Geometry of foliations on the Wiener space
and stochastic calculus of variations

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Abstract

Stochastic Calculus of variations deals with maps defined on the Wiener space, with finite dimensional range; within this context appears the notion of non-degenerate map, which corresponds roughly speaking to some kind of infinite dimensional ellipticity; a non-degenerate map has a smooth law; by conditioning, it generates a continuous desintegration of the Wiener measure. Infinite dimensional Stochastic Analysis and particularly SPDE theory raise the natural question of what can be done for maps with an infinite dimensional range; our approach to this problem emphasizes an intrinsic geometric aspect, replacing range by generated σ-field and its associated foliation of the Wiener space.

Résumé

Géométrie des foliations sur l’espace de Wiener et calcul des variations stochastiques. Le Calcul Stochastique des variations considère classiquement des applications de l’espace de Wiener dans un espace de dimension finie; dans ce contexte s’inscrit la théorie des applications non dégénérées pour lesquelles on peut établir la régularité des lois ainsi que l’existence de désintégrations continues. L’Analyse stochastique en dimension infinie et singulièrement la théorie des SPDE, pose la question naturelle de l’étude des applications de l’espace de Wiener dans un espace de dimension infinie. Nous approchons ce problème de manière intrinsèque, privilégiant l’étude géométrique des sous tribus à travers leurs foliations associées.

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1. Introduction: the theory of non-degenerate maps

We work on an abstract Wiener space $X$, $\mu$ is its canonical Wiener measure; denote by $H$ the Cameron–Martin space, that is the Hilbert space of constant vector fields for which the measure $\mu$ is quasi-invariant under translation by a vector of $H$. Denote by $D^2_\mu(X)$ the Sobolev space of functions which are $r$-times differentiable relatively to $H$, with all their derivatives belonging to $L^\infty_\mu$. Let $D_\mu^{\infty-}(X) := \bigcap_{r<\infty} D^2_\mu(X)$. Consider a map $g : X \mapsto \mathbb{R}^d$, $g = (g_1, \ldots, g_d)$. Assume that $g_s \in D_\mu^{\infty-}(X)$; then define the $d \times d$ covariance matrix $\gamma_{i,j} = (\nabla g_i | \nabla g_j)$; the map $g$ is said to be non-degenerate, if almost surely the matrix $\gamma_{i,j}$ is invertible and if, denoting $\gamma^{\cdot,\cdot}$ its inverse matrix, then $\gamma^{i,j} \in L^{\infty-}_\mu$, an hypothesis which implies that $\gamma^{i,j} \in D_\mu^{\infty-}(X)$. The canonical lift to $X$ of the coordinate vector field $\frac{\partial}{\partial x_i}$ is defined as

$$Z^i_{\text{can}}(\omega) = \sum_{i=1}^d \gamma^{i,i}(\omega)(\nabla g_i)(\omega), \quad \text{we have } (Z^i_{\text{can}}|Z^j_{\text{can}}) = \gamma^{i,j}. \quad (1)$$

The hypothesis: $g$ non-degenerate implies that $Z^i \in D^{\infty-}_1(X; H)$ and $Z^i$ has a divergence relatively to the Gaussian measure $\mu$, let $\delta(\mu) \in L^{\infty-}_\mu(X)$.

$$\int \delta(\mu)(Z^i) \psi \mathbb{d}\mu = -\int (Z^i|\nabla \psi) \mathbb{d}\mu \quad \text{for any differentiable function } \psi \text{ on } X. \quad (2)$$

Denote $\nu = g_*(\mu)$ the law of $g$, $g_*(\mu)(A) = \mu(g^{-1}(A))$. Then the divergence of $\frac{\partial}{\partial x_i}$ relative to $\nu$ is the conditional expectation of the divergence of $Z^i$ relatively to $\mu$,

$$\delta(\nu)(\frac{\partial}{\partial x_i}) = E^\nu(\delta(\mu)(Z^i)) \quad (3)$$

consequently $\delta(\nu)(\frac{\partial}{\partial x_i}) \in L^{\infty-}_\nu$, and $\nu$ has an Hölderian density relatively to the Lebesgue measure. More generally, we call a lift by $g$ of $\frac{\partial}{\partial x_i}$, a vector field $Z^i$ such that

$$Z^i(h \circ g) = \left(\frac{\partial}{\partial x_i} h \right) \circ g. \quad (4)$$

It satisfies (4) if and only if $(Z^i|\nabla g_s) = 1$ and $(Z^i|\nabla g_k) = 0$ if $k \neq s$. Moreover, there is a unique lift by $g$ of $\frac{\partial}{\partial x_i}$, in the space generated by $(\nabla g_j)$, $j=1,\ldots,d$, this is given by $Z^i_{\text{can}}$.

Conditioning by a non-degenerate map preserves differentiability ([5]; page 82). For any lift $Z^i$ by $g$ of $\frac{\partial}{\partial x_i}$, and such that the divergence $\delta(\mu)(Z^i)$ exists, we have

$$\frac{\partial}{\partial x_i} E^\nu(f) - E^\nu(DZ^i f) = E^\nu(\delta(\mu)(Z^i)f) - E^\nu(\delta(\mu)(Z^i)) \times E^\nu(f), \quad \forall f \in D_\mu^{\infty-}(X). \quad (5)$$

With (3) and (5), divergences of lifts play a key role. An analytic expression of the divergence of the canonical lift has been given in coordinates in [5], page 71. In this Note, we give alternative geometric expressions free from coordinates which could be used in the case of $\sigma$-fields (see [3,4,6]).

2. Lifting and Lie derivatives

A vector field $Z$ on $X$ is called basic if $Z \in D^{\infty-}(X; H)$ and if $DZg_s = (Z|\nabla g_s)$ is $g$-measurable for all $s \in [1,d]$. Thus, if $Z$ is basic,

$$(Z(\omega)|\nabla g_s(\omega)) = z_s(\gamma^{i,i}(\omega)), \quad s = 1, \ldots, d. \quad (6)$$
2.1. The duality

The unnormalized conditional expectation is given by Proposition 2.2.

Denote $\|\omega\| = \sqrt{\text{det}(\nabla g_1, \nabla g_k))}$. The $\infty - d$ Gaussian area $da$ satisfies $da \wedge \frac{\gamma}{|\gamma|} = \Omega$. Thus $\frac{da}{|\gamma|} \wedge \gamma = \Omega$. As $\frac{da}{|\gamma|}$ corresponds to the conditional law, the proposition is proved.

Theorem 2.3. Let $Z$ be a basically constant vector field, then the Lie derivative $\mathcal{L}_Z \mathcal{C} = \delta_\mu(Z) \times \mathcal{C}$.

Proof. $\delta_\mu(Z) \times \Omega = \mathcal{L}_Z(\Omega) = \mathcal{L}_Z(\mathcal{C}) \wedge \gamma + \mathcal{C} \wedge \mathcal{L}_Z \gamma = \mathcal{L}_Z(\mathcal{C}) \wedge \gamma$.

Remark 1. Theorem (2.3) reduces the derivative of a conditional expectation to a Skorokhod integral on the Wiener space. With Proposition 2.2, it gives a geometrical proof of (5).

3. Canonical forms of a foliation: curvature and proximity

To avoid the difficulty of selecting versions of maps, we consider the case where $X$ is a finite dimensional Euclidean space with its canonical Gaussian measure; then by Sobolev embedding, $D^{\infty-}(X)$ is contained in the set of continuous functions; we always take the continuous versions. Since the estimates that we obtain are independent of the dimension of $X$, they stay valid for the infinite dimensional case, that is the Wiener space. Let $g = (g_1, g_2, \ldots, g_d): X \to R^d$ be a non-degenerate map. On $X$, we define the equivalence relation $\omega_1 T \omega_2$ if $g(\omega_1) = g(\omega_2)$. Since $g(\omega_1) = g(\omega_2)$ implies that $\phi(g(\omega_1)) = \phi(g(\omega_2))$ for any diffeomorphism $\phi$ of $R^d$, the relation $T$ depends only on the $\gamma$-field generated by the $\phi g$, $\phi \in \text{diff}(R^d)$. Let $N_\omega$ be the normal suspace at $\omega$: it is the $d$-dimensional subspace of $X$ generated by $(\nabla g_j(\omega))_{j=1,\ldots,d}$. Since (1), the canonical lifts $(Z^k_{\text{can}})_{j=1,\ldots,d}$ form a basis of $N_\omega$. 

A vector field $Z$ on $X$ is called $g$-basically constant if $D_Z g_s$ is constant for any $s \in [1, d]$. In particular, any lift by $g$ of a constant vector field is $g$-basically constant. Let $\phi \in R^d$, then a lift by $\phi \circ g$ of a constant vector field is basic. If a basic vector field $Z$ is in the normal space, then (compare with (1)),

$$Z(\omega) = \sum_k z_k(g(\omega))Z^k_{\text{can}}(\omega).$$

In the following, unless it is precised, for lift, we mean a lift by $g$. Let $\gamma = d g_1 \wedge \cdots \wedge d g_d$.

Proposition 2.1. The Lie derivative of $\gamma$ relatively to any basically constant vector field vanishes.

Proof. We use the Cartan formula $\mathcal{L}_Z = i(Z) d + di(Z)$, where $i(Z)$ denotes the interior product by $Z$. We have $\mathcal{L}_Z(dg) = d(\mathcal{L}_Z g)$. As $d\gamma = 0$, the first term of Cartan formula disappears. We obtain $i(Z)\gamma = (D_Z g_1) \wedge dz_1 \wedge \cdots \wedge d g_d$, expression which has a vanishing boundary. 

2.1. The duality

We denote by $\Omega$ the differential form of degree $\infty$ which is the ‘volume form’ associated to the Gaussian measure $\mu$. Let $\mathcal{C}$ be the differential form of degree $\infty - d$ given by $\mathcal{C} = i(\gamma)\Omega$, i.e. $\mathcal{C} \wedge \gamma = \Omega$.

Proposition 2.2. The unnormalized conditional expectation is given by $E^\gamma(f)(\xi) = \int_{(g^{-1}(\xi))} f \times \mathcal{C}$.

Proof. For $f : X \to R$, the unnormalized conditional expectation is given by (see [1])

$$E^\gamma(f)(\alpha) = \int_{g^{-1}(\alpha)} f(\omega) \sqrt{\text{det}(\gamma^{**}(\omega))} \, da(\omega) , \quad \alpha \in R^d.$$ 

Denote $|\gamma| = \sqrt{\text{det}(\nabla g_1, \nabla g_k))}$. The $\infty - d$ Gaussian area $da$ satisfies $da \wedge \frac{\gamma}{|\gamma|} = \Omega$. Since $\frac{da}{|\gamma|}$ corresponds to the conditional law, the proposition is proved. 

Theorem 2.3. Let $Z$ be a basically constant vector field, then the Lie derivative $\mathcal{L}_Z \mathcal{C} = \delta_\mu(Z) \times \mathcal{C}$.

Proof. $\delta_\mu(Z) \times \Omega = \mathcal{L}_Z(\Omega) = \mathcal{L}_Z(\mathcal{C}) \wedge \gamma + \mathcal{C} \wedge \mathcal{L}_Z \gamma = \mathcal{L}_Z(\mathcal{C}) \wedge \gamma$. 

Remark 1. Theorem (2.3) reduces the derivative of a conditional expectation to a Skorokhod integral on the Wiener space. With Proposition 2.2, it gives a geometrical proof of (5).
**Proposition 3.1.** Consider the normal spaces \( N_{\omega_1} \) and \( N_{\omega_2} \) at two different points \( \omega_1 \) and \( \omega_2 \). There exists a unique linear mapping \( \tau_{\omega_2 \rightarrow \omega_1} : N_{\omega_1} \rightarrow N_{\omega_2} \) such that \( \tau_{\omega_2 \rightarrow \omega_1}(\omega_1) = \omega_2 \) is given by

\[
\langle dg_k(\omega_1), n_1 \rangle = \langle dg_k(\omega_2), n_2 \rangle \quad \text{for} \quad k = 1, \ldots, d.
\]

(8)

With the basis \((\nabla g_j(\omega_1))_{j=1,\ldots,d}\) for \( N_{\omega_1} \) and \((\nabla g_j(\omega_2))_{j=1,\ldots,d}\) for \( N_{\omega_2} \), the matrix of \( \tau_{\omega_2 \rightarrow \omega_1} \) is \( \gamma^{**}(\omega_2)\gamma^{**}(\omega_1) \). Because of (4), the image by \( \tau_{\omega_2 \rightarrow \omega_1} \) of the canonical lift \( Z^i_{\text{can}}(\omega_1) \) is \( Z^i_{\text{can}}(\omega_2) \). If \( g(\omega_1) = g(\omega_2) \), then \( \tau_{\omega_2 \rightarrow \omega_1} \) is the same for any \( \phi \circ g \), where \( \phi \in \mathcal{R}^d \). For \( h \in H \), let

\[
\frac{d}{d\epsilon} Z^i_{\text{can}}(\omega + \epsilon h) = \frac{d}{d\epsilon} \frac{\gamma^{**}(\omega + \epsilon h)\gamma^{**}(\omega)}{\epsilon h} = \frac{\gamma^{**}(\omega + \epsilon h)}{\epsilon h} \nabla g_j(\omega + \epsilon h).
\]

(9)

Since \( (Z^i_{\text{can}})_{\nabla g_i} = 1 \), we obtain

\[
2\sum_s \left( \frac{d}{d\epsilon} Z^i_{\text{can}}(\omega + \epsilon h) \right|_{\epsilon = 0} \nabla g_s(\omega) = \sum_{s,j} \frac{d}{d\epsilon} \gamma^{**}(\omega + \epsilon h)\gamma^{**}(\omega) \frac{d}{d\epsilon} \nabla g_j(\omega + \epsilon h).
\]

(10)

For \( \omega \in X, X = \mathcal{R}^m \) and \( h \in H, H = \mathcal{R}^m \), consider the matrix

\[
\beta(\omega; h) = \left[ \frac{d}{d\epsilon} \gamma^{**}(\omega + \epsilon h) \right|_{\epsilon = 0} \gamma^{**}(\omega) - \frac{d}{d\epsilon} \gamma^{**}(\omega + \epsilon h) \right|_{\epsilon = 0}.
\]

(11)

The map \( (\omega, h) \rightarrow \beta(\omega; h) \) from \( X \times H \rightarrow \text{End}(\mathcal{R}^d) \) is called the proximity form of the foliation \( g \). We define \( \nabla^2 g_j(\omega) : H \times H \rightarrow \mathcal{R} \) by \( \nabla^2 g_j(\omega)(Y, Z) = \sum_{k,p} \frac{\partial^2 g_j}{\partial g_k \partial g_p}(\omega) Y_k Z_p \). We have

\[
\frac{d}{d\epsilon} \gamma^{**}(\omega + \epsilon h) \|_{\epsilon = 0} = \{ \nabla^2 g_j(\omega)(h, \nabla g_j(\omega)) + \nabla^2 g_j(\omega)(h, \nabla g_i(\omega)) \}_{ij}.
\]

(12)

The orthogonal projection on \( N_{\omega} \) of (9) is \( \text{Proj}_{N_{\omega}} \left( \frac{d}{d\epsilon} Z^i_{\text{can}}(\omega + \epsilon h) \right) = \sum_r \alpha_{sr}(\omega) Z^r_{\text{can}}(\omega) \) with

\[
\alpha_{sr}(\omega) = \beta(\omega; h)_{sr} \|_{\epsilon = 0} + \sum_j \nabla^2 g_j(\omega, \nabla g_r(\omega)) \gamma^{**} = -\nabla^2 g_r(\omega, Z^r_{\text{can}}).
\]

(13)

The tangent space \( T_{\omega} \) is the subspace of \( h \in H \) such that \( \left( \nabla g_k(\omega) \right| h = 0 \) for \( k = 1, \ldots, d \).

**Theorem 3.2.** For \( h \in H \),

\[
\sum_s \left( \frac{d}{d\epsilon} Z^i_{\text{can}}(\omega + \epsilon h) \right|_{\epsilon = 0} \nabla g_s(\omega) = -\sum_j \nabla^2 g_j(\omega)(Z^j_{\text{can}}(\omega), h) = \frac{1}{2} \text{trace} \beta(\omega; h).
\]

(14)

and for \( h \in T_{\omega} \), (14) is the same for any \( \phi \circ g \), when \( \phi \) varies in \( \text{diff}(\mathcal{R}^d) \).

From (13), we see that taking the derivative of the map \( \tau_{\omega_2 \rightarrow \omega_1} \) tangentially to a leaf, defines a connection on the normal fiber bundle of the foliation. This connection has a curvature equal to zero and it is integrable. The proximity form is the sum of this connection with the transposed connection.

**Remark 2.** In (14), if \( h \notin T_{\omega} \), then for \( \phi = (\phi_1, \phi_2, \ldots, \phi_d) \in \text{diff}(\mathcal{R}^d) \),

\[
\sum_j \nabla^2(\phi \circ g)(\omega)(Z^j_{\text{can}}(\omega), h) = \sum_j \nabla^2 g_j(\omega)(Z^j_{\text{can}}(\omega), h) + \left( \nabla \log(\det \text{jac} \phi) \right)(g(\omega)) \| h.
\]

For non-degenerate \( g : X \rightarrow \mathcal{R}^d \), the second fundamental form at \( \omega \) is (see [2])

\[
l^0_{\omega}(h_1, h_2) = \sum_{r,s} \gamma^{rs} \nabla^2 g_r(h_1, h_2) \nabla g_s = \sum_r \nabla^2 g_r(h_1, h_2) Z^r_{\text{can}}.
\]

(15)
If $h_1$ or $h_2$ is in the tangent space $T_{\omega}$, then $l_{\omega}^{\phi, g}(h_1, h_2) = l_{\omega}^g(h_1, h_2)$ for any diffeomorphism $\phi$ of $\mathbb{R}^d$. In that case, we denote $l_{\omega}(h_1, h_2)$. Consider

$$L_{\omega}^g = \operatorname{trace}_{H \times H} l_{\omega}^g(\bullet, \bullet) = \sum_r \left( \sum_k \frac{\partial^2 g_r}{\partial \xi_k^2} \right) T_{\omega}^g. \tag{16}$$

From (1) and (16), it holds $(L_{\omega}^g|Z_{\omega}^\text{can}(\omega)) = \sum_{i,k} \gamma^{ii}(\omega) \frac{\partial^2 g_r}{\partial \xi_k^2}(\omega)$ and $(L_{\omega}^g|\nabla g^i(\omega)) = \sum_k \frac{\partial^2 g_r}{\partial \xi_k^2}(\omega)$.

**Theorem 3.3.** $\sum_{j,k} \nabla^2 g_k(\omega)(Z_{\omega}^j, \nabla g_j(\omega)) Z_{\omega}^k = \operatorname{trace}_{N_\omega \times N_\omega} l_{\omega}^{\phi, g}(\bullet, \bullet)$.

We define the total curvature of the leaf at $\omega$ as

$$C_{\omega} = \operatorname{trace}_{T_{\omega} \times T_{\omega}} l_{\omega}(\bullet, \bullet) = L_{\omega}^g - \operatorname{trace}_{N_\omega \times N_\omega} l_{\omega}^g(\bullet, \bullet) \tag{17}$$

The proximity form is given in terms of the trilinear form

$$\alpha_{\omega}^g(h_1, h_2, h_3) = -\sum_{p,q} \gamma^{pq}(\omega) |l_{\omega}^g|_{\nabla g_p(\omega)}|h_2)(\nabla g_q(\omega)|h_3). \tag{18}$$

If $h_1 \in T_{\omega}$, then for any diffeomorphism $\phi$ of $\mathbb{R}^d$, $\alpha_{\omega}^{\phi, g}(h_1, h_2, h_3) = \alpha_{\omega}^g(h_1, h_2, h_3)$.

### 4. The canonical form of a foliation

For $m > d$, we consider differentiable maps $g = (g_1, g_2, \ldots, g_d) : \mathbb{R}^m \to \mathbb{R}^d$, non-degenerate at 0, i.e. the matrix $\gamma_{\omega}(0)$ is invertible. We define the equivalence relations:

(1) $g_1 \sim g_2$ if there exists $\phi$, a diffeomorphism of $\mathbb{R}^d$, $\phi \in \operatorname{diff}(\mathbb{R}^d)$ such that $g_2 = \phi \circ g_1$.

(2) $g_1 \vee g_2$ if $\|g_1(\xi) - g_2(\xi)\| \leq \text{constant} \times \|\xi\|^3$ when $\|\xi\| \to 0$.

Let $(\sim, \vee)$ be the equivalence relation on the set of differentiable maps from $\mathbb{R}^m$ to $\mathbb{R}^d$, obtained by superposing the two equivalence relations $\sim$ and $\vee$.

**Theorem 4.1.** In any equivalence class for $(\sim, \vee)$, there is a unique map $\xi \to \eta(\xi)$ from $\mathbb{R}^m$ to $\mathbb{R}^d$ with the following properties: There exist an orthonormal basis $(e_j)_{j=1,\ldots,m}$ of $\mathbb{R}^m$, for $\xi \in \mathbb{R}^m$, $\xi = \sum_{j=1}^m \hat{\xi}_j e_j + \sum_{j=d+1,\ldots,m} \hat{\xi}_j e_j$, and an orthonormal basis $(f_j)_{j=1,\ldots,d}$ of $\mathbb{R}^d$, for $\eta \in \mathbb{R}^d$, $\eta = \sum_{j=1,\ldots,d} \hat{\eta}_j f_j$ such that

for $i = 1, \ldots, d$, we have $\eta_i(\xi) = \hat{\eta}_i + q_i(\hat{\xi}) + B_i(\hat{\xi})$ where $\hat{\xi} = (0, \ldots, 0, \xi_{d+1}, \ldots, \xi_m)$ and $\hat{\xi} = (\xi_1, \ldots, \xi_d, 0, \ldots, 0), q_i$ is an homogeneous polynomial of degree 2 and $B_i$ is a bilinear form.

This reduction is called the canonical form of the foliation. The equivalence relation $(\sim, \vee)$ respects the Euclidean structure of $\mathbb{R}^m$ and that of $\mathbb{R}^d$. For the study of laws of random variables defined on the Wiener space $X$, the Euclidean structure of $H$ is relevant, while on the image space $\mathbb{R}^d$, only the volume structure is needed. For $\omega \in \mathbb{R}^m$, let

$$g_k(\omega) = \hat{\xi}_k + q_k(\hat{\xi}) + B_k(\hat{\xi}), \quad k = 1, \ldots, d, \quad \omega = (\hat{\xi}, \hat{\xi}), \quad \hat{\xi} \in \mathbb{R}^d, \quad \hat{\xi} \in \mathbb{R}^{m-d}. \tag{19}$$

The vector field $\nabla g_k$ has for components

$$[\nabla g_k]^s(\omega) = \frac{\partial}{\partial \xi^s} q_k(\hat{\xi}) + B_k(\hat{\xi}, e_s), \quad k = 1, \ldots, d, \quad s \in [1, d]; \quad [\nabla g_k]^s(\omega) = \frac{\partial}{\partial \xi^s} q_k(\hat{\xi}) + B_k(\hat{\xi}, e_s), \quad s \in [d, m]. \tag{20}$$

where $(e_1, \ldots, e_d, e_{d+1}, \ldots, e_m)$ is the canonical basis of $\mathbb{R}^m$. Thus $\gamma_{\omega}(0) = I_{\mathbb{R}^d}$. The divergence $\delta \nabla g_k$ of $\nabla g_k$ relatively to the Lebesgue volume is
\[ - \text{trace}_{H \times H} \nabla^2 g_k(0)(\bullet, \bullet) = - \sum_s \frac{\partial}{\partial \xi_s} [\nabla g_k]^s = - \sum_{s > d} \frac{\partial^2 g_k}{\partial \xi_s^2} \]  

(21)

The normal space at 0 is the set of \( \hat{\xi} \in \mathbb{R}^d \) and the tangent space at 0 is the set of \( \hat{\xi} \in \mathbb{R}^{m-d} \). The second fundamental form of \( g \) at \( \xi = 0 \) is \( \{ g_k(\hat{\xi}) \}_{i=1,\ldots,d} \). The curvature \( C_0 \) of the leaf \( g(\omega) = g(0) \) is given by (21). We define \( \rho(\hat{\xi}) \in \text{End}(\mathbb{R}^d) \) by

\[ \rho(\hat{\xi})(\hat{\xi}) = \left( B_j (\hat{\xi}, \hat{\xi}) \right)_{j=1,\ldots,d} = \frac{d}{d\epsilon |\epsilon=0} \tau_{0+\epsilon h-0} \quad \text{where } h = (\hat{\xi}, \hat{\xi}). \]  

(22)

Then \( (\beta(0; \hat{\xi}))_{jk} = -[\rho(\hat{\xi})_{jk} + \rho(\hat{\xi})_{kj}] \) and \( (\rho(\hat{\xi}))_{jk} = B_j (e_k, \hat{\xi}) = \nabla^2 g_j (\hat{\xi}, \nabla g_k(0)) \).

**Remark 3.** For \( j = 1, \ldots, d \), we can obtain a lift \( Z^j_{\text{prox}} \) of \( \frac{\partial}{\partial \eta_j} \) using the proximity endomorphism: Let \( g : X \to \mathbb{R}^d \) defined by \( \eta_i(\xi) = \hat{\xi}_i + q_i(\hat{\xi}) + [\rho_i(\hat{\xi})]_i \), for \( i = 1, \ldots, d \) and consider the matrix \( m_{kj} = ((I + \rho(\xi))^{-1})_{kj} \), then

\[ Z^j_{\text{prox}} = \sum_{k=1}^d m_{kj} (\hat{\xi}) \frac{\partial}{\partial \xi_k} \]  

are respectively lifted vector fields of \( \partial \eta_j \), \( j = 1, \ldots, d \).

(23)

The vector fields \((Z^j_{\text{prox}}(\xi))\) have a simple expression, but they are not in the normal space \( N_\xi \) and they are not defined everywhere since the matrix \((I + \rho(\xi))^{-1}\) is not always invertible. For the canonical lifts, this difficulty does not occur.

With (11)–(17), and (21), (22), we obtain the divergences of canonical lifts as trace of the curvature and proximity forms.

**Theorem 4.2.** For the gradient vector fields, it holds \( \delta_\mu(\nabla g^*(\omega)) = (\omega - C_\omega |\nabla g^*(\omega)) \) and for the canonical lifts (1),

\[ \delta_\mu(Z_{\text{can}}^s(\omega)) = (\omega - C_\omega |Z_{\text{can}}^s(\omega)) = \frac{1}{2} \text{trace} \beta(\omega; Z_{\text{can}}^s), \]  

(24)

where \( \text{trace} \beta(\omega; Z_{\text{can}}^s) \) is given by (14).

For a basic vector field \( Z \) in the normal space, see (7), the formula (24) extends with the additional term \( \delta_\nu(z) \), the divergence with respect to \( \nu = g \ast \mu \) of the vector field \( z = \sum_j z_j \partial_j \) on \( \mathbb{R}^d \) deduced from (7). In infinite dimension, the scalar product \( (x | Z) \) involves a Skorokhod stochastic integral.

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