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On the diffusive behavior of isotropic diffusions in a random environment

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Abstract

We present here results concerning the asymptotic behavior of isotropic diffusions in random environment that are small perturbations of Brownian motion. When the space dimension is three or more we prove an invariance principle as well as transience. Our methods also apply to questions of homogenization in random media. ***To cite this article: A.-S. Sznitman, O. Zeitouni, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Comportement asymptotique de diffusions isotropes en milieu aléatoire. Nous présentons ici des résultats sur le comportement asymptotique de diffusions isotropes en milieu aléatoire, qui sont de petites perturbations du mouvement brownien. Lorsque la dimension de l'espace est trois ou plus nous prouvons un principe d'invariance et la transience de la diffusion. Nos méthodes s'appliquent aussi à des problèmes d'homogénéisation en milieu aléatoire. ***Pour citer cet article : A.-S. Sznitman, O. Zeitouni, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Le comportement asymptotique d'un mouvement brownien perturbé par une dérive dépendant de l'environnement qui n'est ni de type gradient ni incompressible reste à ce jour mal compris. Nous présentons dans cette note des résultats sur le comportement diffusif et la transience de diffusions isotropes en environnement aléatoire qui

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sont de petites perturbations du mouvement brownien en dimension au moins trois. Le modèle considéré peut être vu comme le pendant en espace et temps continus du modèle de marche aléatoire en milieu aléatoire regardé par Bricmont–Kupiainen [2]. La stratégie de nos démonstrations est cependant différente.

Les caractéristiques locales, i.e. matrices de covariance et dérive de la diffusion, sont des fonctions stationnaires bornées $a(x,\omega)$, $b(x,\omega)$, $x \in \mathbb{R}^d$, $\omega \in \Omega$, où Ω est muni d'un groupe $(t_x)_{x\in\mathbb{R}^d}$ de transformations conjointement mesurables préservant la probabilité $\mathbb{P}$ sur Ω. On suppose $a(\cdot,\cdot)$ uniformément elliptique, et que pour tout $\omega \in \Omega$, $a(\cdot,\omega)$, $b(\cdot,\omega)$ satisfont la condition de Lipschitz (6) de constante K. On suppose que les caractéristiques locales satisfont aussi une condition de dépendance à portée finie R, cf. (8), et d'isotropie restreinte, cf. (9). On dénote par $P_{x,\omega}$ la loi sur $C(\mathbb{R}_+,\mathbb{R}^d)$ de la solution du problème de martingales associé à x et L, cf. (7). On note $X_{\textbf{.}}$ le processus canonique sur $C(\mathbb{R}_+,\mathbb{R}^d)$. Nos résultats principaux sont

Théorème 0.1 ($d \geqslant 3$). *Il existe $\eta_0(d,K,R) > 0$ tel que lorsque*

$$\text{pour tout } x \in \mathbb{R}^d,\ \omega \in \Omega, \quad \left|a(x,\omega) - I\right| \leqslant \eta_0, \quad \left|b(x,\omega)\right| \leqslant \eta_0, \tag{1}$$

alors pour $\mathbb{P}$-presque tout ω,

$$\text{sous } P_{0,\omega},\ \frac{1}{\sqrt{t}} X_{\cdot t} \text{ converge en loi lorsque } t \to \infty, \text{ vers un mouvement brownien de variance déterministe } \sigma^2 > 0, \tag{2}$$

$$\text{pour tout } x \in \mathbb{R}^d,\ P_{x,\omega}\text{-}p.s., \quad \lim_{t\to\infty} |X_t| = \infty. \tag{3}$$

Théorème 0.2 ($d \geqslant 3$). *La constante $\eta_0(d,K,R)$ peut être choisie de manière à ce que lorsque* (1) *est vérifiée, pour $\mathbb{P}$-presque tout ω, pour toutes fonctions bornées f, g sur $\mathbb{R}^d$ respectivement continue et satisfaisant une condition de Hölder, la solution du problème de Cauchy*

$$\begin{cases} \partial_t u_\epsilon = L_\epsilon u_\epsilon + g, & \text{dans } (0,\infty) \times \mathbb{R}^d, \\ u_\epsilon|_{t=0} = f, \end{cases} \tag{4}$$

avec $\epsilon > 0$ et L_ϵ défini en (14)*, converge uniformément sur les compacts de $\mathbb{R}_+ \times \mathbb{R}^d$, lorsque $\epsilon \to 0$, vers la solution de*

$$\begin{cases} \partial_t u_0 = \dfrac{\sigma^2}{2} \Delta u_0 + g, & \text{dans } (0,\infty) \times \mathbb{R}^d, \\ u_0|_{t=0} = f, \end{cases} \tag{5}$$

avec σ^2 comme en (2)*.*

1. Setting and main results

The mathematical investigation of transport in random media has been an active field of research for a number of years, see for instance [1,3,5–10,12,13]. However, the occurrence of diffusive behavior for Brownian motion perturbed by an environment-dependent drift that is neither of gradient-type nor incompressible remains poorly understood. We present here results concerning the diffusive character and the transience of isotropic diffusions in random environment that are small perturbations of Brownian motion, when the space dimension is three or more. The model under consideration is a continuous counterpart of the model studied by Bricmont–Kupiainen [2]. Our strategy of proof is, however, different.

The local characteristics, i.e. the covariance matrix and the drift of the diffusion, are bounded stationary functions $a(x,\omega)$, $b(x,\omega)$, $x \in \mathbb{R}^d$, $\omega \in \Omega$, and Ω is endowed with a group $(t_x)_{x\in\mathbb{R}^d}$ of jointly measurable transformations preserving the probability $\mathbb{P}$ on Ω. We assume that $a(\cdot,\cdot)$ is uniformly elliptic and for some $K > 0$,

$$\left|a(x,\omega) - a(y,\omega)\right| + \left|b(x,\omega) - b(y,\omega)\right| \leqslant K|x-y|, \quad \text{for } x, y \in \mathbb{R}^d,\ \omega \in \Omega. \tag{6}$$

We denote by $P_{x,\omega}$ the unique probability on $C(\mathbb{R}_+, \mathbb{R}^d)$ that is the solution of the Martingale problem attached to $x \in \mathbb{R}^d$ and L, with

$$L = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(y,\omega)\partial_{ij}^2 + \sum_{i=1}^{d} b_i(y,\omega)\partial_i. \tag{7}$$

We let $X_{\textstyle .}$ stand for the canonical process on $C(\mathbb{R}_+, \mathbb{R}^d)$. The random characteristics are further assumed to have finite range dependence, namely for some $R > 0$, under $\mathbb{P}$

$$\begin{aligned} &\sigma\big(a(x,\cdot), b(x,\cdot),\ x \in A\big) \quad \text{and } \sigma\big(a(y,\cdot), b(y,\cdot), y \in B\big) \text{ are} \\ &\text{independent when } A, B \subseteq \mathbb{R}^d \text{ lie at mutual distance at least } R. \end{aligned} \tag{8}$$

They are also supposed to fulfill a restricted isotropy condition, namely for any rotation matrix r preserving the union of coordinate axes of $\mathbb{R}^d$:

$$\big(a(rx,\omega), b(rx,\omega)\big)_{x\in\mathbb{R}^d} \quad \text{has the same law as } \big(ra(x,\omega)r^T,\ rb(x,\omega)\big)_{x\in\mathbb{R}^d}. \tag{9}$$

We can now state our main results

Theorem 1.1 ($d \geqslant 3$). *There is $\eta_0(d,K,R) > 0$, such that when*

$$\big|a(x,\omega) - I\big| \leqslant \eta_0, \big|b(x,\omega)\big| \leqslant \eta_0, \quad \textit{for all } x \in \mathbb{R}^d,\ \omega \in \Omega, \tag{10}$$

then for $\mathbb{P}$-a.e. ω,

$$\begin{aligned} &\frac{1}{\sqrt{t}} X_{\cdot t} \textit{ converges in } P_{0,\omega}\textit{-law as } t \to \infty, \textit{to a Brownian motion on } \mathbb{R}^d \textit{ with deterministic} \\ &\textit{variance } \sigma^2 > 0, \end{aligned} \tag{11}$$

$$\textit{for all } x \in \mathbb{R}^d,\ P_{x,\omega}\textit{-a.s.},\ \lim_{t\to\infty} |X_t| = \infty. \tag{12}$$

In other words for diffusions in random environment that are small perturbation of Brownian motion and satisfy the restricted isotropy condition (9), we prove transience and diffusive behavior when $d \geqslant 3$. Our results also apply to homogenization in random media, namely

Theorem 1.2 ($d \geqslant 3$). *One can choose $\eta_0(d,K,R) > 0$ so that when* (10) *holds, on a set of full $\mathbb{P}$-probability, for any bounded functions f, g on $\mathbb{R}^d$ that are respectively continuous and Hölder continuous, the solution of the Cauchy problem*:

$$\begin{cases} \partial_t u_\epsilon = L_\epsilon u_\epsilon + g, & \textit{in } (0,\infty) \times \mathbb{R}^d, \\ u_\epsilon|_{t=0} = f, \end{cases} \tag{13}$$

where for $\epsilon > 0$,

$$L_\epsilon = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}\left(\frac{x}{\epsilon},\omega\right)\partial_{ij}^2 + \sum_{i=1}^{d} \frac{1}{\epsilon} b_i\left(\frac{x}{\epsilon},\omega\right)\partial_i, \tag{14}$$

converges uniformly on compact subsets of $\mathbb{R}_+ \times \mathbb{R}^d$, as $\epsilon \to 0$, to the solution of the Cauchy problem

$$\begin{cases} \partial_t u_0 = \dfrac{\sigma^2}{2} \Delta u_0 + g, & \textit{in } (0,\infty) \times \mathbb{R}^d, \\ u_0|_{t=0} = f. \end{cases} \tag{15}$$

2. The main renormalization step

The proofs of the results in the previous section are based on a renormalization scheme that we describe here. One picks a number $\beta \in (0, \frac{1}{2}]$ that will control Hölder regularity, cf. (24) below. One introduces for $L_0 \geqslant 10^{a^{-1}}$, see (17), with L_0 integer multiple of 5, the sequence of length scales $L_n, D_n, \widetilde{D}_n$, $n \geqslant 0$, via

$$L_{n+1} = \ell_n L_n, \quad \text{with } \ell_n = 5[L_n^a/5], \quad \text{where} \tag{16}$$

$$a = \beta/(1000d), \tag{17}$$

$$D_n = L_n \exp\{c_0(\log\log L_n)^2\}, \quad \widetilde{D}_n = L_n \exp\{2c_0(\log\log L_n)^2\}, \quad \text{with} \tag{18}$$

$$c_0 \log\left(1 + \frac{a}{2}\right) = 1. \tag{19}$$

One defines for $n \geqslant 0$, the stopping times

$$T_n = \inf\Big\{u \geqslant 0, \sup_{0 \leqslant s \leqslant u} |X_s - X_0| \geqslant \widetilde{D}_n\Big\}, \tag{20}$$

and the probability kernels

$$R_n(x, dy) = P_{x,\omega}[X_{L_n^2} \in dy], \qquad \widetilde{R}_n(x, dy) = P_{x,\omega}[X_{L_n^2 \wedge T_n} \in dy], \tag{21}$$

$$R_n^0(x, dy) = (2\pi\alpha_n L_n^2)^{-d/2} \exp\left\{-\frac{|y-x|^2}{2\alpha_n L_n^2}\right\} dy, \tag{22}$$

with

$$\alpha_n \, d \, L_n^2 = \mathbb{E}E_{0,\omega}\big[|X_{L_n^2 \wedge T_n}|^2\big]. \tag{23}$$

One denotes with $|\cdot|_{(n)}$ the sequence of norms on the space of β-Hölder continuous functions on $\mathbb{R}^d$:

$$|h|_{(n)} = \sup_{x \in \mathbb{R}^d} \big|h(x)\big| + L_n^\beta \sup_{x \neq y} \frac{|h(x) - h(y)|}{|x-y|^\beta}, \quad n \geqslant 0, \tag{24}$$

and with $\|\cdot\|_n$ the corresponding operator-norm. These operator-norms play an important role in measuring the closeness of R_n or $\widetilde{R}_n$ to R_n^0, cf. (29), (34).

To measure traps arising in various scales in the medium, one introduces non-negative variables as follows. For $n \geqslant 0$, $x \in L_n\mathbb{Z}^d$, one defines the concentric cubes with respective side-length L_n and $3/2L_n$

$$C_n(x) = x + L_n[0,1]^d, \qquad C_n'(x) = x + L_n\left(-\frac{1}{4}, \frac{5}{4}\right)^d. \tag{25}$$

One 'chops' each of the $2d$ faces of $\partial C_n(x)$ into $5^{(d-1)}$ closed $(d-1)$-dimensional cubes of side-length $L_n/5$, and denotes with $C_{n,\gamma}(x)$, $1 \leqslant \gamma \leqslant 2d\,5^{(d-1)}$, the resulting closed d-dimensional cubes in the outwards normal direction to $\partial C_n(x)$. One sets

$$\zeta = 2(1 + 2d3^{d+1})^{-1}, \tag{26}$$

and introduces for $n \geqslant 0$, $x \in L_n\mathbb{Z}^d$, $A \subseteq C_n(x)$, $1 \leqslant \gamma \leqslant 2d\,5^{(d-1)}$, $\omega \in \Omega$,

$$J_{n,x,A,\gamma}(\omega) = \inf\Big\{u \geqslant 0, \inf_{y \in A} P_{y,\omega}\big[H_{C_{n,\gamma}(x)} \leqslant L_n^2 \wedge T_{C_n'(x)}\big] \geqslant c_1 L_n^{-\zeta u}\Big\}, \tag{27}$$

where $H_{C_{n,\gamma}(x)}$ and $T_{C_n'(x)}$ respectively stand for the entrance time of the diffusion in $C_{n,\gamma}(x)$ and its exit time from $C_n'(x)$, and where $c_1 \in (0,1)$ is a certain constant depending on d. The variables in (27) control how well the diffusion starting in the smaller box $C_n(x)$ travels to the boundary boxes $C_{n,\gamma}(x)$ without leaving the larger box $C'_n(x)$, within time L_n^2. Intuitively they measure the strength of traps at level n in $C_n(x)$.

We call *n-admissible family*, for $n \geqslant 0$, an arbitrary collection

$$\begin{aligned}&(u_x, A_x, \gamma_x)_{x\in\mathcal{A}},\ \text{where } \mathcal{A} \text{ is a finite subset of } L_n\mathbb{Z}^d,\ \text{and for } x\in\mathcal{A},\\ &u_x>0,\ \gamma_x\in\{1,\dots,2d\,5^{(d-1)}\},\ \text{and } A_x\subseteq C_n(x) \text{ is a union of boxes}\\ &C_{n-1}(z) \text{ (with the convention } L_{-1}=1,\ \text{when } n=0\text{), such that}\\ &d_\infty(A_x,A_{x'})\geqslant 10dL_{n-1},\ \text{when } x\neq x'.\end{aligned}\tag{28}$$

In the induction step we will propagate an upper bound on $\mathbb{P}[$ for $x\in\mathcal{A}$, $J_{n,x,A_x,\gamma_x}\geqslant u_x]$ for n-admissible families that will show that with overwhelming probability the variables in (27) vanish.

One also introduces for $n\geqslant 0$, $\omega\in\Omega$, cf. (20)–(22) and below (24), the set

$$\mathcal{B}_n(\omega)=\Big\{x\in L_n\mathbb{Z}^d;\ \text{for } |y-x|\leqslant 30\sqrt{d}L_n,\ P_{y,\omega}\Big[\sup_{0\leqslant s\leqslant L_n^2}|X_s-X_0|\geqslant v\Big]\leqslant \mathrm{e}^{-v/D_n},$$

$$\text{for all } v\geqslant D_n,\ \text{and } \big\|\chi_{n,x}(\widetilde{R}_n-R_n^0)\big\|_n\leqslant L_n^{-\delta}\Big\},\tag{29}$$

where the number δ and the cut-off functions $\chi_{n,x}$ are respectively defined by

$$\delta=\frac{5}{32}\beta,\quad \text{and}\tag{30}$$

$$\chi_{n,x}(\cdot)=\chi\left(\frac{\cdot-x}{10\sqrt{d}\,L_n}\right),\quad \text{with } \chi(z)=1\wedge\big(2-|z|\big)_+.\tag{31}$$

In the induction step we will also propagate an upper bound on $\mathbb{P}[0\notin\mathcal{B}_n(\omega)]$ of central importance, that will show that any given site of $L_n\mathbb{Z}^d$ with high probability lies in $\mathcal{B}_n(\omega)$. Finally, one defines the integer $m_0\geqslant 2$, such that

$$(1+a)^{m_0-2}\leqslant 100<(1+a)^{m_0-1},\tag{32}$$

as well as the numbers

$$M_0=100d(1+a)^{m_0+2},\qquad M=1000M_0.\tag{33}$$

The main renormalization step is then described by

Theorem 2.1. *There are positive constants c_2, c such that for $L_0\geqslant c$ and $n_0\geqslant m_0+1$, if for all $0\leqslant n\leqslant n_0$,*

$$\mathbb{P}[0\notin\mathcal{B}_n(\omega)]\leqslant L_n^{-M_0},\tag{34}$$

and for all n-admissible families $(u_x,A_x,\gamma_x)_{x\in\mathcal{A}}$,

$$\mathbb{P}[\textit{for all } x\in\mathcal{A},\ J_{n,x,A_x,\gamma_x}\geqslant u_x]\leqslant L_n^{-\overline{M}_n\sum_{x\in\mathcal{A}}(u_x+1)},\quad \textit{with } \overline{M}_n=M\prod_{0\leqslant j<n}\left(1-\frac{c_2}{\log L_j}\right),\tag{35}$$

and if

$$\frac{1}{2}\leqslant\alpha_n\leqslant 2,\quad \textit{for } 0\leqslant n\leqslant n_0,\qquad |\alpha_{n+1}-\alpha_n|\leqslant L_n^{-(1+9/10)\delta},\quad \textit{for } 0\leqslant n<n_0,\tag{36}$$

then estimates (34), (35) *hold with n_0+1 in place of n_0 and*

$$|\alpha_{n_0+1}-\alpha_{n_0}|\leqslant L_{n_0}^{-(1+9/10)\delta}.\tag{37}$$

Once Theorem 2.1 is proved one sees that choosing the local characteristics of the diffusion in random environment sufficiently close to that of Brownian motion, see (10), one can start the induction and verify (34)–(36) for all $n_0 \geqslant m_0 + 1$. With Borel–Cantelli's lemma, one then sees that on a set of full $\mathbb{P}$-measure, for $n \geqslant N(\omega)$, each x in $L_n\mathbb{Z}^d \cap (-L_{n+3}^2, L_{n+3}^2)^d$ belongs to $\mathcal{B}_n(\omega)$. Using the Kantorovich–Rubinstein Theorem, cf. Dudley [4], the controls on Hölder norms and on the displacements of the trajectory, cf. the definition of $\mathcal{B}_n(\omega)$ in (29), enable to construct good couplings of the diffusion in random environment with Brownian motion of variance α_n. With (36) the numbers α_n converge to a positive limit, namely the σ^2 of (15). The results of Section 1 are then derived with the help of these coupling measures. Detailed proofs will be found in [11].

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