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C. R. Acad. Sci. Paris, Ser. I 339 (2004) 359–364
COMPTES RENDUS MATHEMATIQUE

Dynamical Systems

Time and entry–exit relation near a planar turning point

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Received 17 June 2004; accepted 22 June 2004
Available online 10 August 2004
Presented by Jean-Christophe Yoccoz

Abstract

Following the geometric approach for studying singular perturbation problems in the plane at turning points, and considering a very general setting where canard solutions are shown to exist, we study the transition time of orbits passing near the turning point, as well as the entry–exit relation at such turning points. The manifolds of canard solutions are in general only C^0 at the turning point, making the classical asymptotic approach impossible. The method involves a (family) blow up of the turning point and the use of C^k-normal forms and center manifolds. ***To cite this article: P. De Maesschalck, F. Dumortier, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Temps et relation entrée–sortie proche d'un point tournant planaire. Suivant l'approche géométrique dans l'étude de problèmes de perturbations singulières dans le plan aux points tournants, et travaillant dans un cadre très général dans lequel apparaissent des solutions canards, nous étudions le temps de passage des orbites proche des points tournants, tout comme la relation entrée–sortie à tel point. Les variétés de solutions canards rencontrées ne sont en général que C^0 à un point tournant, ne permettant pas une approche asymptotique classique. L'approche est basée sur l'éclatement et l'utilisation de variétés centrales et de formes normales C^k. ***Pour citer cet article : P. De Maesschalck, F. Dumortier, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Nous présentons ici des résultats concernant les solutions canards réels des champs de vecteurs lents-rapides. Il s'agit de familles $X_{\epsilon,a,\lambda}$ définies sur une variété M de dimension 2, avec $\epsilon \in [0, \epsilon_0[$, $a \in]-a_0, a_0[$ et $\lambda \in \Lambda$. Le système réduit $X_{0,a,\lambda}$ a une courbe de singularités γ (« *courbe critique* »), qui disparaît pour $\epsilon > 0$. Ici, on s'intéresse aux points tournants, c'est-à-dire des points où la courbe critique se decompose en $\gamma_- \cup \{p_*\} \cup \gamma_+$, de

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2004.06.020

façon que $X_{0,a,\lambda}$ est normalement attractive par rapport à γ_-, et normalement répulsive par rapport à γ_+. On dit que p_* est un point tournant. Dans une telle situation, l'existence d'une trajectoire au niveau $\epsilon > 0$ petit, longeant aussi bien γ_- que γ_+, est exceptionnel ; une telle solution est appelée une solution canard. En effet, la majorité des trajectoires qui démarrent dans le voisinage de γ_- suivent la branche γ_- jusqu'à un voisinage $O(\epsilon)$ du point tournant p_*, et s'éloignent de la courbe γ_+ immédiatement.

La question suivante se pose : considérant un point $s_-(\epsilon, \lambda)$ dans le voisinage de γ_- et un point $s_+(\epsilon, \lambda)$ dans le voisinage de γ_+, pour quelles valeurs de a y-a-t'il une trajectoire au niveau ϵ allant de $s_-(\epsilon, \lambda)$ à $s_+(\epsilon, \lambda)$? Suivant le point de vue géométrique, nous considérons plutôt deux « courbes de conditions » Σ_- et Σ_+, de classe C^∞, dans l'espace $M \times [0, \epsilon_0[$ ainsi que l'ensemble W de toutes les trajectoires avec point initial en Σ_- et nous cherchons pour quelles valeurs $a = \mathcal{A}(\epsilon, \lambda)$ cette variété W rencontre la courbe Σ_+. Nous avons démontré [2] que, sous des hypothèses très générales, pour toutes paires de courbes (Σ_-, Σ_+) il existe une « *courbe de contrôle* » $a = \mathcal{A}(\epsilon, \lambda)$ dans l'espace de paramètres, pour laquelle les trajectoires par rapport à la sous-famille $X_{\epsilon,\mathcal{A}(\epsilon,\lambda),\lambda}$ relient Σ_- à Σ_+ (Théorème 1.1). Cette courbe contrôle est C^∞ en $\epsilon^{1/m}$ pour un $m \in \mathbf{N}_1$, précisé dans le texte.

L'ensemble des trajectoires de Σ_- définit une variété W dans $M \times [0, \epsilon_0[$, qui est invariante par rapport à la sous-famille $X_{\epsilon,\mathcal{A}(\epsilon,\lambda),\lambda}$. En effet, W est une variété centrale C^∞ aux points de $\gamma_- \cup \gamma_+$. A p_*, la variété W n'est en général que continue. Alors, une telle variété ne se développe pas nécessairement en puissances de ϵ [2,3]. On appelle W une variété de solutions canards (ou simplement *variété canard*).

Dans la Note présente, nous explorons quelques propriétés des courbes de contrôle et des variétés canards. Tout d'abord, nous analysons le temps de passage de Σ_- à Σ_+. Nous obtenons la structure précise pour cette fonction temps $T(\epsilon, \lambda)$ ce qui permet entre autre de démontrer que ce temps tend de façon monotone vers l'infini lorsque $\epsilon \to 0$. Une formulation précise des résultats est donnée dans le Théorème 2.1. Un résultat analogue peut être obtenue pour l'intégrale de la divergence.

Après, nous considérons deux paires de courbes (Σ_-, Σ_+) et (Σ'_-, Σ'_+) et nous déduisons la distance entre les courbes de contrôle $\mathcal{A}$ et $\mathcal{A}'$ associées à ces deux paires. Cette distance est exponentiellement petite par rapport à ϵ (Théorème 3.1). Ceci mène à un résultat concernant la distance entre deux variétés canards W et W' associées à ces deux paires de courbes de conditions (Théorème 3.2). La distance est la somme de deux contributions : une partie qui est liée à la distance entre deux variétés centrales d'une même sous-famille, et une partie qui est liée au fait que les variétés centrales sont liées à deux sous-familles différentes $X_{\epsilon,\mathcal{A},\lambda}$ et $X_{\epsilon,\mathcal{A}',\lambda}$. Une formule explicite est obtenue, donnée par une intégrale de divergence du champ de vecteurs $X_{0,a,\lambda}$, le long d'une partie de la courbe critique γ. Les résultats des Théorèmes 3.1 et 3.2 mènent à une rélation entrée–sortie (Théorème 4.1).

Il convient de mentionner que des résultats analogues à ceux des Théorèmes 3.1, 3.2 et 4.1 ont été obtenus par des techniques non-standard, ainsi qu'en utilisant des techniques asymptotiques Gevrey. L'étude ici est non seulement basée sur des méthodes géométriques, mais a l'avantage de la généralité. Les solutions canards que nous considérons ne sont pas nécessairement prolongeables dans un voisinage complexe du point tournant, et de même, l'existence d'un développement asymptotique des variétés canards n'est pas exigée.

Les démonstrations reposent d'un côté sur l'utilisation de techniques courantes dans l'étude des systèmes dynamiques, comme les formes normales C^k, les variétés centrales C^∞ et l'éclatement (blow-up) de la famille $X_{\epsilon,a,\lambda}$ au point tournant p_*, et de l'autre côté sur de théorèmes d'analyse comme le théorème des fonctions implicites et le théorème de la convergence dominée de Lebesgue. Les démonstrations complètes aussi que des résultats supplémentaires peuvent se trouver dans [3].

1. Introduction

We study singular perturbation problems at turning points in the plane, or more generally, on a smooth 2-manifold M. Let $X_{\epsilon,a,\lambda}$ be a C^∞ family of vector fields on M, with $\epsilon \in [0, \epsilon_0[$, $a \in [-a_0, a_0]$ and $\lambda \in \Lambda$. We assume that $\gamma := \gamma_{a,\lambda}$ is a smooth curve of singularities for $X_{0,a,\lambda}$ and call this (family of) curve(s) the *critical curve*. Typically in turning point problems, γ contains a normally attracting part γ_- (normal hyperbolicity w.r.t. $X_{0,a,\lambda}$)

and a normally repelling part γ_+. We write $\gamma = \gamma_- \cup \{p_*\} \cup \gamma_+$. p_* is the *turning point*. Our aim is to study canard manifolds W and control curves $\mathcal{A}$. Such a canard manifold is a manifold inside $M \times [0, \epsilon_0[$ that is a collection of orbits for $\epsilon > 0$ that stay o(1)-close to γ. The a-typical behaviour of orbits to stay near γ_+ despite the latter's repelling character is in general only found in exponentially small regions in parameter space. The well-known geometric point of view in this matter is (see [4,5]) that in general, center manifolds of points of γ_- do not coincide with center manifolds of points of γ_+, but intersect transversally in a section above the turning point. With an implicit function argument, a curve in parameter space is chosen along which the center manifolds do coincide, and along which a canard manifold is formed. In [2], a general set of conditions is formulated where this geometric approach leads to canard manifolds. We will briefly discuss them here, and formulate the result in Theorem 1. In a chart of M where $p_* = (0, 0)$, we assume that the family is of the form

$$X_{\epsilon,a,\lambda} = f(x, y, \epsilon, a, \lambda)\frac{\partial}{\partial x} + \epsilon g(x, y, \epsilon, a, \lambda)\frac{\partial}{\partial y}$$

with f and g C^∞, and where, locally near p_*, γ is a graph of the form $y = \varphi(x, a, \lambda)$, with $\varphi(0, \lambda) = 0$.

We put an extra condition on γ: the order of contact of γ with the fast flow (the flow for $\epsilon = 0$, i.e. $\dot{y} = 0$) should be independent of λ at the turning point. For the Van der Pol vector field, this order is 2, but here we allow more degenerate turning points, where the order of contact is $2p$, with $p \in \mathbf{N}_1$.

Remark 1. We have chosen for a representation where the critical curve not only persists for $a \neq 0$, but does not even depend on a. In practice, prior to the study of the turning point, the parameter space (ϵ, a) is rescaled with $(\epsilon, a) = (v^k, v^\ell A)$ for well-chosen weights $(k, \ell) \in \mathbf{N}_1^2$ and one continues with the parameters (v, A) instead of (ϵ, a). For more information, see [2]. We assume that any such rescalings have been done in advance.

Instead of describing the exact conditions that are necessary to be able to prove the results, we describe near all points of γ the normal forms that we can need. These normal forms are either standard Takens normal forms, or are applications of results of Bonckaert [1]. Let p be a normally hyperbolic point of γ, then we assume that near p, there exist C^k-normal forms for conjugacy of the form

$$f(x, y, \epsilon, a, \lambda)\left(-x\frac{\partial}{\partial x} + \epsilon^\sigma h(y, \epsilon, a, \lambda)\frac{\partial}{\partial y}\right), \tag{1}$$

where f and g are C^k and strictly positive. The strength of this normal form is in the fact that h is independent of x, making an explicit integration possible. Near p, we define the slow vector field along γ (which in these coordinates is $\{x = \epsilon = 0\}$) as the vector field given by

$$\frac{\mathrm{d}y}{\mathrm{d}s} = h(y, 0, 0, \lambda) f(0, y, 0, 0, \lambda). \tag{2}$$

These local vector fields can be glued together to form a vector field along γ, defined for all $p \in \gamma_- \cup \gamma_+$. The slow vector field may or may not have a smooth extension at the turning point.

The study at the turning point relies on a family blow up of the turning point in $M \times [0, \epsilon_0[$. In a chart where $p_* = (0, 0)$, such a blow up is of the form $(x, y, \epsilon) = (u^\mu \bar{x}, u^\nu \bar{y}, u^m \bar{\epsilon})$, for some weights $(\mu, \nu, m) \in \mathbf{N}_1^3$, and with $(\bar{x}, \bar{y}, \bar{\epsilon}) \in S^2$. The blow up procedure replaces the origin $(0, 0, 0)$ by a sphere (the blow up locus). The blown up vector field is the pull back of the original vector field under the blow up map $\Phi: (u, (\bar{x}, \bar{y}, \bar{\epsilon})) \mapsto (u^\mu \bar{x}, u^\nu \bar{y}, u^m \bar{\epsilon})$, divided by some power of u:

$$\overline{X}_{a,\lambda} = u^{-\alpha} X_{a,\lambda}, \quad \text{where } X_{a,\lambda} = X_{\epsilon,a,\lambda} + 0\frac{\partial}{\partial \epsilon}.$$

We assume that $\gamma_\pm$ has a well-defined limit point $P_\pm$ on this sphere that is partially hyperbolic, and we assume that at this point the blow up vector field has C^k-normal forms for conjugacy of the form

$$f(y,\bar{\epsilon},z,a,\lambda)\left(-u\bar{\epsilon}^{\sigma}h(u,\bar{\epsilon},a,\lambda)\frac{\partial}{\partial u}+m\bar{\epsilon}^{\sigma+1}h(u,\bar{\epsilon},a,\lambda)\frac{\partial}{\partial\bar{\epsilon}}-z\frac{\partial}{\partial z}\right),\tag{3}$$

where $P_{\pm}$ is given by $(u,\bar{\epsilon},z)=(0,0,0)$, and with $h(0,0,a,\lambda)>0$ at P_- and $h(0,0,a,\lambda)<0$ at P_+, and with f strictly positive. The factor f may be removed if a C^k-normal form for equivalence is needed. Notice that in (3) the invariant foliation $\mathrm{d}\epsilon=0$ is replaced by the foliation $\mathrm{d}(u^m\bar{\epsilon})=0$.

From this normal form, we deduce the existence of an invariant separatrix on $\{u=0\}$: in this normal form it is locally given by $\{z=u=0\}$. In general however, the separatrix at $P_{\pm}$ are orbits $\Gamma^{\pm}_{a,\lambda}$ that depend on (a,λ). We assume that $\Gamma:=\Gamma^-_{0,\lambda}=\Gamma^+_{0,\lambda}$, i.e. for $a=0$ there is a regular connection on the blow up locus, connecting P_- to P_+. At any point of Γ, we hence have normal forms

$$f(\bar{x},\bar{y},u,a,\lambda)\left(\frac{\partial}{\partial\bar{x}}+0\frac{\partial}{\partial\bar{y}}+0\frac{\partial}{\partial u}\right)\tag{4}$$

with $f(\bar{x},\bar{y},0,a,\lambda)>0$ and so that $\Gamma^{\pm}_{a,\lambda}$ is given by $\{y=g_{\pm}(a,\lambda)\}$, with

$$g_-(0,\lambda)=g_+(0,\lambda),\qquad \frac{\partial}{\partial a}\big(g_-(a,\lambda)-g_+(a,\lambda)\big)\bigg|_{a=0}\neq 0.$$

Theorem 1.1 [2]. *Let the entry-boundary curve Σ_- be the C^{∞} graph of $(\epsilon,\lambda)\mapsto s_-(\epsilon,\lambda)\in M$ with $s_-(0,\lambda)$ in the basin of attraction of γ_-, and let the exit-boundary curve Σ_+ be defined similarly w.r.t. the basin of repulsion of γ_+. There exists a unique control curve $a=\mathcal{A}(\epsilon,\lambda)$ that is smooth w.r.t. $\epsilon^{1/m}$ and a canard manifold W (manifold with boundary) that contains both Σ_- and Σ_+ and that is invariant under the flow of $X_{\epsilon,\mathcal{A}(\epsilon,\lambda),\lambda}$. Outside p_* the manifold W is smooth except at the points $c_-\in\gamma_-$ and $c_+\in\gamma_+$ which are resp. the ω-limit and α-limit of the orbits through $s_-(0,\lambda)$ and $s_+(0,\lambda)$. At p_* the manifold is in general only C^0, but in blow up coordinates, the manifold W is smooth on the blow up locus.*

Note: it is clear that if the canard manifold is C^{∞} at the turning point, then an asymptotic expansion at this turning point exists. The converse is also true: if an asymptotic expansion at the turning point exists so that as a formal object it is formally invariant under the flow of $X_{\epsilon,a,\lambda}$, then any canard manifold is C^{∞} at the turning point, and their Taylor expansions coincide with the given asymptotic expansion [2].

2. Transition time

Consider a pair of entry–exit boundary curves (Σ_-,Σ_+), and associated to this pair the control curve $a=\mathcal{A}(\epsilon,\lambda)$ and the canard manifold W. We want to determine the transition time for Σ_- to Σ_+.

Theorem 2.1. *Let $c_{\pm}\in\gamma_{\pm}$ be the corner points as defined in Theorem* 1.1. *Then, the integral*

$$\int_{c_-}^{c_+}\mathrm{d}s:=\lim_{\substack{p\to p_*\\ p\in\gamma_-}}\int_{c_-}^{p}\mathrm{d}s+\lim_{\substack{p\to p_*\\ p\in\gamma_+}}\int_{p}^{c_+}\mathrm{d}s$$

converges (integration w.r.t. time of the slow vector field (2)), provided $\sigma>\alpha$ (see remark below the theorem). Furthermore, the transition time of the point $s_-(\epsilon,\lambda)\in\Sigma_-$ to $s_+(\epsilon,\lambda)\in\Sigma_+$ w.r.t. the vector field $X_{\epsilon,\mathcal{A}(\epsilon,\lambda),\lambda}$, and for $\epsilon>0$, is given by

$$T(\epsilon,\lambda)=\epsilon^{-\sigma}\left(\int_{c_-}^{c_+}\mathrm{d}s+\varphi(\epsilon,\lambda)+\tilde{\varphi}(\epsilon,\lambda)\epsilon^{\sigma-\alpha}\log\epsilon\right),\tag{5}$$

where both φ and $\tilde{\varphi}$ are C^∞ and $\varphi(\epsilon,\lambda) = \mathrm{O}(\epsilon)$. In particular, it follows that $T(\epsilon,\lambda)$ tends monotonously to ∞ as $\epsilon \to 0$, for $\epsilon > 0$ small enough. Like all other asymptotic properties in this note, they are uniform w.r.t. λ in compact sets. If the canard manifold is smooth at the turning point, one can take $\tilde{\varphi} = 0$.

Remark 2. In the previous theorem we have restricted to the case $\sigma > \alpha$, which is the only relevant one for "nilpotent" turning points. For a description of the precise results in the other case, we refer to [3].

Remark 3. A similar structure theorem for the divergence integral can be obtained [3]. We can even change the coefficient $\sigma - \alpha$ in expression (5) by σ and the results hold without any restriction on $\sigma - \alpha$.

Using the different normal forms for conjugacy (1), (3) and (4), and remembering that the last two normal forms are normal forms for the blown up vector field, and are written down in a faster time scale (the vector field is divided by u^α), the proof of this theorem can be done locally. The treatment near P_- and P_+ is the most delicate one.

3. Distance between canard manifolds

It is well-known that the difference between two center manifolds along a normally attracting critical curve can be expressed as $\exp(-I/\epsilon^\sigma)$, where I is an integral of the divergence along a compact piece of the critical curve. Measuring the difference between two canard manifolds W^1 and W^2 is more involved for several reasons. First, one needs to overcome the turning point and prove that the passage along the blow up locus is negligible in the study of the distance. Second, the canard manifold W^i is, locally near any point of $\gamma_- \cup \gamma_+$, a center manifold at that point for the family $X_{\epsilon,\mathcal{A}^i(\epsilon,\lambda),\lambda}$, meaning that we have to compare center manifolds of two different subfamilies of $X_{\epsilon,a,\lambda}$. Therefore, a knowledge of $|\mathcal{A}^2(\epsilon,\lambda) - \mathcal{A}^1(\epsilon,\lambda)|$ is necessary.

We define the *slow integral of the divergence* as the integral along the slow vector field (2) of the divergence of the reduced vector field $X_{0,0,\lambda}$. The divergence is calculated w.r.t. any chosen volume form Ω on the manifold M; the results presented below are independent of Ω. For points of γ we write

$$I_\lambda(p) := \lim_{\substack{q\to p_*\\ q\in\gamma_-}} \int_p^q \operatorname{div} X_{0,0,\lambda}\,\mathrm{d}s < 0, \quad \forall p \in \gamma_-; \qquad I_\lambda(p) := \lim_{\substack{q\to p_*\\ q\in\gamma_+}} \int_p^q \operatorname{div} X_{0,0,\lambda}\,\mathrm{d}s < 0, \quad \forall p \in \gamma_+.$$

The value $-I_\lambda(p)$ is a measure of how far p is away from the turning point.

Let E be a finite set of points on $\gamma_- \cup \gamma_+$, we say that E is an *admissible entry–exit configuration* if the point of E that is closest to p_*, denoted p_E, is unique.

Theorem 3.1. *For $i = 1, 2$ let $(W^i, \mathcal{A}^i)$ be the canard manifold and control curve for the pair of entry–exit curves (Σ_-^i, Σ_+^i). Let $c_\pm^i$ be the corner points as defined in Theorem 1.1, and $E := \{c_-^1, c_+^1, c_-^2, c_+^2\}$ be an admissible entry–exit configuration. Define p_E as the unique point of E closest p_*. Then, there exist smooth functions φ, $\tilde{\varphi}$ with $\varphi = \mathrm{O}(\epsilon)$, so that*

$$\left|\mathcal{A}^1(\epsilon,\lambda) - \mathcal{A}^2(\epsilon,\lambda)\right| = \exp\left(\epsilon^{-\sigma}\left(I_\lambda(p_E) + \varphi(\epsilon,\lambda) + \tilde{\varphi}(\epsilon,\lambda)\epsilon^\sigma \log\epsilon\right)\right) \quad \text{as } \epsilon \to 0.$$

Theorem 3.2. *Let S be a section of $M \times [0,\epsilon_0[$ intersecting γ_- transversally at a point q. Assume that q is strictly closer to p_* than $p_{\{c_-^1,c_-^2\}}$. Let p_E be defined as in the previous theorem (and let E be admissible). In any coordinate system (z,ϵ) for S so that $z = 0$ corresponds to q, the canard manifold W^i intersects S in a C^∞ curve $z = \varphi^i(\epsilon,\lambda)$, with $\varphi^i(0,\lambda) = 0$. One has*

$$\left|\varphi^1(\epsilon,\lambda) - \varphi^2(\epsilon,\lambda)\right| = \exp\left(\frac{1}{\epsilon^\sigma}\left(I_\lambda(p_{\{c_-^1,c_-^2\}}) - I_\lambda(q) + \mathrm{o}(1)\right)\right) + f(\epsilon,\lambda)\exp\left(\frac{1}{\epsilon^\sigma}\left(I_\lambda(p_E) + \mathrm{o}(1)\right)\right),$$

as $\epsilon \to 0$, *for some smooth function* f (*that may have zeros*). *The* $\mathrm{o}(1)$-*contributions in both terms are functions that are smooth in* (ϵ, λ), *and linear in* $\epsilon^\sigma \log \epsilon$.

The first term comes from the contribution of the difference between two center manifolds. The second term comes from the fact that the two center manifolds are center manifolds for two different subfamilies, and relies on Theorem 3.1. A similar result is valid for sections S intersecting γ at a normally repelling point. The method of proof is similar to the study of the transition time, but is more involved. The study can be done locally, at any point of γ (and at any point of Γ on the blow up locus), and is based on the use of normal forms for C^k-equivalence. See also [3].

4. Entry–exit relation

Theorem 4.1. *Let* $(W, \mathcal{A})$ *be a canard manifold and control curve for the entry–exit boundary curves* (Σ_-, Σ_+), *having an admissible entry–exit configuration* $\{c_-, c_+\}$ (*with* $c_\pm$ *as in Theorem* 1.1), *and suppose* $p_{\{c_-,c_+\}} = c_+$ (*i.e.* c_+ *lies closest*). *Let* Σ'_- *be another entry boundary curve, with corner point* c'_-. *The orbits w.r.t.* $X_{\epsilon,\mathcal{A}(\epsilon,\lambda),\lambda}$ *form a manifold* W' *that leaves* γ_+ *at some point* c'_+. *This point*:

(i) *is the unique point on* γ_+ *for which* $I_\lambda(c'_-) = I_\lambda(c'_+)$ *if* c'_- *is strictly closer to* p_* *than* c_+ (*tunnel*);
(ii) *is equal to* c_+ *if* c'_- *if* c_+ *is strictly closer to* p_* *than* c'_- (*funnel*);
(iii) *lies at least beyond* c_+ *if* $I_\lambda(c'_-) = I_\lambda(c_+)$ (*inverse funnel*).

Choosing a section near c'_+ *transverse to the fast fibers, the manifold* W' *intersects this section in a curve that is smooth in* $(\epsilon, \lambda, \epsilon^\sigma \log \epsilon)$. *If moreover* W' *is smooth at* p_*, W' *exits* γ *in a smooth curve w.r.t.* (ϵ, λ).

In these circumstances, one says c_+ is a point of maximum bifurcation delay; c_+ is called a *buffer point*. The case in which c_- is closer to p_* than c_+ can be treated identically, after reversing time. If $\{c_-, c_+\}$ is not an admissible configuration, then Theorem 4.1 is not valid (as can be shown by counter examples).

It is important to realize that such results have been obtained before, using nonstandard analysis, or using complex techniques (Gevrey techniques). Both techniques however were based on the existence of formal power series solutions (and hence implicitly on the smoothness of the canard manifolds at the turning point), which in general is not present. The geometric method overcomes this problem, by blowing up the turning point and only pursuing smoothness in blow up coordinates.

References

[1] P. Bonckaert, Partially hyperbolic fixed points with constraints, Trans. Amer. Math. Soc. 348 (3) (1996) 997–1011.
[2] F. Dumortier, P. De Maesschalck, Topics on singularities and bifurcations of vector fields, in: Normal Forms, Bifurcations and Finiteness Problems in Differential Equations, in: NATO Sci. Ser. II: Math. Phys. Chem., vol. 137, 2004.
[3] F. Dumortier, P. De Maesschalck, Time analysis and entry–exit relation near planar turning points, 2004, preprint.
[4] F. Dumortier, R. Roussarie, Canard cycles and center manifolds, Mem. Amer. Math. Soc. 121 (1996) 577.
[5] F. Dumortier, R. Roussarie, Multiple Canard cycles in generalized Liénard equations, J. Differential Equations 174 (1) (2001) 1–29.