Abstract

If $O_S$ is the ring of $S$-integers of an algebraic number field $F$, and $O_S$ has infinitely many units, we show that no finite-index subgroup of $\text{SL}(2, O_S)$ is left orderable. (Equivalently, these subgroups have no nontrivial orientation-preserving actions on the real line.) This implies that if $G$ is an isotropic $F$-simple algebraic group over an algebraic number field $F$, then no nonarchimedean $S$-arithmetic subgroup of $G$ is left orderable. Our proofs are based on the fact, proved by D. Carter, G. Keller, and E. Paige, that every element of $\text{SL}(2, O_S)$ is a product of a bounded number of elementary matrices.

Résumé

Les groupes $S$-arithmétiques non-archimédiens isotropes ne sont pas ordonnables à gauche. Si $O_S$ est l’anneau des $S$-entières d’un corps de nombres $F$, et $O_S$ a une infinité d’unités, nous prouvons qu’aucun sous-groupe d’indice fini de $\text{SL}(2, O_S)$ n’est ordonnable à gauche. (En d’autres termes, les sous-groupes d’indice fini de $\text{SL}(2, O_S)$ ne possèdent pas d’action non triviale sur la droite réelle respectant l’orientation.) Cela implique que si $G$ est un groupe algébrique $F$-simple isotrope, défini sur un corps de nombres $F$, alors aucun sous-groupe $S$-arithmétique non-archimédien de $G$ n’est ordonnable à gauche. La démonstration est fondée sur le fait, dû à D. Carter, G. Keller, et E. Paige, que chaque élément de $\text{SL}(2, O_S)$ est le produit d’un nombre borné de matrices élémentaires.

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1. Introduction

It is known [9] that finite-index subgroups of \( \text{SL}(3, \mathbb{Z}) \) or \( \text{Sp}(4, \mathbb{Z}) \) are not left orderable. (That is, there does not exist a total order \( \prec \) on any finite-index subgroup, such that \( ab \prec ac \) whenever \( b \prec c \).) More generally, if \( G \) is a \( \mathbb{Q} \)-simple algebraic \( \mathbb{Q} \)-group, with \( \mathbb{Q} \)-rank \( G \geq 2 \), then no finite-index subgroup of \( G_{\mathbb{Z}} \) is left orderable. It has been conjectured that the restriction on \( \mathbb{Q} \)-rank can be replaced with the same restriction on \( \mathbb{R} \)-rank, which is a much weaker hypothesis:

**Conjecture 1.** If \( G \) is a \( \mathbb{Q} \)-simple algebraic \( \mathbb{Q} \)-group, with \( \mathbb{R} \)-rank \( G \geq 2 \), then no finite-index subgroup \( \Gamma \) of \( G_{\mathbb{Z}} \) is left orderable.

In other words, if \( H \) is a connected, semisimple real Lie group, with \( \mathbb{R} \)-rank \( H \geq 2 \), and \( \Gamma \) is an irreducible lattice in \( H \), then \( \Gamma \) is not left orderable.

It is natural to propose an analogous conjecture that replaces \( \mathbb{Z} \) with a ring of \( S \)-integers, and weakens the restriction on \( \mathbb{R} \)-rank. For simplicity, let us state it only in the case where \( \mathbb{R} \)-rank \( G \geq 1 \):

**Conjecture 2.** If \( G \) is a \( \mathbb{Q} \)-simple algebraic \( \mathbb{Q} \)-group, with \( \mathbb{R} \)-rank \( G \geq 1 \), and \( \{ p_1, \ldots, p_n \} \) is any nonempty set of prime numbers, then no finite-index subgroup \( \Gamma' \) of \( G_{\mathbb{Z}[1/p_1,\ldots,1/p_n]} \) is left orderable.

In other words, if \( H \) is a product of noncompact real and \( p \)-adic simple Lie groups, with at least one real factor and at least one \( p \)-adic factor, and \( \Gamma' \) is any irreducible lattice in \( H \), then \( \Gamma' \) is not left orderable.

We prove Conjecture 2 under the additional assumption that \( \mathbb{Q} \)-rank \( G \geq 1 \):

**Theorem 1.1.** If \( G \) is a \( \mathbb{Q} \)-simple algebraic \( \mathbb{Q} \)-group, with \( \mathbb{Q} \)-rank \( G \geq 1 \), and \( \{ p_1, \ldots, p_n \} \) is any nonempty set of prime numbers, then no finite-index subgroup \( \Gamma' \) of \( G_{\mathbb{Z}[1/p_1,\ldots,1/p_n]} \) is left orderable.

More generally, if \( H \) is a product of real and \( p \)-adic simple Lie groups, with at least one \( p \)-adic factor, and \( \Gamma' \) is any irreducible lattice in \( H \), such that \( H/\Gamma' \) is not compact, then \( \Gamma' \) is not left orderable.

We also prove some cases of Conjecture 1 (with \( \mathbb{Q} \)-rank \( G = 1 \)). For example, we consider the case where every simple factor of \( G_{\mathbb{R}} \) (or of \( H \)) is isomorphic to \( \text{SL}(2, \mathbb{R}) \) or \( \text{SL}(2, \mathbb{C}) \):

**Theorem 1.2.** If \( \mathcal{O} \) is the ring of integers of a number field \( F \), and \( F \) is neither \( \mathbb{Q} \) nor an imaginary quadratic extension of \( \mathbb{Q} \), then no finite-index subgroup \( \Gamma' \) of \( \text{SL}(2, \mathcal{O}) \) is left orderable.

In geometric terms, the theorems can be restated as the nonexistence of orientation-preserving actions on the line:

**Corollary 1.3.** If \( \Gamma' \) is as described in Theorem 1.1 or Theorem 1.2, then there does not exist any nontrivial homomorphism \( \varphi: \Gamma' \to \text{Homeo}^+(\mathbb{R}) \).

Combining this corollary with an important theorem of Ghys [4] yields the conclusion that every orientation-preserving action of \( \Gamma' \) on the circle \( S^1 \) is of an obvious type; any such action is either virtually trivial or semiconjugate to an action by linear-fractional transformations, obtained from a composition \( \Gamma' \to \text{PSL}(2, \mathbb{R}) \to \text{Homeo}^+(S^1) \). See [5] for a discussion of the general topic of group actions on the circle.

It has recently been proved that certain individual arithmetic groups are not left orderable (see, e.g., [3]), but our results apparently provide the first new examples in more than ten years of arithmetic groups that have no left-orderable subgroups of finite index. They are also the only known such examples that have \( \mathbb{Q} \)-rank 1.

If \( \Gamma' \) is as described in Theorem 1.1 or Theorem 1.2, then \( \Gamma' \) contains a finite-index subgroup of \( \text{SL}(2, \mathcal{O}_S) \), where \( S \) is a finite set of places of some algebraic number field \( F \) (containing all the archimedean places), such
that the corresponding ring \( O_S \) of \( S \)-integers has infinitely many units. The theorems are obtained by reducing to the fact, proved by Carter, Keller, and Paige [1], that \( \text{SL}(2, O_S) \) has bounded generation by unipotent elements. (That is, the fact that \( \text{SL}(2, O_S) \) is the product of finitely many of its unipotent subgroups. See [7] for a recent discussion of bounded generation. Partial results were proved previously in [2] and [6].) We are also able to prove this reduction for noncocompact lattices in \( \text{SL}(3, \mathbb{R}) \):

**Theorem 1.4.** Suppose \( \Gamma \) is a finite-index subgroup of either

(i) \( \text{SL}(2, \mathbb{Z}[1/r]) \), for some natural number \( r > 1 \), or, more generally,
(ii) \( \text{SL}(2, O_S) \), where \( S \) is a finite set of places of an algebraic number field \( F \) (containing all the archimedean places), such that the corresponding ring \( O_S \) of \( S \)-integers has infinitely many units, or
(iii) an arithmetic subgroup of a quasi-split \( \mathbb{Q} \)-form of the \( \mathbb{R} \)-algebraic group \( \text{SL}(3, \mathbb{R}) \).

If \( \varphi : \Gamma \to \text{Homeo}^+(\mathbb{R}) \) is any homomorphism, and \( U \) is any unipotent subgroup of \( \Gamma \), then every \( \varphi(U) \)-orbit on \( \mathbb{R} \) is bounded.

**Corollary 1.5.** Suppose

- \( \Gamma \) is as described in Theorem 1.4, and
- \( \Gamma \) is commensurable to a group that has bounded generation by unipotent elements.

Then every homomorphism \( \varphi : \Gamma \to \text{Homeo}^+(\mathbb{R}) \) is trivial. Therefore, \( \Gamma \) is not left orderable.

2. Proof of Theorem 1.4(i)

**Notation 1.** For convenience, let

\[
\begin{bmatrix}
1 & u \\
0 & 1
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 \\
v & 1
\end{bmatrix}, \quad
\begin{bmatrix}
s & 0 \\
0 & 1/s
\end{bmatrix}
\]

for \( u, v \in \mathbb{Z}[1/r] \) and \( s \in \{ r^n \mid n \in \mathbb{Z} \} \).

Suppose some \( \varphi(U) \)-orbit on \( \mathbb{R} \) is not bounded above. (This will lead to a contradiction.) Let us assume \( U \) is a maximal unipotent subgroup of \( \Gamma \).

Let \( V \) be a subgroup of \( \Gamma \) that is conjugate to \( U \), but is not commensurable to \( U \). Then \( V_Q \neq U_Q \). Because \( \mathbb{Q} \)-rank \( \text{SL}(2, \mathbb{Q}) = 1 \), this implies that \( V_Q \) is opposite to \( U_Q \). Therefore, after replacing \( U \) and \( V \) by a conjugate under \( \text{SL}(2, \mathbb{Q}) \), we may assume

\[
U = \left\{ \begin{bmatrix} u & \cdot \\ \cdot & \cdot \end{bmatrix} \mid u \in \mathbb{Z}[1/r] \right\} \cap \Gamma \quad \text{and} \quad V = \left\{ \begin{bmatrix} \cdot & \cdot \\ v & \cdot \end{bmatrix} \mid v \in \mathbb{Z}[1/r] \right\} \cap \Gamma.
\]

Because \( V \) is conjugate to \( U \), we know that some \( \varphi(V) \)-orbit is not bounded above. Let

\[
\begin{align*}
x_U &= \sup\{ x \in \mathbb{R} \mid \text{the } \varphi(U)\text{-orbit of } x \text{ is bounded above} \} < \infty \quad \text{and} \quad \\
x_V &= \sup\{ x \in \mathbb{R} \mid \text{the } \varphi(V)\text{-orbit of } x \text{ is bounded above} \} < \infty.
\end{align*}
\]

Assume, without loss of generality, that \( x_U \geq x_V \).

Fix some \( s = r^n > 1 \), such that \( s \in \Gamma \), and let \( B = \langle \tilde{s} \rangle U \). Because \( \langle \tilde{s} \rangle \) normalizes \( U \), this is a subgroup of \( \Gamma \). Note that \( \varphi(B) \) fixes \( x_U \), so it acts on the interval \( (x_U, \infty) \). Since \( \varphi(B) \) is nonabelian, it is well known (see, e.g., [5, Thm. 6.10]) that some nontrivial element of \( \varphi(B) \) must fix some point of \( (x_U, \infty) \). In fact, it is not difficult to see that each element of \( \varphi(B)/\varphi(U) \) fixes some point of \( (x_U, \infty) \). In particular, \( \varphi(\tilde{s}) \) fixes some point \( x \) of \( (x_U, \infty) \).
The left-ordering of any additive subgroup of \( \mathbb{Q} \) is unique (up to a sign), so we may assume that
\[
\varphi(\bar{u}_1)x < \varphi(\bar{u}_2)x \Leftrightarrow u_1 < u_2 \quad \text{and} \quad \varphi(v_1)x < \varphi(v_2)x \Leftrightarrow v_1 < v_2.
\]
The \( \varphi(U) \)-orbit of \( x \) is not bounded above (because \( x > x_U \)), so we may fix some \( u_0, v_0 > 0 \), such that
\[
\varphi(v_0)x < \varphi(u_0)x.
\]
For any \( \bar{v} \in V \), there is some \( k \in \mathbb{Z}^+ \), such that \( v < s^{2k}v_0 \). Then, because \( \varphi(\hat{s}) \) fixes \( x \) and \( s^{-2k} < 1 \), we have
\[
\varphi(v)x < \varphi(s^{2k}v_0)x = \varphi(\hat{s}^{-k}u_0s^k)x = \varphi(\hat{s}^{-k})\varphi(u_0)x
\]
\[
< \varphi(\hat{s}^{-k})\varphi(\bar{u}_0)x = \varphi(\hat{s}^{-k}\bar{u}_0s^k)x = \varphi(s^{-2k}u_0)x < \varphi(u_0)x.
\]
So the \( \varphi(V) \)-orbit of \( x \) is bounded above by \( \varphi(\bar{u}_0)x \). This contradicts the fact that \( x > x_U \geq x_V \).

3. Other parts of Theorem 1.4

(ii) The above proof of case (i) needs only minor modifications to be applied with a more general ring \( O_S \) of \( S \)-integers in the place of \( \mathbb{Z}[1/\omega] \). (We choose \( s = \omega^\rho \), where \( \omega \) is a unit of infinite order in \( O_S \).) The one substantial difference between the two cases is that the left-ordering of the additive group of \( O_S \) is far from unique—there are usually infinitely many different orderings. Fortunately, we are interested only in left-orderings of \( U = \{ \bar{u} \mid u \in O \} \cap \Gamma \) that arise from an unbounded \( \varphi(U) \)-orbit, and it turns out that any such left-ordering must be invariant under conjugation by \( \hat{s} \). The left-ordering must, therefore, arise from a field embedding \( \sigma \) of \( F \) in \( \mathbb{C} \) (such that \( \sigma(s) \) is real whenever \( \hat{s} \in \Gamma \)), and there are only finitely many such embeddings. Hence, we may replace \( U \) and \( V \) with two conjugates of \( U \) whose left-orderings come from the same field embedding (and the same choice of sign).

(iii) A serious difficulty prevents us from applying the above proof to quasi-split \( \mathbb{Q} \)-forms of \( SL(3, \mathbb{R}) \). Namely, the reason we were able to obtain a contradiction is that if \( \bar{u}_0 \) is upper triangular, \( v \) is lower triangular, \( \hat{s} \) is diagonal, and \( \lim_{k \to \infty} \hat{s}^{-k}\bar{u}_0s^k = \infty \) under an ordering of \( \Gamma \), then \( \lim_{k \to -\infty} \hat{s}^{-k}\bar{u}_0s^k = e \). Unfortunately, the “opposition involution” of \( SL(3, \mathbb{R}) \) causes the calculation to result in a different conclusion in case (iii): if \( \hat{s}^{-k}\bar{u}_0s^k \) tends to \( \infty \), then \( \hat{s}^{-k}\bar{u}_0s^k \) also tends to \( \infty \). Thus, the above simple argument does not immediately yield a contradiction.

Instead, we employ a lemma of Raghunathan [8, Lem. 1.7] that provides certain nontrivial relations in \( \Gamma \). These relations involve elements of both \( U \) and \( V \); they provide the crucial tension that leads to a contradiction.

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References