where $(m_*,\beta_*,\omega_*)=(m,\beta,\omega)_*$.

The proof of Theorem 3.1 is somewhere similar to that of Theorem 1.1 and is based on properties of Lizorkin spaces. One can start the proof by assuming that (m,β,ω) is $\mathscr{F}$-coherent.

Now, let us apply Theorem 3.1 to some explicit spaces. In all the examples $1<p\leqslant 2$.

Example 1. If $\omega=(0,\dots,0)$, $\beta=(0,\dots,0)$, then $\mathscr{W}^p(m,\beta,\omega)$ is nothing but the usual Sobolev space $W^{m,p}(\mathbb{R}^n)$. In this case $(m,\beta,\omega)=(m_*,\beta_*,\omega_*)$ with $m_*=0$, $\omega_*=(m)$, $\beta_*=(0)$. Theorem 3.1 states the well known property $\mathscr{F}(W^{m,p}(\mathbb{R}^n))\hookrightarrow\mathscr{V}_m^{0,p^*}(\mathbb{R}^n)$.

Example 2. If $\omega=(k,\dots,k)$, $\beta=(0,\dots,0)$, then $\mathscr{W}^p(m,\beta,\omega)$ is nothing but the space $\mathscr{V}_k^{m,p}(\mathbb{R}^n)$ of Theorem 1.1. In this case $m_*=k$, $\omega_*=(m,\dots,m)$, $\beta_*=(0,\dots,0)$ and we retrieve assertion of Theorem 1.1 when s and r are integers.

Example 3. If $\omega=(k,k+1,\dots,k+m)$, $k\geqslant 0$, $\beta=(0,\dots,0)$, then $\mathscr{W}^p(m,\beta,\omega)$ is nothing but the space $W_{k+m}^{m,p}(\mathbb{R}^n)$, defined by $\langle x\rangle^{k+|\mu|}D^\lambda u\in L^p(\mathbb{R}^n)$ for each $|\lambda|\leqslant m$. This kind of spaces are often used for solving PDEs in unbounded domains (see, e. g., [5] or [1]). Theorem 3.1 asserts that the Fourier transform of a function $u\in W_{k+m}^{m,p}(\mathbb{R}^n)$ satisfies $\langle\xi\rangle^m D^\mu\hat{u}\in L^{p^*}(\mathbb{R}^n)$ if $|\mu|\leqslant k$, and $\langle\xi\rangle^{m+k-|\mu|}\xi^\lambda D^\mu\hat{u}\in L^{p^*}(\mathbb{R}^n)$ if $k\leqslant|\mu|\leqslant k+m$ and $|\lambda|=|\mu|-k$.

Among the consequences of Theorem 3.1, let us underline the *invariance of some weighted Sobolev spaces* under Fourier transform. More precisely, let us write $(m',\beta',\omega')\hookrightarrow(m,\beta,\omega)$ when $m'\geqslant m$, $\beta'\leqslant\beta$ and $\omega'\geqslant\omega$. An $\mathscr{F}$-coherent triplet (m,β,ω) is said *invariant* if $(m,\beta,\omega)_*\hookrightarrow(m,\beta,\omega)$. One can prove that an $\mathscr{F}$-coherent triplet (m,β,ω) with ω concave is invariant if and only if $(m,\beta,\omega)_*=(m,\beta,\omega)$. Moreover in that case, Theorem 3.1 implies that necessarily $\mathscr{W}^2(m,\beta,\omega)$ *is invariant under Fourier transform*. In other words, to get invariant spaces, it suffices to characterize invariant triplets. Moreover, the identity $\mathscr{W}^2(m,\beta,\omega)=\mathscr{W}^2(m,\beta,\omega_{**})$ proved in Corollary 3.2 allows one to suppose, without loss of generality, that ω is concave. We state the following:

Theorem 3.5. *Let* $m\geqslant 0$ *be an integer and* $\omega\in\mathscr{M}_m$ *be a concave* $(m+1)-uplet$ *such that* $\omega(i)\leqslant m$, $\omega(\omega(i))\geqslant i$, $\forall 0\leqslant i\leqslant m$, *and* $\omega(i+1)\leqslant\omega(i)+1$ $\forall 0\leqslant i<m$. *Then,* $\mathscr{F}(\mathscr{W}^2(m,\omega^-,\omega))=\mathscr{W}^2(m,\omega^-,\omega)$.

Let us sketch some examples of invariant weighted Sobolev spaces. Firstly, one can consider the spaces $\mathscr{V}_m^{m,2}(\mathbb{R}^n)=\mathscr{W}^2(m,\beta,\omega)$ with $\beta=(0,\dots,0)$ and $\omega=(m,\dots,m)$. One can consider also the space of all functions whose generalized derivatives of order $|\mu|\leqslant m$ satisfy $x^\lambda D^\mu u\in L^2(\mathbb{R}^n)$ for each λ such that $|\lambda|=|\mu|$. This space corresponds to the triplet (m,β,ω) with $\omega=\beta=(0,1,\dots,m)$.

In a forthcoming paper [3], the author will give some applications of Theorem 3.1. An interpretation of the multi-indices calculus in terms of Galois connections will also be given.

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