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Another approach to linearized elasticity and Korn's inequality

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Abstract

We describe and analyze an approach to the pure traction problem of three-dimensional linearized elasticity, whose novelty consists in considering the linearized strain tensor as the 'primary' unknown, instead of the displacement itself as is customary. This approach leads to a well-posed minimization problem, constrained by a weak form of the St Venant compatibility conditions. It also provides a new proof of Korn's inequality. ***To cite this article: P.G. Ciarlet, P. Ciarlet Jr., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

On décrit et analyse une approche du problème de traction pure en élasticité linéarisée tridimensionnelle, dont la nouveauté consiste à considérer le tenseur linéarisé des déformations comme l'inconnue principale, au lieu du déplacement lui-même selon l'habitude. Cette approche conduit à un problème bien posé de minimisation sous contraintes, celles-ci consistant en une forme affaiblie des conditions de compatibilité de St Venant. Cette approche conduit aussi à une nouvelle démonstration de l'inégalité de Korn. ***Pour citer cet article : P.G. Ciarlet, P. Ciarlet Jr., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

1. The classical approach to existence theory in linearized elasticity

Latin indices range over the set $\{1, 2, 3\}$ and the summation convention with respect to repeated indices is used in conjunction with this rule. The Euclidean and exterior products of $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ are denoted $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{a} \wedge \boldsymbol{b}$. The matrix inner product of two 3×3 matrices $\boldsymbol{\varepsilon}$ and $\boldsymbol{e}$ is denoted $\boldsymbol{\varepsilon} : \boldsymbol{e} = \operatorname{tr} \boldsymbol{\varepsilon}^{\mathrm{T}} \boldsymbol{e}$. The identity mapping of a set X is denoted $\boldsymbol{id}_X$. The restriction of a mapping f to a set X is denoted $f|_X$.

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Given an open subset Ω of $\mathbb{R}^3$, spaces of vector-valued or matrix-valued functions or distributions defined on Ω are denoted by boldface letters. The norm in the space $\boldsymbol{L}^2(\Omega)$ is denoted $\|\cdot\|_{0,\Omega}$ and that in the space $\boldsymbol{H}^1(\Omega)$ is denoted $\|\cdot\|_{1,\Omega}$. If V is a vector space and R a subspace of V, the quotient space of V modulo R is denoted V/R and the equivalence class of $v \in V$ modulo R is denoted $\dot{v}$.

Let x_i denote the coordinates of a point $x \in \mathbb{R}^3$, let $\partial_i := \partial/\partial x_i$ and $\partial_{ij} := \partial^2/\partial x_i \partial x_j$. Given a vector field $\boldsymbol{v} = (v_i)$, the 3×3 matrix with $\partial_j v_i$ as its element at the i-th row and j-th column is denoted $\boldsymbol{\nabla v}$.

Let Ω be an open, bounded, and connected subset of $\mathbb{R}^3$ whose boundary Γ is Lipschitz-continuous in the sense of Nečas [13] or Adams [1]. Assume that the set $\overline{\Omega}$ is the *reference configuration* occupied by a *linearly elastic body* in the absence of applied forces. The elastic material constituting the body, which may be nonhomogeneous and anisotropic, is thus characterized by its *elasticity tensor* $\boldsymbol{A} = (A_{ijkl}) \in \boldsymbol{L}^\infty(\Omega)$, whose elements possess the symmetries $A_{ijkl} = A_{jikl} = A_{klij}$, and which is uniformly positive-definite a.e. in Ω, in the sense that there exists a constant $\alpha > 0$ such that $\boldsymbol{A}(x)\boldsymbol{t} : \boldsymbol{t} \geqslant \alpha \boldsymbol{t} : \boldsymbol{t}$ for almost all $x \in \Omega$ and all 3×3 symmetric matrices $\boldsymbol{t} = (t_{ij})$, where $(\boldsymbol{A}(x)\boldsymbol{t})_{ij} := A_{ijkl}(x)t_{kl}$.

The body is assumed to be subjected to *applied body forces* in its interior with density $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$. Hence the linear form $L : \boldsymbol{H}^1(\Omega) \to \mathbb{R}$ defined by $L(\boldsymbol{v}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}x$ for all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ is continuous.

Then the associated *pure traction problem of linearized elasticity* classically consists in finding a displacement vector field $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ that satisfies $J(\boldsymbol{u}) = \inf_{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)} J(\boldsymbol{v})$, where the quadratic functional J is defined by

$$J(\boldsymbol{v}) = \frac{1}{2} \int_\Omega \boldsymbol{A e}(\boldsymbol{v}) : \boldsymbol{e}(\boldsymbol{v})\, \mathrm{d}x - L(\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}^1(\Omega),$$

$$\boldsymbol{e}(\boldsymbol{v}) := \frac{1}{2}(\boldsymbol{\nabla v}^{\mathrm{T}} + \boldsymbol{\nabla v}) = \left(\frac{1}{2}(\partial_i v_j + \partial_j v_i)\right) \in \boldsymbol{L}^2_{\mathrm{sym}}(\Omega)$$

denotes the *linearized strain tensor field* associated with an arbitrary vector field $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, and

$$\boldsymbol{L}^2_{\mathrm{sym}}(\Omega) := \{\boldsymbol{e} = (e_{ij}) \in \boldsymbol{L}^2(\Omega);\ e_{ij} = e_{ji} \text{ in } \Omega\}.$$

Let

$$\boldsymbol{R}(\Omega) := \{\boldsymbol{r} \in \boldsymbol{H}^1(\Omega);\ \boldsymbol{e}(\boldsymbol{r}) = 0 \text{ in } \Omega\} = \{\boldsymbol{r} = \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{id}_\Omega;\ \boldsymbol{a} \in \mathbb{R}^3,\ \boldsymbol{b} \in \mathbb{R}^3\}$$

denote the *space of infinitesimal rigid displacements of the set* Ω. The applied forces are also assumed to be such that the associated linear form L satisfies the (clearly necessary) relation $L(\boldsymbol{r}) = 0$ for all $\boldsymbol{r} \in \boldsymbol{R}(\Omega)$. Hence the above minimization problem is equivalent to finding $\dot{\boldsymbol{u}} \in \dot{\boldsymbol{H}}^1(\Omega) := \boldsymbol{H}^1(\Omega)/\boldsymbol{R}(\Omega)$ such that

$$J(\dot{\boldsymbol{u}}) = \inf_{\dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega)} J(\dot{\boldsymbol{v}}),$$

where $J(\dot{\boldsymbol{v}}) := J(\boldsymbol{v})$ for all $\dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega)$. In order to apply the Lax–Milgram lemma, it suffices to show that the mapping $\dot{\boldsymbol{v}} \to \|\boldsymbol{e}(\dot{\boldsymbol{v}})\|_{0,\Omega}$ is a norm over the quotient space $\dot{\boldsymbol{H}}^1(\Omega)$ equivalent to the quotient norm, defined by

$$\|\dot{\boldsymbol{v}}\|_{1,\Omega} := \inf_{\boldsymbol{r} \in \boldsymbol{R}(\Omega)} \|\boldsymbol{v} + \boldsymbol{r}\|_{1,\Omega} \quad \text{for all } \dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega).$$

The proof comprises two stages, whose proofs are well known. We nevertheless record these here (see Theorems 1.1 and 1.2) for the sake of comparison with those found in the present approach. The first stage consists in establishing the classical *Korn inequality in the space* $\boldsymbol{H}^1(\Omega)$:

Theorem 1.1. *There exists a constant C such that*

$$\|\boldsymbol{v}\|_{1,\Omega} \leqslant C\big\{\|\boldsymbol{v}\|^2_{0,\Omega} + \|\boldsymbol{e}(\boldsymbol{v})\|^2_{0,\Omega}\big\}^{1/2} \quad \textit{for all } \boldsymbol{v} \in \boldsymbol{H}^1(\Omega).$$

Proof. As shown in Theorem 3.2, Chapter 3 of Duvaut and Lions [10], the essence of this remarkable inequality is that the *two Hilbert spaces* $\boldsymbol{H}^1(\Omega)$ *and* $\boldsymbol{K}(\Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ \boldsymbol{e}(\boldsymbol{v}) \in \boldsymbol{L}^2_{\mathrm{sym}}(\Omega)\}$ *coincide*. This property relies on a fundamental *lemma of J.L. Lions* that asserts that, if a distribution $v \in H^{-1}(\Omega)$ is such that $\partial_j v \in H^{-1}(\Omega)$, then $v \in L^2(\Omega)$ (see Theorem 3.2, Chapter 3 of Duvaut and Lions [10] for domains with smooth boundaries and Amrouche and Girault [2] for Lipschitz-continuous boundaries). The Korn inequality in $\boldsymbol{H}^1(\Omega)$ then becomes a consequence of the *closed graph theorem* applied to the identity mapping from $\boldsymbol{H}^1(\Omega)$ into $\boldsymbol{K}(\Omega)$, which is thus surjective and otherwise clearly continuous. □

The second stage consists in establishing the (equally classical) *Korn inequality in the quotient space* $\dot{\boldsymbol{H}}^1(\Omega)$ as a corollary to Theorem 1.1, a proof of which can be found in Theorem 3.4, Chapter 3 of Duvaut and Lions [10]:

Theorem 1.2. *There exists a constant* $\dot{C}$ *such that*

$$\|\dot{\boldsymbol{v}}\|_{1,\Omega} \leqslant \dot{C}\,\|\boldsymbol{e}(\dot{\boldsymbol{v}})\|_{0,\Omega} \quad \text{for all } \dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega).$$

Interestingly, our subsequent analysis will provide 'as a by-product' an essentially different proof of Korn inequalities in both spaces $\boldsymbol{H}^1(\Omega)$ and $\dot{\boldsymbol{H}}^1(\Omega)$ (see Corollary 3.2).

2. Weak versions of a classical theorem of Poincaré and of St Venant compatibility conditions

A classical *theorem of Poincaré* (see, e.g., page 235 in Schwartz [14]) asserts that, if functions $h_k \in \mathcal{C}^1(\Omega)$ satisfy $\partial_l h_k = \partial_k h_l$ in a simply-connected open subset Ω of $\mathbb{R}^3$ (or $\mathbb{R}^n$ for that matter), then there exists a function $p \in \mathcal{C}^2(\Omega)$ such that $h_k = \partial_k p$ in Ω. This theorem was extended by Girault and Raviart [12] (see Theorem 2.9 in Chapter 1), who showed that, if functions $h_k \in L^2(\Omega)$ satisfy $\partial_l h_k = \partial_k h_l$ in $H^{-1}(\Omega)$ on a bounded, connected and simply-connected open subset Ω of $\mathbb{R}^3$ with a Lipschitz-continuous boundary, then there exists $p \in H^1(\Omega)$ such that $h_k = \partial_k p$ in $L^2(\Omega)$. In fact, this extension can be carried out one step further:

Theorem 2.1. *Let* Ω *be a bounded, connected, and simply-connected open subset of* $\mathbb{R}^3$ *with a Lipschitz-continuous boundary. Let* $h_k \in H^{-1}(\Omega)$ *be distributions that satisfy*

$$\partial_l h_k = \partial_k h_l \quad \text{in } H^{-2}(\Omega).$$

Then there exists a function $p \in L^2(\Omega)$*, unique up to an additive constant, such that*

$$h_k = \partial_k p \quad \text{in } H^{-1}(\Omega).$$

Idea of the proof. Given *any* $\boldsymbol{h} \in \boldsymbol{H}^{-1}(\Omega)$, Theorem 5.1, Chapter 1 of Girault and Raviart [12] shows that there exist $\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega)$ and $p \in L^2(\Omega)$ such that (the assumptions that Ω is bounded and has a Lipschitz-continuous boundary are used here) $-\Delta\boldsymbol{u} + \mathbf{grad}\,p = \boldsymbol{h}$ in $\boldsymbol{H}^{-1}(\Omega)$ and $\operatorname{div}\boldsymbol{u} = 0$ in Ω.

It then suffices to show that, if *in addition* $\mathbf{curl}\ \boldsymbol{h} = \mathbf{0}$ in $\boldsymbol{H}^{-2}(\Omega)$, then $\boldsymbol{u} = \mathbf{0}$. The proof of this crucial implication relies on several results, which include in particular an extension result of Girault [11, Theorem 3.2] and a representation theorem of Girault and Raviart [12, Theorem 2.9, Chapter 1], the assumption of simple-connectedness being essential here (as in the 'classical' version of this theorem). See [4] for a complete proof. □

In 1864, A.J.C.B. de Saint Venant showed that, if functions $e_{ij} = e_{ji} \in \mathcal{C}^3(\Omega)$ satisfy in Ω ad hoc *compatibility relations* that since then bear his name, then there exists a vector field $(v_i) \in \mathcal{C}^4(\Omega)$ such that $e_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$ in Ω. Thanks to Theorem 2.1, these *St Venant compatibility relations* are also sufficient conditions *in the sense of distributions*, according to the following result:

Theorem 2.2. *Let Ω be a bounded, connected, and simply-connected open subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. Let $\boldsymbol{e} = (e_{ij}) \in \boldsymbol{L}^2_{\text{sym}}(\Omega)$ be a symmetric matrix field that satisfies the following compatibility relations*:

$$\mathcal{R}_{ijkl}(\boldsymbol{e}) := \partial_{lj} e_{ik} + \partial_{ki} e_{jl} - \partial_{li} e_{jk} - \partial_{kj} e_{il} = 0 \quad \textit{in } H^{-2}(\Omega).$$

Then there exists a vector field $\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega)$ such that

$$e_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j) \quad \textit{in } L^2(\Omega),$$

and all other solutions $\tilde{\boldsymbol{v}} = (\tilde{v}_i) \in \boldsymbol{H}^1(\Omega)$ of the equations $e_{ij} = \frac{1}{2}(\partial_j \tilde{v}_i + \partial_i \tilde{v}_j)$ are of the form $\tilde{\boldsymbol{v}} = \boldsymbol{v} + \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{id}$, with $\boldsymbol{a} \in \mathbb{R}^3$ and $\boldsymbol{b} \in \mathbb{R}^3$.

Idea of the proof. The proof consists in showing that the classical proof can be re-interpreted in such a way that it still holds in the sense of distributions (see [4]). That all other solutions are of the indicated form is well known. □

Remark 2.3. A different necessary and sufficient condition for a tensor $\boldsymbol{e} \in \boldsymbol{L}^2_{\text{sym}}(\Omega)$ to be of the form $\boldsymbol{e} = \frac{1}{2}(\boldsymbol{\nabla} \boldsymbol{v}^{\mathrm{T}} + \boldsymbol{\nabla} \boldsymbol{v})$ for some $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ has been given by Ting [16].

Remark 2.4. The assumption that Ω is simply-connected can be disposed of with some extra care; see [9].

3. A basic isomorphism and a new proof of Korn's inequality

Let a symmetric matrix field $\boldsymbol{e} = (e_{ij}) \in \boldsymbol{L}^2_{\text{sym}}(\Omega)$ satisfy $\mathcal{R}_{ijkl}(\boldsymbol{e}) = 0$ in $H^{-2}(\Omega)$, i.e., the weak form of St Venant's compatibility conditions considered in Theorem 2.2. There then exists a unique equivalence class $\dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega) = \boldsymbol{H}^1(\Omega)/\boldsymbol{R}(\Omega)$ such that $\boldsymbol{e} = \boldsymbol{e}(\dot{\boldsymbol{v}})$ in $\boldsymbol{L}^2_{\text{sym}}(\Omega)$. We now show the mapping $\boldsymbol{\mathcal{F}} : \boldsymbol{e} \to \dot{\boldsymbol{v}}$ defined in this fashion has a remarkable property.

Theorem 3.1. *Let Ω be a bounded, connected, and simply-connected open subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. Define the space*

$$\boldsymbol{E}(\Omega) := \left\{\boldsymbol{e} = (e_{ij}) \in \boldsymbol{L}^2_{\text{sym}}(\Omega);\ \mathcal{R}_{ijkl}(\boldsymbol{e}) = 0 \quad \textit{in } H^{-2}(\Omega)\right\},$$

and let $\boldsymbol{\mathcal{F}} : \boldsymbol{E}(\Omega) \to \dot{\boldsymbol{H}}^1(\Omega)$ be the linear mapping defined for each $\boldsymbol{e} \in \boldsymbol{E}(\Omega)$ by $\boldsymbol{\mathcal{F}}(\boldsymbol{e}) = \dot{\boldsymbol{v}}$, where $\dot{\boldsymbol{v}}$ is the unique element in the quotient space $\dot{\boldsymbol{H}}^1(\Omega)$ that satisfies $\boldsymbol{e}(\dot{\boldsymbol{v}}) = \boldsymbol{e}$; see Theorem 2.2. *Then $\boldsymbol{\mathcal{F}}$ is an isomorphism between the Hilbert spaces $\boldsymbol{E}(\Omega)$ and $\dot{\boldsymbol{H}}^1(\Omega)$.*

Proof. It is easily seen that the mapping $\boldsymbol{\mathcal{F}}$ is injective and surjective and that the inverse mapping $\boldsymbol{\mathcal{F}}^{-1} : \dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega) \to \boldsymbol{e}(\dot{\boldsymbol{v}}) \in \boldsymbol{E}(\Omega)$ is continuous. The conclusion thus follows from the *closed graph theorem*. □

Remarkably, the classical Korn's inequalities of Section 1 can now be very simply recovered:

Corollary 3.2. *That the mapping $\boldsymbol{\mathcal{F}} : \boldsymbol{E}(\Omega) \to \dot{\boldsymbol{H}}^1(\Omega)$ is an isomorphism implies Korn's inequalities in both spaces $\boldsymbol{H}^1(\Omega)$ and $\dot{\boldsymbol{H}}^1(\Omega)$ (see Theorems* 1.1 *and* 1.2).

Proof. (i) Since $\boldsymbol{\mathcal{F}}$ is an isomorphism, there exists a constant $\dot{C}$ such that $\|\boldsymbol{\mathcal{F}}(\boldsymbol{e})\|_{1,\Omega} \leqslant \dot{C}\|\boldsymbol{e}\|_{0,\Omega}$ for all $\boldsymbol{e} \in \boldsymbol{E}(\Omega)$, or equivalently such that $\|\dot{\boldsymbol{v}}\|_{1,\Omega} \leqslant \dot{C}\|\boldsymbol{e}(\dot{\boldsymbol{v}})\|_{0,\Omega}$ for all $\dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega)$. But this is exactly *Korn's inequality in the quotient space* $\dot{\boldsymbol{H}}^1(\Omega)$, obtained by different means in Theorem 1.2.

(ii) One then shows by means of standard arguments that *Korn's inequality in the quotient space* $\dot{\boldsymbol{H}}^1(\Omega)$ *implies Korn's inequality in the space* $\boldsymbol{H}^1(\Omega)$ (see [4]). □

4. Another approach to existence theory in linearized elasticity

Thanks to the isomorphism $\boldsymbol{\mathcal{F}} : \boldsymbol{E}(\Omega) \to \dot{\boldsymbol{H}}^1(\Omega)$ introduced in Theorem 3.1, the pure traction problem of linearized elasticity problem can be recast as *another minimization problem*, this time in terms of an unknown that lies in the space $\boldsymbol{E}(\Omega)$:

Theorem 4.1. *Let Ω be a bounded, connected, and simply-connected open subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. The minimization problem: Find $\boldsymbol{\varepsilon} \in \boldsymbol{E}(\Omega)$ such that*

$$j(\boldsymbol{\varepsilon}) = \inf_{\boldsymbol{e}\in \boldsymbol{E}(\Omega)} j(\boldsymbol{e}), \quad \textit{where } j(\boldsymbol{e}) := \frac{1}{2}\int_\Omega \boldsymbol{A}\boldsymbol{e} : \boldsymbol{e}\,\mathrm{d}x - \Lambda(\boldsymbol{e}),$$

the linear form $\Lambda : \boldsymbol{E}(\Omega) \to \mathbb{R}$ being defined by $\Lambda := L \circ \boldsymbol{\mathcal{F}}$, has one and only one solution $\boldsymbol{\varepsilon}$. Besides, $\boldsymbol{\varepsilon} = \boldsymbol{e}(\dot{\boldsymbol{u}})$ where $\dot{\boldsymbol{u}}$ is the unique solution to the 'classical' variational formulation of the pure traction problem of linearized elasticity.

Proof. By assumption (Section 1), there exists $\alpha > 0$ such that $\int_\Omega \boldsymbol{A}\boldsymbol{e} : \boldsymbol{e}\,\mathrm{d}x \geqslant \alpha \|\boldsymbol{e}\|^2_{0,\Omega}$ for all $\boldsymbol{e} \in \boldsymbol{L}^2_{\mathrm{sym}}(\Omega)$. The linear form Λ is continuous since L and $\boldsymbol{\mathcal{F}}$ are continuous. Finally, $\boldsymbol{E}(\Omega)$ is a closed subspace of $\boldsymbol{L}^2_{\mathrm{sym}}(\Omega)$. Consequently, there exists one, and only one, minimizer of the functional j over $\boldsymbol{E}(\Omega)$. That $\dot{\boldsymbol{u}}$ minimizes the functional J over $\dot{\boldsymbol{H}}^1(\Omega)$ implies that $\boldsymbol{e}(\dot{\boldsymbol{u}})$ minimizes j over $\boldsymbol{E}(\Omega)$. Hence $\boldsymbol{\varepsilon} = \boldsymbol{e}(\dot{\boldsymbol{u}})$ since the minimizer is unique. □

5. Concluding remarks

(a) While the minimization problem over the space $\dot{\boldsymbol{H}}^1(\Omega)$ is an *unconstrained one* with three unknowns, that found in Theorem 4.1 over the space $\boldsymbol{E}(\Omega)$ is in effect a *constrained minimization problem* over the space $\boldsymbol{L}^2_{\mathrm{sym}}(\Omega)$ with six unknowns, the constraints (in the sense of optimization theory) being the compatibility relations $\mathcal{R}_{ijkl}(\boldsymbol{e}) = 0$ in $H^{-2}(\Omega)$ that the matrix fields $\boldsymbol{e} \in \boldsymbol{E}(\Omega)$ satisfy (it is easily seen that these compatibility relations reduce in fact to six independent ones).

(b) As recalled in the proof of Theorem 1.1, the *lemma of J.L. Lions* is the keystone of the classical proof of Korn's inequality. In a sense, the same role is played in the present approach by the '*H^{-2}-version of a classical theorem of Poincaré*' established in Theorem 2.1.

(c) In linearized elasticity, the *stress tensor field* $\boldsymbol{\sigma} \in \boldsymbol{L}^2_{\mathrm{sym}}(\Omega)$ is given in terms of the displacement field by $\boldsymbol{\sigma} = \boldsymbol{A}\boldsymbol{e}(\boldsymbol{v})$. Since the elasticity tensor $\boldsymbol{A}$ is assumed to be uniformly positive-definite a.e. in Ω, the minimization problem of Theorem 4.1 can thus be immediately recast as a *constrained minimization problem with the stress tensor as the primary unknown*.

(d) Various attempts to consider the 'fully nonlinear' *Green–St Venant strain tensor* $\boldsymbol{E}(\boldsymbol{v}) = \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{\nabla}\boldsymbol{v})$, or equivalently the *Cauchy–Green tensor* $\boldsymbol{I} + 2\boldsymbol{E}(\boldsymbol{v})$, as the 'primary' unknown in *three-dimensional nonlinear elasticity* (this idea goes back to Antman [3]) have been recently undertaken in the same spirit; see [5–8,15]. These attempts have met only partial success, however, since nonlinearity *per se* creates specific challenging difficulties.

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