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C. R. Acad. Sci. Paris, Ser. I 339 (2004) 839-842



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# A strongly degenerate elliptic equation arising from the semilinear Maxwell equations ☆

Partial Differential Equations

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Received and accepted 2 July 2004

Presented by Haïm Brezis

In memory of Guido Stampacchia

#### Abstract

We study a nonlinear equation arising from a semilinear perturbation of the Maxwell equations. The presence of the curl operator makes this equation strongly degenerate. A new variational approach, related to the Hodge decomposition of the vector potential **A**, is developed. *To cite this article: V. Benci, D. Fortunato, C. R. Acad. Sci. Paris, Ser. I 339 (2004).* © 2004 Académie des sciences. Published by Elsevier SAS. All rights reserved.

#### Résumé

Une équation elliptique fortement dégénérée provenant des équations de Maxwell semilinéaires. On étudie une équation nonlinéaire provenant d'une perturbation semilinéaire des équations de Maxwell. La présence du rotationnel rend l'équation fortement dégénérée. On propose une nouvelle approche liée à la décomposition de Hodge du potentiel vecteur A. *Pour citer cet article : V. Benci, D. Fortunato, C. R. Acad. Sci. Paris, Ser. I 339 (2004).* © 2004 Académie des sciences. Published by Elsevier SAS. All rights reserved.

#### 1. Introduction

In [1] a semilinear perturbation of Maxwell equations has been introduced. The Semilinear Maxwell Equations (SME) are the Euler–Lagrange equation relative to the functional

$$\frac{1}{2}\int \left[\langle \mathsf{d}\omega,\mathsf{d}\omega\rangle - W\big(\ast\langle\omega,\omega\rangle\big)\sigma\right]$$

(1)

<sup>&</sup>lt;sup>A</sup> Conference given by the first author during the meeting, Journées à la mémoire de Guido Stampacchia, Paris, 31 March and 1 April 2003. *E-mail addresses:* benci@dma.unipi.it (V. Benci), fortunat@dm.uniba.it (D. Fortunato).

<sup>1631-073</sup>X/ – see front matter © 2004 Académie des sciences. Published by Elsevier SAS. All rights reserved. doi:10.1016/j.crma.2004.07.029

where  $\omega = \sum_{i=0}^{3} A_i dx^i$  is the gauge-potential 1-form defined on the Minkowski space, \* is the Hodge operator relative to the Minkowski metric,  $\langle \cdot, \cdot \rangle$  is the Hodge pairing, W is a strictly convex smooth function, and  $\sigma$  is the volume form. In this model the particles are finite energy solitary waves due to the presence of the nonlinearity. In this respect the matter and the electromagnetic field have the same nature and their relation can be interpreted from a *unitarian standpoint* (see [4]). Finite energy means that particle have finite mass and this makes electrodynamics consistent.

In the magnetostatic case (i.e. when the electric field  $\mathbf{E} = \mathbf{0}$  and the magnetic field  $\mathbf{H}$  does not depend on time) SME are reduced to the semilinear elliptic degenerate equation

$$\nabla \times \nabla \times \mathbf{A} = f'(\mathbf{A}) \tag{2}$$

where  $\nabla \times$  denotes the *curl* operator, f' is the gradient of a strictly convex smooth function  $f : \mathbf{R}^3 \to \mathbf{R}$  and  $\mathbf{A} : \mathbf{R}^3 \to \mathbf{R}^3$  is the gauge potential related to the magnetic field  $\mathbf{H} (\mathbf{H} = \nabla \times \mathbf{A})$ .

The main difficulty in dealing with Eq. (2) relies in the fact that the energy related to it

$$\mathcal{E}(\mathbf{A}) = \int \left(\frac{1}{2}|\nabla \times \mathbf{A}|^2 - f(\mathbf{A})\right) \mathrm{d}x \tag{3}$$

does not yield a priori bound on  $\|\nabla \mathbf{A}\|_{L^2}$ . In particular the functional (3) is strongly indefinite in the sense that it is not bounded from above or from below and any possible critical point **A** has infinite Morse index; namely, the second variation of (3)

$$\mathcal{E}''(\mathbf{A})[v,v] = \int \left( |\nabla \times v|^2 - f''(\mathbf{A})[v,v] \right) dx$$

is negative definite on the infinite dimensional space

$$\{v = \nabla \varphi: \varphi \in C_0^\infty(\mathbf{R}^3, \mathbf{R})\}.$$

On the other hand, the nonlinearity  $f'(\mathbf{A})$  destroys the gauge invariance of (2).<sup>1</sup> So it is not possible to choose the Coulomb gauge (where  $\nabla \cdot \mathbf{A} = 0$ ) to avoid this indefiniteness.

We will use a min-max argument. Strongly indefinite functional have been largely considered in relation to other problems arising in mathematical physics and treated with min-max methods; we recall just few of them: periodic solutions of Hamiltonian system [3], Lorentzian geometry [2], Dirac equation [6]. By its nature, (3) cannot be studied by a direct application of the theories previously developed and a new approach is required. In particular we will use a global Liapunov–Schmidt type argument (which is a standard tool for this type of problems since the pioneering paper [5]) combined with a new functional framework related to the Hodge splitting of the vector field  $\mathbf{A}$ .

### 2. Statement of the result

Let  $f : \mathbf{R}^3 \to \mathbf{R}$  be a  $C^2$  function satisfying the following assumptions:

$$f(0) = 0$$
 and  $f$  strictly convex. (4)

There are positive constants  $c_1, c_2, p, q$  with 2 such that

$$c_1|\xi|^p \leqslant f(\xi) \quad \text{for } |\xi| \ge 1,\tag{5}$$

$$c_1|\xi|^q \leqslant f(\xi) \quad \text{for } |\xi| \leqslant 1,\tag{6}$$

 $<sup>^{1}</sup>$  The fact that this model is not gauge invariant deserves some discussion on physical grounds, for which we refer to a forthcoming paper. Here, we are mainly interested in the existence of solutions of (2).

$$\left|f'(\xi)\right| \leqslant c_2 |\xi|^{p-1} \quad \text{for } |\xi| \ge 1,\tag{7}$$

$$\left|f'(\xi)\right| \leqslant c_2 |\xi|^{q-1} \quad \text{for } |\xi| \leqslant 1. \tag{8}$$

The following theorem holds

**Theorem 2.1.** If f satisfies assumptions (4)–(8), Eq. (2) has at least a nontrivial weak solution having finite and positive energy. Moreover this solution has radial symmetry, namely

$$\mathbf{A}(x) = g^{-1}\mathbf{A}(gx) \quad \text{for all } g \in \mathcal{O}(3).$$

It can be shown that the functional (3) is the energy obtained by applying the classical Noether theorem to SME in the magnetostatic case. A simple rescaling argument shows that (3) on the solutions of (2) becomes (see [1])

$$\mathcal{E}(\mathbf{A}) = \frac{1}{3} \int |\nabla \times \mathbf{A}|^2 \, \mathrm{d}x.$$

- If we interpret  $f'(\mathbf{A})$  as the current density, (2) are formally the Maxwell equations (Ampere's law) for the magnetostatic. Notice that the current density is function of the gauge potential  $\mathbf{A}$  and

$$\Omega(\mathbf{A}) = \left\{ x \in \mathbf{R}^3 \colon \left| \mathbf{A}(x) \right| \ge 1 \right\}$$

represents the portion of space filled with matter. The growth assumptions on the nonlinear term f have not been suggested by technical mathematical reasons and existence results could be obtained under different assumptions. The assumptions on f have been suggested by physical reasons and they make the properties of the solutions consistent with the observed phenomena. In particular (8) implies that the current density  $\mathbf{J} = f'(\mathbf{A})$  becomes negligible in  $\mathbf{R}^3 \setminus \Omega(\mathbf{A})$  where (2) can be interpreted as the Maxwell equations in the empty space.

- It can be shown [1] that any static solution  $(\mathbf{A}, \varphi)$  of (SME) with positive energy carries a nontrivial magnetic potential  $\mathbf{A} \neq \mathbf{0}$  even when the electric potential  $\varphi$  vanishes. So any static solution carries a magnetic field and it is sensitive to external magnetic field even if it has no charge. This can be interpreted as the classical analogous of the spin.

The proof of Theorem 2.1 is quite involved. Here we give only an heuristic idea of the proof. By the Hodge decomposition theorem the vector field  $\mathbf{A} : \mathbf{R}^3 \to \mathbf{R}^3$  in (2) can be split as follows

$$\mathbf{A} = u + v = u + \nabla w \tag{9}$$

where  $u : \mathbf{R}^3 \to \mathbf{R}^3$  is a divergence free vector field  $(\nabla \cdot u = 0)$  and  $v : \mathbf{R}^3 \to \mathbf{R}^3$  is a potential vector field,  $v = \nabla w$  (*w* scalar field).

Since f is strictly convex, for every u with  $\nabla \cdot u = 0$ , we can find a scalar field  $w_0$  which minimizes the functional

$$w \mapsto \int f(u + \nabla w).$$

Set  $w_0 = \Phi(u)$ . Replacing (9) in (3) with  $w = \Phi(u)$ , we get a new functional

$$J(u) := \mathcal{E}(u, \Phi(u)) = \int \left(\frac{1}{2}|\nabla u|^2 - f(u + \nabla \Phi(u))\right) dx$$

which depends only on u. This functional has the mountain pass geometry. Then, we expect the existence of a nontrivial critical point  $u_0$ . Now, if J and the map  $u \to \Phi(u)$  were sufficiently smooth in suitable function spaces, the field

$$\mathbf{A} = u_0 + \nabla \big[ \boldsymbol{\Phi}(u_0) \big]$$

would solve Eq. (2). However, the lack of smoothness does not allow us to carry out a direct simple proof.

841

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