On the multidimensional stochastic equation $Y_{n+1} = A_n Y_n + B_n$

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Abstract
We study the behavior at infinity of the tail of the stationary solution of a multidimensional linear auto-regressive process with random coefficients. We exhibit an extended class of multiplicative coefficients satisfying a condition of irreducibility and proximality that yield to a heavy tail behavior.

Résumé
Sur l’équation vectorielle stochastique $Y_{n+1} = A_n Y_n + B_n$. On étudie le comportement à l’infini de la queue de la solution stationnaire d’un processus auto-régressif linéaire multidimensionnel à coefficients aléatoires. On donne une vaste classe de coefficients multiplicatifs vérifiant une condition d’irréductibilité et de proximalité qui conduisent à un comportement de type queue polynomiale.

1. Introduction
We study the following stochastic difference equation

$$Y_{n+1} = A_n Y_n + B_n, \quad n \in \mathbb{N}, \quad Y_n \in \mathbb{R}^d,$$

(1)

where $(A_n, B_n)$ is an iid sequence of random variables, $A_n$ is in $G$ the linear group of invertible square matrices of size $d$, and $B_n$ is a vector of $\mathbb{R}^d$. Here we restrict ourselves to $d \geq 2$ (see [8] and [4] for the one-dimensional case).

Under weak assumptions, the corresponding Markov process has a unique stationary solution. The behavior of its tail at infinity has been investigated by Kesten [8], when the coefficients are non-negative matrices and vectors.

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LePage [10] gave another result for a class of non-singular matrices. This note extends the latter result to a wide class of multiplicative coefficients, namely a class with a property of irreducibility and proximality.

2. Definitions and notation

For $s \geq 0$, we denote $k(s) = \lim_n (\mathbb{E}\|A_1 \cdots A_n\|^s)^{1/n}$, and $\sigma = \sup\{s \geq 0; \ k(s) < +\infty\}$. Throughout this note, we assume that (see [2,7])

$$\sigma > 0, \quad \mathbb{E}\log\|A_1^{-1}\| < \infty, \quad \mathbb{E}\log\|B_1\| < \infty, \quad \alpha = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}\left[\log\|A_1 A_2 \cdots A_n\|\right] < 0. \tag{C}$$

Then, Eq. (1) has a unique stationary solution (see [1]) that has the same law as the random variable

$$R = \sum_{k=1}^{\infty} A_1 A_2 \cdots A_{k-1} B_k.$$

Let $\eta$ denote the law of $(A_1, B_1)$, $S_\eta$ its support in the group $A = \mathcal{G} \ltimes \mathbb{R}^d$ of affine transformations $x \mapsto Ax + B$ on $\mathbb{R}^d$, and $\Gamma_\mu$ be the semi-group generated by $S_\eta$. Similarly, let $\mu$ be the law of $A_1$ ($\mu$ is the projection of $\eta$ on $\mathcal{G}$). $S_\mu$ its support and $\Gamma_\mu$ the semi-group it generates.

Following [8], we consider the row vectors of $\mathbb{R}^d$ and the right-hand side action of $\mathcal{G}$ on the unit sphere $S^{d-1}$: for all $x \in S^{d-1}$ and $a \in \mathcal{G}$, the action of $a$ on $x$ is denoted by $x \cdot a$ that is equal to $x a \|x a\|^{-1}$.

The semi-group $\Gamma_\mu$ is said to be irreducible if it has no invariant non-trivial sub-space. It is said to be proximal (see [3]) if for all $v$ and $v'$ in the projective space $\mathcal{P}^{d-1}$ (corresponding to row vectors) there is a sequence $(a_n)$ in $\Gamma_\mu$ such that $\lim_n \delta(v a_n, v' a_n) = 0$, where $\delta$ is a distance on $\mathcal{P}^{d-1}$. Finally, $\Gamma_\mu$ is said to be expanding or contracting if it has at least one element with spectral radius greater or less than one, respectively. If $\Gamma_\mu$ is all at once irreducible, proximal and expanding, it is said to satisfy Condition i-p-e.

3. The main theorem

Theorem 3.1. Let $d \geq 2$ and $(A_n, B_n)$ in $A$ be a sequence of iid random variable satisfying Condition (C). Suppose in addition that

(i) The semi-group $\Gamma_\mu$ generated by the support of the law $\mu$ of $A_1$ satisfies condition i-p-e.
(ii) The semi-group $\Gamma_\mu$ has no invariant salient closed convex cone with non empty interior.
(iii) The semi-group $\Gamma_\eta$ generated by the support of the law $\eta$ of $(A_1, B_1)$ has no fixed point in $\mathbb{R}^d$.

Then equation $k(s) = 1$ has a unique positive solution $\kappa$ on $[0, \sigma]$.

If in addition $\mathbb{E}[\|A_1\|^s \log |A_1|] > -\infty$ and there is a $\delta > 0$ such that $\mathbb{E}\|B_1\|^s < \infty$, then for all $x \in S^{d-1}$ we have

$$\lim_{t \to +\infty} t^\delta \mathbb{P}(x R > t) = 0, \quad \lim_{t \to +\infty} t^\delta \mathbb{P}(x R > t) = \varepsilon \varepsilon(x), \tag{2}$$

where $\varepsilon > 0$ and $\varepsilon$ is a positive symmetric continuous function on $S^{d-1}$.

In [8], a similar result is proved for non-negative matrices. This case is out of the scope of our theorem because of assumption (ii). Actually, the proof of [8] can be extended to the case when the semi-group $\Gamma_\mu$ has an invariant cone. Therefore our result is the complement of that of [8].

In [10], the assumption made on the coefficient $A_n$ is that the Markov chain $X_n = X_0 A_1 \cdots A_n$ on $S^{d-1}$ must hit any open subset for any starting point $X_0 = x$. Our conditions (i) and (ii) are much weaker. Indeed take for
instance a probability \( \mu \) with two atoms \( a \) and \( a' \), \( a' \) being a positive matrix and \( a' \) a negative matrix. Then the Markov chain \((X_n)\) starting from any positive or negative vector will never hit the set of vectors that are neither negative nor positive. It is not difficult to exhibit such examples satisfying conditions (i) and (ii). For instance, set \( d = 2, \mu = (\delta_2 + \delta_3)/2 \) and

\[
\begin{pmatrix}
2 & 1 \\
0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
-1/5 & -1/5 \\
-1/5 & 0
\end{pmatrix}.
\]

Then the semi-group \( \Gamma_\mu \) satisfies our hypotheses but not that of [10].

Our theorem also enables us to answer an open problem stated by Kesten in [8], namely: Let \( d = 2, \mu_1 \) and \( m_2 \) two positive matrices and \( m_3 \) a rotation. Take \( \mu = \mu_1 \delta_{m_1} + \mu_2 \delta_{m_2} + \mu_3 \delta_{m_3} \) with \( \mu_1 > 0 \) and \( \mu_1 + \mu_2 + \mu_3 = 1 \).

4. Sketch of the proof

Our proof follows the same steps as in [10] but uses the new tools given in [6]. The key point is to derive a renewal equation satisfied by \( z(x,t) = e^{-t} \int_0^t \mathbb{P}(x R > u) \, du \) and to prove that the renewal theorem for functionals of a Markov chain given in [9] applies.

The first step is to study the operator \( \mathcal{P} \) defined on the projective space \( \mathcal{P}^{d-1} \) by

\[
\mathcal{P} f(v) = \mathbb{E}[\|v A_1\|^\alpha f(v A_1)].
\]

It is proved in [6] that under the assumptions of our theorem, its spectral radius is 1 and it has a unique corresponding continuous eigenfunction \( \epsilon_\alpha \), which is positive. Hence we can define a Markovian operator on \( \mathcal{P}^{d-1} \) by:

\[
Q f(v) = \frac{1}{\epsilon_\alpha(v)} \mathbb{E}[\|v A_1\|^\alpha \epsilon_\alpha(v A_1) f(v A_1)].
\]

Under our assumption, \( Q \) has a spectral gap on a space of Hölder functions.

The second step is to prove that the operator \( Q \) defined on \( \mathcal{S}^{d-1} \) by:

\[
Q f(x) = \frac{1}{\epsilon_\alpha(\bar{x})} \mathbb{E}[\|x A_1\|^\alpha \epsilon_\alpha(\bar{x} A_1) f(\bar{x} \cdot A_1)],
\]

where \( \bar{x} \) is the projective image of \( x \), has the same properties as \( Q \), and in particular that it has a unique invariant probability. Assumption (ii) is essential for this uniqueness.

Then we prove that the renewal theorem of [9] applies to the following operator on \( \mathcal{S}^{d-1} \times \mathbb{R} \):

\[
Q f(x,t) = \frac{1}{\epsilon_\alpha(\bar{x})} \mathbb{E}[\|x A_1\|^\alpha \epsilon_\alpha(\bar{x} A_1) f(\bar{x} \cdot A_1, t - \log\|x A_1\|)].
\]
This gives us Eq. (2) with a non-negative constant $\ell$. To prove that $\ell$ is actually positive requires a detailed study of the operator defined by $Q$ on spaces of functions with controlled growth at infinity. Here again, we follow the original idea of [10].

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