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Logic

Antidirected paths in 5-chromatic digraphs

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Abstract

Let T_5 be the regular 5-tournament. B. Grünbaum proved that T_5 is the only 5-tournament which contains no copy of the antidirected path P_4. In this Note, we prove that, except for T_5, any connected 5-chromatic oriented digraph in which each vertex has out-degree at least two contains a copy of P_4. It will be shown, by an example, that the condition that each vertex has out-degree at least two is indispensable. ***To cite this article: A. El Sahili, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Chemins antidirigés dans les graphes 5-chromatiques. Soit T_5 le tournoi régulier contenant cinq sommets. B. Grünbaum a prouvé que T_5 est le seul 5-tournoi qui ne contient pas le chemin antidirigé P_4. Nous prouvons dans cette Note que T_5 est le seul graphe orienté 5-chromatique dans lequel tout sommet a un degré extérieur au moins deux qui ne contient pas le chemin antidirigé P_4. On prouve à l'aide d'un exemple que la condition « tout sommet a un degré exterieur au moins deux » est indispensable. ***Pour citer cet article : A. El Sahili, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

1. Introduction

The digraphs considered here have no loops or multiple edges. An *oriented graph* is a digraph in which, for every two vertices x and y, at most one of (x, y), (y, x) is an edge. The digraphs used in this Note are all oriented graphs. By $G(D)$ we denote the underlying graph of a digraph D. The chromatic number of a digraph is the chromatic number of its underlying graph. A graph G is *k-critical* if $\chi(G) = k$ and $\chi(G - v) = k - 1$ for any vertex v in $V(G)$.

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A block of an oriented path is a maximal directed subpath. We recall that the *length* of a path is the number of its edges. The antidirected path is an oriented path in which each block is of length 1. We denote by P_n the antidirected path of length n, beginning with a backward edge.

The problem of determining which oriented paths lie in a given n-chromatic digraph D is a well-known problem. When D is an n-tournament, the problem has been completely resolved (Havet and Thomassé [6]). However, the case of an arbitrary n-chromatic digraph is still an open question. We know only that an n-chromatic digraph contains a directed path of length $n-1$ (Roy [7], Gallai [4]), and a path of length $n-1$ formed by two blocks, one of which has length 1 [2]. In this Note, we will be interested in the antidirected paths. In order to generalize the results found on tournaments to arbitrary digraphs, and as a first step in this direction, we generalize to 5-chromatic digraphs a particular result of Grünbaum on 5-tournaments: any 5-tournament, except for the regular tournament T_5, contains a copy of P_4.

2. The main result

Theorem 2.1. *Let D be a 5-chromatic connected digraph, distinct from T_5, in which each vertex has out-degree at least two. Then D contains a copy of P_4.*

To prove this theorem, we need several lemmas.

Lemma 2.2 (Grünbaum [5]). *Except for T_5, any 5-tournament contains a copy of P_4.*

Corollary 2.3. *Let D be as in the above theorem. If D contains T_5, then D contains a copy of P_4.*

In the sequel, D will denote an oriented digraph as described in Theorem 2.1; by the above corollary we may assume that D contains no 5-tournament as a subdigraph. Moreover, we suppose to the contrary that D contains no copy of P_4. Let D' be a 5-critical subdigraph of D and let D° be the subdigraph of D' induced by the vertices of out-degree at least three in D'.

Let G be a graph which contains no K_{2n+1}, where $n \geqslant 2$. Suppose that we can orient G in such a way that each vertex has in-degree at most n. It is shown in [1] that $\chi(G) \leqslant 2n$. We have then the following lemma

Lemma 2.4. *The set $V(D^\circ)$ is not empty.*

Lemma 2.5. *Let v be a vertex of D and let x, y be two vertices in $N^-(v)$. If $x \in V(D^\circ)$, then $y \notin V(D^\circ)$.*

Corollary 2.6. *For every vertex v in D°, $d^-_{D^\circ}(v) \leqslant 1$.*

Lemma 2.7. *Let H be a connected digraph in which each vertex has in-degree at most one. Then H contains at most one cycle.*

Lemma 2.8. *Let v be a vertex of D such that $d^+(v) \geqslant 3$ and let x, y and z be three distinct vertices in $N^+(v)$. Suppose that $x \to y$. Then $N^-(y) = N^-(z) = \{v, x\}$.*

We may easily deduce that $x \to z$ and $yz \notin E(G(D))$ in this case.

Corollary 2.9. *Let x and y be two adjacent vertices of D. Suppose that there exist two vertices v and v' of D such that $\{x, y\} \subseteq N^+(v) \cap N^-(v')$. Then $N^+(v) = \{x, y\}$.*

Lemma 2.10. *The set $V(D^\circ)$ is independent in D.*

Claim 1. *Any connected component L of D° contains a vertex v such that $N^+(v) \cap (V(D') \setminus V(D^\circ))$ has at least two vertices.*

Proof. If L is a cycle, then each vertex of L satisfies the claim; otherwise L contains a vertex v of out-degree zero in D°, and so $N^+(v) \subseteq V(D') \setminus V(D^\circ)$. □

Proof of Lemma 2.10. Suppose to the contrary that D° is not an independent set, then there is a connected component L of D° containing at least two vertices. We can choose a vertex v in L satisfying the claim such that $d_L^-(v) = 1$. Let v' be a vertex in L such that $v' \to v$ and let v_1, v_2 and v_3 be three vertices in $N_{D'}^+(v)$ such that $\{v_1, v_2\} \subseteq V(D') \setminus V(D^\circ)$. The digraph D' is 5-critical, so any vertex has degree at least 4 in D'. Since for any $i \in \{1, 2\}$, $d_{D'}^+(v_i) \leqslant 2$, we have $d_{D'}^-(v_i) \geqslant 2$. Therefore, there is a vertex u of D' and $j \in \{1, 2\}$ such that $u \notin \{v, v_1, v_2\}$ and $u \to v_j$; we have either $u \notin \{v, v_1, v_2, v_3\}$ or $u = v_3$. In the latter case $v_3 \notin V(D^\circ)$ by Lemma 2.5. We have $d_{D'}^-(v_3) \geqslant 2$, so there is a vertex w of D' such that $w \notin \{v, v_1, v_2, v_3\}$ and $w \to v_3$, thus we may assert that there exists a vertex u of D' and $j \in \{1, 2, 3\}$ such that $u \notin \{v, v_1, v_2, v_3\}$, $v_j \notin D^\circ$ and $u \to v_j$. Let u' be a vertex of D distinct from v_j such that $u \to u'$. If $u' \neq v$, the path $u'uv_jvv_h$ is a copy of P_4, where $h \in \{1, 2, 3\} \setminus \{j\}$ is chosen such that $u' \neq v_h$, a contradiction. Otherwise, let w be a vertex in $N^+(v') \setminus \{v, v_j, u\}$. Such a vertex exists since $d^+(v') \geqslant 3$ and $v_j \notin N^+(v')$ by Lemma 2.5. The path $v_juvv'w$ is a copy of P_4, a contradiction. □

In the sequel, we will need the following theorem proved by Gallai [3].

Theorem 2.11. *Let G be a k-critical graph, where k is a positive integer. Let G_m be the subgraph of G induced by the vertices of degree $k-1$. Then each block of G_m is either complete or a chordless odd cycle.*

D_4 will denote the subdigraph of D' induced by the vertices of degree 4.

Lemma 2.12. *Any vertex of D' has in-degree (in D') at least 2.*

We now associate to each vertex v in D° the set

$$S(v) = \{t(v), t'(v), v_0, \ldots, v_{g(v)}, v_{g(v)+1}\}, \quad 0 \leqslant g(v) \leqslant 5,$$

defined as follows (see Fig. 1): $\{v_0, t(v), t'(v)\} = N_{D'}^+(v)$ where $v_0 \to t(v)$ and $v_0 \to t'(v)$, $v_1 = v$. Set $T(v) = \{t(v), t'(v)\}$. If $d_{D'}^-(v_0) \geqslant 3$, put $g(v) = 0$; if not, let v_2 be the unique vertex of D' distinct from v_1 such that $v_2 \to v_0$. We have $v_2 \to v_1$. Again, if $d_{D'}^-(v_1) \geqslant 3$, put $g(v) = 1$; otherwise, let v_3 be the unique vertex of D' distinct from v_2 such that $v_3 \to v_1$; such a vertex exists by Lemma 2.12. We have $v_3 \to v_2$, since otherwise we have either a path P_4 in D or $d_{D'}^-(v_0) \geqslant 3$. We may continue this process until meeting the first vertex of in-degree at least three in D'; call this vertex $v_{g(v)}$, where $g(v)$ is the number of iterations required. Such a vertex exists and $g(v) \leqslant 5$. In fact, suppose that $v_1, \ldots, v_5$ are defined as above and $d_{D'}^+(v_i) = 2$, $i = 1, \ldots, 4$. By Corollary 2.9,

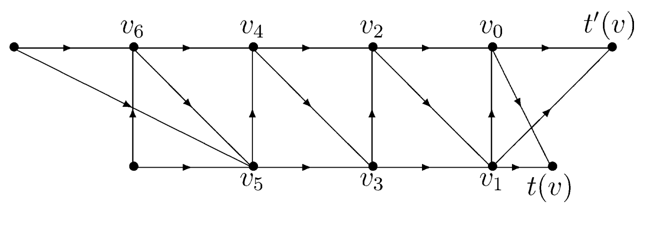

Fig. 1. The case $g(v) = 5$.

we have $d^+_{D'}(v_i) = 2$, $i = 2, \ldots, 5$. If $d^-_{D'}(v_5) = 2$ the vertices $v_2, \ldots, v_5$ will be in the same block of D_4. By Theorem 2.11, $D'[v_2, \ldots, v_5]$ is complete, which is a contradiction since $v_2v_5 \notin E(G(D))$.

Set $O(v) = t\{z \in D'\colon z \neq v_{g(v)+1}$ and $z \to v_{g(v)}\}$; we have $z \to v_{g(v)+1}$ for every z in $O(v)$.

Lemma 2.13. *Let u and v be two distinct vertices of D°. We have*:

$$S(u) \cap S(v) = \phi.$$

Lemma 2.14. *Set $L = \{v_{g(v)}\colon v \in D^\circ\}$. We have*:

(i) $d^-_{D'}(x) = 3$ *for any x in L.*
(ii) $d^-_{D'}(x) = 2$ *otherwise.*

Corollary 2.15. *For any vertex v in D°, $O(v)$ contains exactly two vertices.*

Proof of Theorem 2.1. Define the sets:

$$S = \bigcup_{v \in V(D^\circ)} S(v), \quad O = \bigcup_{v \in V(D^\circ)} O(v), \quad T = \bigcup_{v \in V(D^\circ)} T(v).$$

We have $|O| \leqslant |T|$. If $O = T$, then $N_{D'}(v) \subseteq S$ for every v in S. Since D' is critical, it must be connected and so $D' = D'[S]$. We define a colouring c of D' as follows: Let v be a vertex in D°. Put $c(t(v)) = c(t'(v)) = 1$, $c(v_0) = 2$, $c(v_1) = 3$. If $g(v) = 1$, put $c(v_2) = 4$. If $g(v) > 1$, the colours 1, 2 and 3 suffice to colour $S(v) \setminus \{v_{g(v)}, v_{g(v)+1}\}$. Put $c(v_{g(v)}) = 4$ and $c(v_{g(v)+1}) = i$ where $i \in \{2, 3\}$ is chosen such that $i \neq c(v_{g(v)-1})$. It is clear that c is a proper 4-colouring of the 5-chromatic digraph D', a contradiction.

If $O \neq T$ then, since $|O| \leqslant |T|$, there is a vertex v in D° such that either $t(v) \notin O$ or $t'(v) \notin O$. Suppose, without loss of generality, that $t(v) \notin O$. Then $N^+_{D'}(t(v)) \cap S = \phi$. Let $N^+_{D'}(t(v)) = \{u, u'\}$. We have $\{u, u'\} \cap (D^\circ \cup L) = \phi$, so $d^-_{D'}(u) = d^+_{D'}(u) = d^-_{D'}(u') = d^+_{D'}(u') = 2$ and $d_{D'}(u) = d_{D'}(u') = 4$. On the other hand, there exists a vertex w in D' such that $w \notin \{u, u'\}$ and $N^+_{D'}(w) \cap \{u, u'\} \neq \phi$. We have $N^+_{D'}(w) = \{u, u'\}$ since D' contains no path P_4 and $wt(v)$ cannot be an edge of $G(D')$; thus $d_{D'}(w) = 4$.

Since $d_{D'}(t(v)) = 4$, the vertices $t(v)$, u, u' and w are in a block of D_4 which is neither complete nor a chordless odd cycle, which contradicts Theorem 2.11. This completes the proof of Theorem 2.1. □

An example which shows that the condition that each vertex has out-degree at least two in Theorem 2.1 is indispensable can be constructed from the 5-tournament T_5 with an edge (x, y) such that $x \notin V(T_5)$ and $y \in V(T_5)$.

If H contains a path P_4, x cannot be an interior vertex of P_4 since $d(x) = 1$; furthermore it cannot be an end of P_4 since $d^-(x) = 0$. Thus $P_4 \subseteq T_5$ which contradicts Lemma 2.2.

We conclude this paper by asking the following question: Does there exist a 5-chromatic oriented graph which contains neither a 5-tournament nor P_4?

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