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Probability Theory

Tail of a linear diffusion with Markov switching

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Abstract

Let Y be a Ornstein–Uhlenbeck diffusion governed by a stationary and ergodic Markov jump process X, i.e. $\mathrm{d}Y_t = a(X_t)Y_t\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t$, $Y_0 = y_0$. Ergodicity conditions for Y have been obtained. Here we investigate the tail property of the stationary distribution of this model. A characterization of the only two possible cases is established: light tail or polynomial tail. Our method is based on discretizations and renewal theory. ***To cite this article: B. de Saporta, J.-F. Yao, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Queue d'une diffusion linéaire à régime markovien. Soit Y une diffusion de Ornstein–Uhlenbeck dirigée par un processus Markovien de saut X stationnaire et ergodique : $\mathrm{d}Y_t = a(X_t)Y_t\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t$, $Y_0 = y_0$. On connaît des conditions d'ergodicité pour Y. Ici on s'intéresse à la queue de la loi stationnaire de ce modèle. Par des méthodes de discrétisation et de renouvellement, on donne une caractérisation complète des deux seuls cas possibles : queue polynômiale ou existence de moment à tout ordre. ***Pour citer cet article : B. de Saporta, J.-F. Yao, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

1. Introduction

The discrete time models $Y = (Y_n,\ n \in \mathbf{N})$ governed by a switching process $X = (X_n,\ n \in \mathbf{N})$ fit well to the situations where an autonomous process X is responsible for the dynamic (or *regime*) of Y. These models are parsimonious with regard to the number of parameters, and extend significantly the case of a single regime. Among them, the so-called Markov switching ARMA models are popular in several application fields, e.g., in econometric modeling (see [4]). More recently continuous-time version of Markov-switching models have been proposed in [1] and [3], among others where ergodicity conditions are established. Here we investigate the tail property of the

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stationary distribution of this continuous-time process. One of the main results states that this model can provide heavy tails which is one of the major features required in nonlinear time series modeling.

2. Linear diffusion with Markov switching and main theorems

The *diffusion with Markov switching* Y is constructed in two steps:

First, the *switching process* $X = (X_t)_{t\geqslant 0}$ is a Markov jump process defined on a probability space $(\Omega, \mathcal{A}, Q)$, with a finite state space $E = \{1, \ldots, N\}$, $N > 1$. We assume that the intensity function λ of X is positive and the jump kernel $q(i, j)$ on E is irreducible and satisfies $q(i,i) = 0$, for each $i \in E$. The process X is ergodic and will be taken stationary with an invariant probability measure denoted by μ.

Secondly, let $W = (W_t)_{t\geqslant 0}$ be a standard Brownian motion defined on a probability space $(\Theta, \mathcal{B}, Q')$, and $\mathcal{F} = (\mathcal{F}_t)$ the filtration of the motion. We will consider the product space $(\Omega \times \Theta, \mathcal{A} \times \mathcal{B}, (Q_x \otimes Q'))$, $\mathbb{P} = Q \otimes Q'$ and $\mathbb{E}$ the associated expectation. Conditionally to X, $Y = (Y_t)_{t\geqslant 0}$ is a real-valued diffusion process, defined, for each $\omega \in \Omega$ by:

(i) Y_0 is a random variable defined on $(\Theta, \mathcal{B}, Q')$, $\mathcal{F}_0$-measurable;
(ii) Y is solution of the linear SDE

$$\mathrm{d}Y_t = a(X_t)Y_t\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \quad t \geqslant 0. \tag{1}$$

Thus (Y_t) is a linear diffusion driven by an 'exogenous' jump process (X_t).

We say a continuous or discrete time process $S = (S_t)_{t\geqslant 0}$ is *ergodic* if there exists a probability measure m such that when $t \to \infty$, the law of S_t converges weakly to m independently of the initial condition S_0. The distribution m is then the *limit law* of S. When S is a Markov process, m is its unique invariant law.

In [3], it is proved that the Markov-switching diffusion Y is ergodic under the condition

$$\alpha = \sum_{i\in E} a(i)\mu(i) < 0. \tag{2}$$

Note that Condition (2) will be assumed to be satisfied throughout the Note and we denote by ν the stationary (or limit) distribution of Y.

Theorem 2.1 (light tail case). *If for all i, $a(i) \leqslant 0$, then the stationary distribution ν of the process Y has moments of all order, i.e. for all $s > 0$ we have*:

$$\int_{\mathbb{R}} |x|^s \nu(\mathrm{d}x) < \infty.$$

Theorem 2.2 (heavy tail case). *If there is a i such that $a(i) > 0$, one can find an exponent $s_0 > 0$ and a constant $L > 0$ such that the stationary distribution ν of the process Y satisfies*

$$t^{s_0}\nu\big(]t, +\infty[\big) \underset{t\to+\infty}{\longrightarrow} L,$$
$$t^{s_0}\nu\big(]-\infty, -t[\big) \underset{t\to+\infty}{\longrightarrow} L.$$

Note that the two situations from Theorems 2.1 and 2.2 form a dichotomy. Moreover the characteristic exponent s_0 in the heavy tail case is completely determined as follows. Let

$$s_1 = \min\left\{\frac{\lambda(i)}{a(i)} \;\Big|\; a(i) > 0\right\},$$
$$M_s = \left(q(i,j)\frac{\lambda(i)}{\lambda(i) - sa(i)}\right)_{i,j\in E} \quad \text{for } 0 \leqslant s < s_1.$$

Then s_0 is the unique $s \in]0, s_1[$ such that the spectral radius of M_s equals 1.

3. Discretization of the process

Our study of Y is based on the investigations of its discretization $Y^{(\delta)}$ as in [3]. First we give an explicit formula for the diffusion process. For $0 \leqslant s \leqslant t$, let

$$\Phi(s,t) = \Phi_{s,t}(\omega) = \exp\int_s^t a(X_u)\,\mathrm{d}u.$$

The process Y has the representation:

$$Y_t = Y_t(\omega) = \Phi(0,t)\left[Y_0 + \int_0^t \Phi(0,u)^{-1}\sigma(X_u)\,\mathrm{d}W_u\right],$$

and for $0 \leqslant s \leqslant t$, Y satisfies the recursion equation:

$$Y_t = \Phi(s,t)\left[Y_s + \int_s^t \Phi(s,u)^{-1}\sigma(X_u)\,\mathrm{d}W_u\right] = \Phi(s,t)Y_s + \int_s^t \left[\exp\int_u^t a(X_v)\,\mathrm{d}v\right]\sigma(X_u)\,\mathrm{d}W_u.$$

It is useful to rewrite this recursion as:

$$Y_t(\omega) = \Phi_{s,t}(\omega)Y_s(\omega) + V_{s,t}^{1/2}(\omega)\xi_{s,t}, \tag{3}$$

where $\xi_{s,t}$ is a standard Gaussian variable, function of $(W_u,\ s \leqslant u \leqslant t)$, and

$$V_{s,t}(\omega) = \int_s^t \exp\left[2\int_u^t a(X_v)\,\mathrm{d}v\right]\sigma^2(X_u)\,\mathrm{d}u.$$

For $\delta > 0$, we will call *discretization at step size* δ of Y the discrete time process $Y^{(\delta)} = (Y_{n\delta})_n$, where $n \in \mathbb{N}$. For a fixed $\delta > 0$, the discretization $Y^{(\delta)}$ follows an AR(1) equation with random coefficients:

$$Y_{(n+1)\delta}(\omega) = \Phi_{n+1}(\omega)Y_{n\delta}(\omega) + V_{n+1}^{1/2}(\omega)\xi_{n+1}, \tag{4}$$

with

$$\Phi_{n+1}(\omega) = \Phi_{n+1}(\delta)(\omega) = \exp\left[\int_{n\delta}^{(n+1)\delta} a\big(X_u(\omega)\big)\,\mathrm{d}u\right],$$
$$V_{n+1}(\omega) = \int_{n\delta}^{(n+1)\delta} \exp\left[2\int_u^{(n+1)\delta} a\big(X_v(\omega)\big)\,\mathrm{d}v\right]\sigma^2\big(X_u(\omega)\big)\,\mathrm{d}u,$$

where (ξ_n) is a standard Gaussian i.i.d. sequence defined on $(\Theta, \mathcal{B}, Q')$. Note that under condition (2), all these discretizations are ergodic with the same limit distribution ν (see [3]).

4. Sketch of the proof

The limit distribution ν is also the law of the stationary solution of Eq. (4). To investigate the behaviour of its tail, we use the same renewal-theoretic methods as [5,6] and [2]. In these works, the coefficients (Φ_n) form an i.i.d. sequence. Here the sequence (Φ_n) is neither i.i.d nor a Markov chain. Indeed we know only the conditional independence between Φ_n and Φ_{n+1} given $X_{n\delta}$. We thus need to adapt the mentioned methods to this special situation. Our problem leads to a system of renewal equations, and we use a new renewal theorem for systems of equations reported in [7].

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