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# Isotropic nonarchimedean $S$-arithmetic groups are not left orderable

Lucy Lifschitz[a], Dave Witte Morris[b,c,*]

[a] Department of Mathematics, University of Oklahoma, Norman, OK 73019, USA
[b] Department of Mathematics and Computer Science, University of Lethbridge, Lethbridge, Alberta T1K 3M4, Canada
[c] Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA

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**Abstract**

If $\mathcal{O}_S$ is the ring of $S$-integers of an algebraic number field $F$, and $\mathcal{O}_S$ has infinitely many units, we show that no finite-index subgroup of $\mathrm{SL}(2, \mathcal{O}_S)$ is left orderable. (Equivalently, these subgroups have no nontrivial orientation-preserving actions on the real line.) This implies that if $G$ is an isotropic $F$-simple algebraic group over an algebraic number field $F$, then no nonarchimedean $S$-arithmetic subgroup of $G$ is left orderable. Our proofs are based on the fact, proved by D. Carter, G. Keller, and E. Paige, that every element of $\mathrm{SL}(2, \mathcal{O}_S)$ is a product of a bounded number of elementary matrices. ***To cite this article: L. Lifschitz, D.W. Morris, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***


**Résumé**

**Les groupes $S$-arithmétiques non-archimédiens isotropes ne sont pas ordonnables à gauche.** Si $\mathcal{O}_S$ est l'anneau des $S$-entiers d'un corps de nombres $F$, et $\mathcal{O}_S$ a une infinité d'unités, nous prouvons qu'aucun sous-groupe d'indice fini de $\mathrm{SL}(2, \mathcal{O}_S)$ n'est ordonnable à gauche. (En d'autres termes, les sous-groupes d'indice fini de $\mathrm{SL}(2, \mathcal{O}_S)$ ne possèdent pas d'action non triviale sur la droite réelle respectant l'orientation.) Cela implique que si $G$ est un groupe algébrique $F$-simple isotrope, défini sur un corps de nombres $F$, alors aucun sous-groupe $S$-arithmétique non-archimédien de $G$ n'est ordonnable à gauche. La démonstration est fondée sur le fait, dû à D. Carter, G. Keller, et E. Paige, que chaque élément de $\mathrm{SL}(2, \mathcal{O}_S)$ est le produit d'un nombre borné de matrices élémentaires. ***Pour citer cet article : L. Lifschitz, D.W. Morris, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***


* Corresponding author.
*E-mail addresses:* LLifschitz@math.ou.edu (L. Lifschitz), Dave.Morris@uleth.ca (D.W. Morris).
*URLs:* http://www.math.ou.edu/~llifschitz/ (L. Lifschitz), http://people.uleth.ca/~dave.morris/ (D.W. Morris).

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## 1. Introduction

It is known [9] that finite-index subgroups of $\mathrm{SL}(3,\mathbb{Z})$ or $\mathrm{Sp}(4,\mathbb{Z})$ are not left orderable. (That is, there does not exist a total order $\prec$ on any finite-index subgroup, such that $ab \prec ac$ whenever $b \prec c$.) More generally, if $G$ is a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group, with $\mathbb{Q}\text{-rank}\, G \geqslant 2$, then no finite-index subgroup of $G_{\mathbb{Z}}$ is left orderable. It has been conjectured that the restriction on $\mathbb{Q}$-rank can be replaced with the same restriction on $\mathbb{R}$-rank, which is a much weaker hypothesis:

**Conjecture 1.** *If $G$ is a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group, with $\mathbb{R}\text{-rank}\, G \geqslant 2$, then no finite-index subgroup $\Gamma$ of $G_{\mathbb{Z}}$ is left orderable.*

*In other words, if $H$ is a connected, semisimple real Lie group, with $\mathbb{R}\text{-rank}\, H \geqslant 2$, and $\Gamma$ is an irreducible lattice in $H$, then $\Gamma$ is not left orderable.*

It is natural to propose an analogous conjecture that replaces $\mathbb{Z}$ with a ring of $S$-integers, and weakens the restriction on $\mathbb{R}$-rank. For simplicity, let us state it only in the case where $\mathbb{R}\text{-rank}\, G \geqslant 1$.

**Conjecture 2.** *If $G$ is a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group, with $\mathbb{R}\text{-rank}\, G \geqslant 1$, and $\{p_1,\ldots,p_n\}$ is any nonempty set of prime numbers, then no finite-index subgroup $\Gamma$ of $G_{\mathbb{Z}[1/p_1,\ldots,1/p_n]}$ is left orderable.*

*In other words, if $H$ is a product of noncompact real and $p$-adic simple Lie groups, with at least one real factor and at least one $p$-adic factor, and $\Gamma$ is any irreducible lattice in $H$, then $\Gamma$ is not left orderable.*

We prove Conjecture 2 under the additional assumption that $\mathbb{Q}\text{-rank}\, G \geqslant 1$:

**Theorem 1.1.** *If $G$ is a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group, with $\mathbb{Q}\text{-rank}\, G \geqslant 1$, and $\{p_1,\ldots,p_n\}$ is any nonempty set of prime numbers, then no finite-index subgroup $\Gamma$ of $G_{\mathbb{Z}[1/p_1,\ldots,1/p_n]}$ is left orderable.*

*More generally, if $H$ is a product of real and $p$-adic simple Lie groups, with at least one $p$-adic factor, and $\Gamma$ is any irreducible lattice in $H$, such that $H/\Gamma$ is not compact, then $\Gamma$ is not left orderable.*

We also prove some cases of Conjecture 1 (with $\mathbb{Q}\text{-rank}\, G = 1$). For example, we consider the case where every simple factor of $G_{\mathbb{R}}$ (or of $H$) is isomorphic to $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$:

**Theorem 1.2.** *If $\mathcal{O}$ is the ring of integers of a number field $F$, and $F$ is neither $\mathbb{Q}$ nor an imaginary quadratic extension of $\mathbb{Q}$, then no finite-index subgroup $\Gamma$ of $\mathrm{SL}(2,\mathcal{O})$ is left orderable.*

In geometric terms, the theorems can be restated as the nonexistence of orientation-preserving actions on the line:

**Corollary 1.3.** *If $\Gamma$ is as described in Theorem 1.1 or Theorem 1.2, then there does not exist any nontrivial homomorphism $\varphi : \Gamma \to \mathrm{Homeo}^+(\mathbb{R})$.*

Combining this corollary with an important theorem of Ghys [4] yields the conclusion that every orientation-preserving action of $\Gamma$ on the circle $S^1$ is of an obvious type; any such action is either virtually trivial or semiconjugate to an action by linear-fractional transformations, obtained from a composition $\Gamma \to \mathrm{PSL}(2,\mathbb{R}) \hookrightarrow \mathrm{Homeo}^+(S^1)$. See [5] for a discussion of the general topic of group actions on the circle.

It has recently been proved that certain individual arithmetic groups are not left orderable (see, e.g., [3]), but our results apparently provide the first new examples in more than ten years of arithmetic groups that have no left-orderable subgroups of finite index. They are also the only known such examples that have $\mathbb{Q}$-rank 1.

If $\Gamma$ is as described in Theorem 1.1 or Theorem 1.2, then $\Gamma$ contains a finite-index subgroup of $\mathrm{SL}(2,\mathcal{O}_S)$, where $S$ is a finite set of places of some algebraic number field $F$ (containing all the archimedean places), such that the corresponding ring $\mathcal{O}_S$ of $S$-integers has infinitely many units. The theorems are obtained by reducing to the fact, proved by Carter, Keller, and Paige [1], that $\mathrm{SL}(2, \mathcal{O}_S)$ has bounded generation by unipotent elements. (That is, the fact that $\mathrm{SL}(2, \mathcal{O}_S)$ is the product of finitely many of its unipotent subgroups. See [7] for a recent discussion of bounded generation. Partial results were proved previously in [2] and [6].) We are also able to prove this reduction for noncocompact lattices in $\mathrm{SL}(3, \mathbb{R})$:

**Theorem 1.4.** *Suppose $\Gamma$ is a finite-index subgroup of either*

(i) $\mathrm{SL}(2, \mathbb{Z}[1/r])$, *for some natural number $r > 1$, or, more generally,*
(ii) $\mathrm{SL}(2, \mathcal{O}_S)$, *where $S$ is a finite set of places of an algebraic number field $F$ (containing all the archimedean places), such that the corresponding ring $\mathcal{O}_S$ of $S$-integers has infinitely many units, or*
(iii) *an arithmetic subgroup of a quasi-split $\mathbb{Q}$-form of the $\mathbb{R}$-algebraic group* $\mathrm{SL}(3, \mathbb{R})$.

*If $\varphi : \Gamma \to \mathrm{Homeo}^+(\mathbb{R})$ is any homomorphism, and $U$ is any unipotent subgroup of $\Gamma$, then every $\varphi(U)$-orbit on $\mathbb{R}$ is bounded.*

**Corollary 1.5.** *Suppose*

- *$\Gamma$ is as described in Theorem* 1.4*, and*
- *$\Gamma$ is commensurable to a group that has bounded generation by unipotent elements.*

*Then every homomorphism $\varphi : \Gamma \to \mathrm{Homeo}^+(\mathbb{R})$ is trivial. Therefore, $\Gamma$ is not left orderable.*

## 2. Proof of Theorem 1.4(i)

**Notation 1.** *For convenience, let*

$$\bar{u} = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}, \qquad \underline{v} = \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix}, \qquad \hat{s} = \begin{bmatrix} s & 0 \\ 0 & 1/s \end{bmatrix}$$

*for $u, v \in \mathbb{Z}[1/r]$ and $s \in \{ r^n \mid n \in \mathbb{Z} \}$.*

Suppose some $\varphi(U)$-orbit on $\mathbb{R}$ is not bounded above. (This will lead to a contradiction.) Let us assume $U$ is a maximal unipotent subgroup of $\Gamma$.

Let $V$ be a subgroup of $\Gamma$ that is conjugate to $U$, but is not commensurable to $U$. Then $V_{\mathbb{Q}} \neq U_{\mathbb{Q}}$. Because $\mathbb{Q}\text{-rank}\,\mathrm{SL}(2, \mathbb{Q}) = 1$, this implies that $V_{\mathbb{Q}}$ is opposite to $U_{\mathbb{Q}}$. Therefore, after replacing $U$ and $V$ by a conjugate under $\mathrm{SL}(2, \mathbb{Q})$, we may assume

$$U = \{ \bar{u} \mid u \in \mathbb{Z}[1/r] \} \cap \Gamma \quad \text{and} \quad V = \{ \underline{v} \mid v \in \mathbb{Z}[1/r] \} \cap \Gamma.$$

Because $V$ is conjugate to $U$, we know that some $\varphi(V)$-orbit is not bounded above. Let

$$x_U = \sup\{ x \in \mathbb{R} \mid \text{the } \varphi(U)\text{-orbit of } x \text{ is bounded above} \} < \infty \quad \text{and}$$
$$x_V = \sup\{ x \in \mathbb{R} \mid \text{the } \varphi(V)\text{-orbit of } x \text{ is bounded above} \} < \infty.$$

Assume, without loss of generality, that $x_U \geqslant x_V$.

Fix some $s = r^n > 1$, such that $\hat{s} \in \Gamma$, and let $B = \langle \hat{s} \rangle U$. Because $\langle \hat{s} \rangle$ normalizes $U$, this is a subgroup of $\Gamma$. Note that $\varphi(B)$ fixes $x_U$, so it acts on the interval $(x_U, \infty)$. Since $\varphi(B)$ is nonabelian, it is well known (see, e.g., [5, Thm. 6.10]) that some nontrivial element of $\varphi(B)$ must fix some point of $(x_U, \infty)$. In fact, it is not difficult to see that each element of $\varphi(B) \backslash \varphi(U)$ fixes some point of $(x_U, \infty)$. In particular, $\varphi(\hat{s})$ fixes some point $x$ of $(x_U, \infty)$.

The left-ordering of any additive subgroup of $\mathbb{Q}$ is unique (up to a sign), so we may assume that

$$\varphi(\overline{u_1})x < \varphi(\overline{u_2})x \Leftrightarrow u_1 < u_2 \quad \text{and} \quad \varphi(\underline{v_1})x < \varphi(\underline{v_2})x \Leftrightarrow v_1 < v_2.$$

The $\varphi(U)$-orbit of $x$ is not bounded above (because $x > x_U$), so we may fix some $u_0, v_0 > 0$, such that

$$\varphi(\underline{v_0})x < \varphi(\overline{u_0})x.$$

For any $\underline{v} \in V$, there is some $k \in \mathbb{Z}^+$, such that $v < s^{2k} v_0$. Then, because $\varphi(\hat{s})$ fixes $x$ and $s^{-2k} < 1$, we have

$$\begin{aligned}\varphi(\underline{v})x &< \varphi(\underline{s^{2k} v_0})x = \varphi(\hat{s}^{-k}\underline{v_0}\hat{s}^k)x = \varphi(\hat{s}^{-k})\varphi(\underline{v_0})x \\ &< \varphi(\hat{s}^{-k})\varphi(\overline{u_0})x = \varphi(\hat{s}^{-k}\overline{u_0}\hat{s}^k)x = \varphi(\overline{s^{-2k}u_0})x < \varphi(\overline{u_0})x.\end{aligned}$$

So the $\varphi(V)$-orbit of $x$ is bounded above by $\varphi(\overline{u_0})x$. This contradicts the fact that $x > x_U \geqslant x_V$.

## 3. Other parts of Theorem 1.4

(ii) The above proof of case (i) needs only minor modifications to be applied with a more general ring $\mathcal{O}_S$ of $S$-integers in the place of $\mathbb{Z}[1/r]$. (We choose $s = \omega^n$, where $\omega$ is a unit of infinite order in $\mathcal{O}_S$.) The one substantial difference between the two cases is that the left-ordering of the additive group of $\mathcal{O}_S$ is far from unique—there are usually infinitely many different orderings. Fortunately, we are interested only in left-orderings of $U = \{\bar{u} \mid u \in \mathcal{O}\} \cap \Gamma$ that arise from an unbounded $\varphi(U)$-orbit, and it turns out that any such left-ordering must be invariant under conjugation by $\hat{s}$. The left-ordering must, therefore, arise from a field embedding $\sigma$ of $F$ in $\mathbb{C}$ (such that $\sigma(s)$ is real whenever $\hat{s} \in \Gamma$), and there are only finitely many such embeddings. Hence, we may replace $U$ and $V$ with two conjugates of $U$ whose left-orderings come from the same field embedding (and the same choice of sign).

(iii) A serious difficulty prevents us from applying the above proof to quasi-split $\mathbb{Q}$-forms of $\mathrm{SL}(3,\mathbb{R})$. Namely, the reason we were able to obtain a contradiction is that if $\overline{u_0}$ is upper triangular, $\underline{v}$ is lower triangular, $\hat{s}$ is diagonal, and $\lim_{k\to\infty} \hat{s}^{-k}\overline{u_0}\hat{s}^k = \infty$ under an ordering of $\Gamma$, then $\lim_{k\to\infty} \hat{s}^{-k}\underline{v}\hat{s}^k = e$. Unfortunately, the "opposition involution" of $\mathrm{SL}(3,\mathbb{R})$ causes the calculation to result in a different conclusion in case (iii): if $\hat{s}^{-k}\overline{u_0}\hat{s}^k$ tends to $\infty$, then $\hat{s}^{-k}\underline{v}\hat{s}^k$ also tends to $\infty$. Thus, the above simple argument does not immediately yield a contradiction.

Instead, we employ a lemma of Raghunathan [8, Lem. 1.7] that provides certain nontrivial relations in $\Gamma$. These relations involve elements of both $U$ and $V$; they provide the crucial tension that leads to a contradiction.

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