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Partial Differential Equations

Propagation speed for reaction–diffusion equations in general domains

Henri Berestycki [a], François Hamel [b], Nikolai Nadirashvili [c]

[a] *EHESS, CAMS, 54, boulevard Raspail, 75006 Paris, France*
[b] *Université Aix-Marseille III, LATP, avenue Esc. Normandie-Niemen, 13397 Marseille cedex 20, France*
[c] *University of Chicago, Department of Mathematics, 5734, S. University Avenue, Chicago, IL 60637-1546, USA*

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Abstract

This Note is devoted to the analysis of some propagation phenomena for reaction–diffusion–advection equations with Fisher or Kolmogorov–Petrovsky–Piskunov (KPP) type nonlinearities. Some formulæ for the speed of propagation of pulsating fronts in periodic domains are given. These allow us to describe the influence of the various terms in the equation or of geometry on propagation. We also derive results for propagation speed in more general domains without periodicity. ***To cite this article: H. Berestycki et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Vitesse de propagation pour des équations de réaction–diffusion dans des domaines généraux. Cette Note est consacrée à l'analyse de phénomènes de propagation pour des équations de réaction–diffusion–advection du type Fisher ou Kolmogorov–Petrovsky–Piskunov (KPP). On donne des formules pour les vitesses de propagation de fronts pulsatoires dans des domaines périodiques. Celles-ci permettent de mettre en lumière l'influence des différents termes de l'équation ou de la géométrie sur la propagation. On considère également le cas de domaines généraux. ***Pour citer cet article : H. Berestycki et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

E-mail addresses: hb@ehess.fr (H. Berestycki), francois.hamel@univ.u-3mrs.fr (F. Hamel), nicholas@math.uchicago.edu (N. Nadirashvili).

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Pour l'équation $u_t = \Delta u + f(u)$ dans $\mathbb{R}^N$, un front progressif plan est une solution du type $u(t,x) = \phi(x \cdot e - ct)$, se propageant à vitesse c dans une direction unitaire e donnée, et telle que $\phi(-\infty) = 1$ et $\phi(+\infty) = 0$ (on suppose à normalisation près que $f(0) = f(1) = 0$). Si $f > 0$ dans $(0,1)$, alors, pour toute direction e, on sait qu'il existe des fronts progressifs sous la condition nécessaire et suffisante que $c \geqslant c^*$, où $c^* > 0$ est appelée la vitesse minimale de propagation. Un résultat classique de KPP [11] donne la valeur $c^* = 2\sqrt{f'(0)}$, dans le cas où $0 < f(s) \leqslant f'(0)s$ pour tout $s \in]0,1[$.

Quand le domaine ou les coefficients du milieu sont périodiques, la notion de front progressif est généralisée par celle de front progressif *pulsatoire*. Plus précisément, soit $d \in \{1,\dots,N\}$, soit $L_1,\dots,L_d > 0$ et Ω un ouvert connexe régulier de $\mathbb{R}^N$, de normale extérieure ν, tel que $\Omega = \Omega + k$ pour tout $k \in L_1\mathbb{Z} \times \cdots \times L_d\mathbb{Z} \times \{0\}^{N-d}$. Posons $x = (x_1,\dots,x_d)$, $y = (x_{d+1},\dots,x_N)$. On suppose qu'il existe $R > 0$ tel que $|y| \leqslant R$ pour tout $(x,y) \in \Omega$. Les cylindres infinis droits, ou à frontière périodique, l'espace entier $\mathbb{R}^N$, avec éventuellement des trous périodiques, rentrent dans ce cadre. On étudie les phénomènes de propagation associés à des équations de réaction–diffusion–advection du type

$$\begin{cases} u_t = \operatorname{div}\big(A(x,y)\nabla u\big) + q(x,y) \cdot \nabla u + f(x,y,u), & (t,x,y) \in \mathbb{R} \times \Omega, \\ \nu A(x,y)\nabla u = 0, \quad (t,x,y) \in \mathbb{R} \times \partial\Omega, \end{cases} \tag{1}$$

où le champ de matrices A (elliptiques), le champ de vecteurs q et le terme de réaction f ont la même périodicité que le domaine Ω. La fonction f, positive et régulière sur $\overline{\Omega} \times [0,1]$, vérifie : (i) $f(x,y,0) = f(x,y,1) = 0$ dans $\overline{\Omega}$, (ii) il existe $\rho > 0$ tel que $f(x,y,s) \geqslant f(x,y,s')$ pour tous $(x,y) \in \overline{\Omega}$ et $1 - \rho \leqslant s \leqslant s' \leqslant 1$, (iii) pour tout $s \in (0,1)$, il existe $(x,y) \in \overline{\Omega}$ tel que $f(x,y,s) > 0$, (iv) $\partial_u f(x,y,0) > 0$.

Un front progressif pulsatoire se propageant dans une direction $e \in \mathbb{R}^d$ unitaire donnée, avec une certaine vitesse *effective* $c \neq 0$, est une solution $0 \leqslant u \leqslant 1$ de (1) vérifiant les relations (3) ci-dessous.

Théorème 0.1. *Sous les hypothèses précédentes, pour tout* e ($|e| = 1$)*, il existe une vitesse minimale* $c^*(e) > 0$ *telle que le problème* (1), (3) *a une solution* (c,u) *si, et seulement si,* $c \geqslant c^*(e)$.

De plus, si $0 < f(x,y,s) \leqslant f'_u(x,y,0)s$ *pour tout* $(x,y,s) \in \overline{\Omega} \times (0,1)$*, alors* $c^*(e) = \min_{\lambda>0} k(\lambda)/\lambda$*, où* $k(\lambda)$ *est la valeur propre principale de l'opérateur* $L_\lambda\psi := \operatorname{div}(A\nabla\psi) - 2\lambda eA\nabla\psi + q \cdot \nabla\psi + [-\lambda \operatorname{div}(Ae) - \lambda q \cdot e + \lambda^2 eAe + \partial_u f(x,y,0)]\psi$ *agissant sur les fonctions* $\psi \in C^2(\overline{\Omega})$*, ayant la même périodicité que* Ω *et telles que* $\nu A\nabla\psi = \lambda(\nu Ae)\psi$ *sur* $\partial\Omega$*. On a identifié* e *avec le vecteur* $(e,0,\dots,0) \in \mathbb{R}^N$.

La preuve de ce théorème est faite dans [2] (pour l'existence) et [4] (pour la formule donnant $c^*(e)$), et repose sur la construction de sur- et sous-solutions et sur une méthode de régularisation dans des domaines bornés. Nous montrons également dans [4] que, sous l'hypothèse (4) et sous certaines simplifications supplémentaires, les perforations ralentissent la vitesse minimale de propagation, alors que l'advection l'augmente, et que cette vitesse est croissante par rapport à l'amplitude de la diffusion et de la réaction.

Nous définissons également par la relation (7) ci-dessous la notion de vitesse asymptotique de propagation $w^*(e)$ pour les solutions $u(t,z)$ de $u_t = \Delta u + f(u)$, avec donnée initiale positive u_0 ($u_0 \not\equiv 0$) à support compact, dans un domaine Ω général (non nécessairement périodique). Cette notion est cohérente avec les résultats connus dans les domaines périodiques mais permet de traiter des cas beaucoup plus généraux :

Théorème 0.2. *Si* Ω *est un domaine extérieur* (*complémentaire d'un compact*)*, alors* $w^*(e) = 2\sqrt{f'(0)}$*. Plus précisément,* $\limsup_{|z|\geqslant ct,\ z\in\overline{\Omega}} u(t,z) = 0$ *pour tout* $c > 2\sqrt{f'(0)}$ *et* $\limsup_{|z|\leqslant ct,\ z\in\overline{\Omega}} |1 - u(t,z)| = 0$ *pour tout* $c < 2\sqrt{f'(0)}$.

Nous donnons également une classe générale de domaines pour lesquels $w^*(e) \leqslant 2\sqrt{f'(0)}$.

1. Introduction

Travelling fronts are a special class of global solutions of reaction–diffusion equations, connecting two different states. They arise in many models in biology, population dynamics, ecology or combustion. One of the most important issues concerns the determination of their speed of propagation.

For instance, for the equation

$$u_t = \Delta u + f(u) \quad \text{in } \mathbb{R}^N,$$

a planar travelling front is a solution of the type $u(t,x) = \phi(x \cdot e - ct)$, propagating in a given unit direction e with speed c and connecting two zeroes of f, say 0 and 1: namely, $\phi(-\infty) = 1$ and $\phi(+\infty) = 0$. The function f is assumed to be Lipschitz-continuous, and such that $f(0) = f(1) = 0$. If f is positive in $(0,1)$, then, for any direction e, planar travelling fronts exist for all speed $c \geqslant c^*$, where c^* is called the minimal speed of propagation [10]. A classical result of Kolmogorov–Petrovsky–Piskunov [11] (see also [7]) asserts that if, furthermore, f satisfies $0 < f(s) \leqslant f'(0)s$ for all $s \in (0,1)$, then $c^* = 2\sqrt{f'(0)}$.

When the domain or the coefficients of the equation are not invariant by translation in the direction of propagation, then the notion of travelling fronts is generalized by that of *pulsating* travelling front. The shape of such fronts, in general, is not constant in time in the moving frame along the direction of propagation. More precisely, let $d \in \{1,\dots,N\}$ and call $x = (x_1,\dots,x_d)$ and $y = (x_{d+1},\dots,x_N)$. Let $L_1,\dots,L_d$ be d positive numbers and let Ω be a smooth connected open subset of $\mathbb{R}^N$ such that

$$\forall k \in L_1\mathbb{Z} \times \cdots \times L_d\mathbb{Z} \times \{0\}^{N-d}, \quad \Omega = \Omega + k.$$

Assume there is $R > 0$ such that $|y| \leqslant R$ for all $(x,y) \in \Omega$. Let ν denote the unit outward normal on $\partial\Omega$. In the sequel, a field w is said to be L-periodic with respect to (w.r.t.) x in Ω if $w(x+k,y) = w(x,y)$ in Ω, for all $k \in L_1\mathbb{Z} \times \cdots \times L_d\mathbb{Z}$. This framework includes the case of the whole space $\mathbb{R}^N$, with or without periodic holes, as well as, say, the case of straight or periodically oscillating infinite cylinders.

We are concerned here with propagation phenomena for the following reaction–diffusion–advection equation:

$$\begin{cases} u_t = \operatorname{div}\big(A(x,y)\nabla u\big) + q(x,y)\cdot\nabla u + f(x,y,u), & (t,x,y) \in \mathbb{R}\times\Omega, \\ \nu A(x,y)\nabla u = 0, \quad (t,x,y) \in \mathbb{R}\times\partial\Omega. \end{cases} \tag{2}$$

The diffusion matrix field $A(x,y) = (A_{ij}(x,y))_{1\leqslant i,j\leqslant N}$ is symmetric and uniformly elliptic. The advection field $q(x,y) = (q_1(x,y),\dots,q_N(x,y))$ is divergence free, such that $q\cdot\nu = 0$ on $\partial\Omega$ and all q_i $(1 \leqslant i \leqslant d)$ have zero average over the periodicity cell. Lastly, $f(x,y,u)$ is a nonnegative function defined in $\overline{\Omega}\times[0,1]$ such that: (i) $f(x,y,0) = f(x,y,1) = 0$ in $\overline{\Omega}$, (ii) $f(x,y,s) \geqslant f(x,y,s')$ for all $(x,y) \in \overline{\Omega}$, $1-\rho \leqslant s \leqslant s' \leqslant 1$, for some $\rho \in (0,1)$, (iii) for all $s \in (0,1)$, there is $(x,y) \in \overline{\Omega}$ such that $f(x,y,s) > 0$, (iv) $f'_u(x,y,0) := \lim_{u\to0^+} f(x,y,u)/u > 0$. Both A, q and f are assumed to be smooth and L-periodic with respect to x.

A pulsating (or periodic) travelling front is a classical solution u of (2) satisfying $0 \leqslant u \leqslant 1$ and

$$\begin{cases} \forall k \in L_1\mathbb{Z}\times\cdots\times L_d\mathbb{Z},\ \forall (t,x,y) \in \mathbb{R}\times\overline{\Omega}, \quad u\left(t - \dfrac{k\cdot e}{c}, x, y\right) = u(t, x+k, y), \\ u(t,x,y) \underset{x\cdot e\to+\infty}{\longrightarrow} 0, \quad u(t,x,y) \underset{x\cdot e\to-\infty}{\longrightarrow} 1, \end{cases} \tag{3}$$

where $e = (e^1,\dots,e^d)$ is a given unit vector in $\mathbb{R}^d$. Such a solution propagates in direction e with a (generalized) speed $c \neq 0$. The above limits in (3) hold locally in t and uniformly in y and in the directions of $\mathbb{R}^d$ which are orthogonal to e. In the sequel, we identify e and $(e^1,\dots,e^d,0,\dots,0) \in \mathbb{R}^N$.

Theorem 1.1. *Under the above assumptions, for all direction e $(|e| = 1)$, there exists $c^*(e) > 0$ such that pulsating travelling fronts u in the direction e with the speed c exist if and only if $c \geqslant c^*(e)$; furthermore, $u_t > 0$. If f also fulfills*

$$\forall (x,y,s) \in \overline{\Omega}\times(0,1), \quad 0 < f(x,y,s) \leqslant f'_u(x,y,0)s, \tag{4}$$

then

$$c^*(e) = \min_{\lambda>0} k(\lambda)/\lambda, \tag{5}$$

where $k(\lambda)$ is the first eigenvalue of the operator

$$L_\lambda \psi := \operatorname{div}(A\nabla\psi) - 2\lambda e A\nabla\psi + q\cdot\nabla\psi + \big[-\lambda \operatorname{div}(Ae) - \lambda q\cdot e + \lambda^2 eAe + f_u'(x,y,0)\big]\psi$$

acting on $E = \{\psi \in C^2(\overline{\Omega}),\ \psi$ is L-periodic w.r.t. x and $\nu A\nabla\psi = \lambda(\nu Ae)\psi$ on $\partial\Omega\}$.

Formula (5) extends the KPP formula $c^*(e) = 2\sqrt{f'(0)}$ (independent of e) for the minimal speed of planar fronts for the homogeneous equation $u_t = \Delta u + f(u)$ in $\mathbb{R}^N$. The dependence of $c^*(e)$ on the function f in (5) is only through the derivative of f with respect to u at $u = 0$. If one does not assume (4), but just that $f(x,y,s) \geqslant 0$, then the general inequality

$$c^*(e) \geqslant \min_{\lambda>0} k(\lambda)/\lambda$$

holds [2]. The proof of (5) in [4] is based on the construction of sub- and supersolutions and on regularizing approximations in bounded domains.

2. Influence of the geometry and of the medium

Assume here that f satisfies (4) and call $c^*_{\Omega,A,q,f}(e)$ the minimal speed of pulsating fronts satisfying (2) and (3) in Ω with diffusion A, advection q and reaction term f.

Theorem 2.1. (a) (Influence of geometry) *For the homogeneous equation $u_t = \Delta u + f(u)$ in a periodic domain Ω, then*

$$c^*_{\Omega,I,0,f}(e) \leqslant 2\sqrt{f'(0)}.$$

Furthermore, equality holds if and only if the domain Ω is a straight cylinder in the direction e.

(b) (Influence of reaction) *If f and g satisfy $f_u'(x,y,0) \leqslant g_u'(x,y,0)$ in $\overline{\Omega}$, then*

$$c^*_{\Omega,A,q,f}(e) \leqslant c^*_{\Omega,A,q,g}(e).$$

The function $B \mapsto c^*_{\Omega,A,q,Bf}(e)$ is increasing with $B > 0$ and $c^*_{\Omega,A,q,Bf}(e) = O(\sqrt{B})$ as $B \to +\infty$. Furthermore, if $\Omega = \mathbb{R}^N$ or if $\nu Ae \equiv 0$ on $\partial\Omega$, then $\liminf_{B\to+\infty} c^*_{\Omega,A,q,Bf}(e)/\sqrt{B} > 0$.

(c) (Influence of advection) *If Ω is invariant in the direction e, $A = I$ and $f = f(u)$ only depends on u, then*

$$c^*_{\Omega,I,q,f}(e) \geqslant c^*_{\Omega,I,0,f}(e) \quad \big(= 2\sqrt{f'(0)}\big),$$

and equality holds if and only if $q\cdot e \equiv 0$ in $\overline{\Omega}$.

(d) (Influence of diffusion) *Let $q = 0$ and $f = f(u)$ depend on u only. If $0 < \alpha \leqslant \beta$, then*

$$c^*_{\Omega,\alpha A,0,f}(e) \leqslant c^*_{\Omega,\beta A,0,f}(e).$$

Furthermore, $\limsup_{\alpha\to+\infty} \frac{c^*_{\Omega,\alpha A,0,f}(e)}{\sqrt{\alpha}} < +\infty$ *and* $0 < \liminf_{\alpha\to 0^+} \frac{c^*_{\mathbb{R}^N,\alpha A,0,f}(e)}{\sqrt{\alpha}} \leqslant \limsup_{\alpha\to 0^+} \frac{c^*_{\mathbb{R}^N,\alpha A,0,f}(e)}{\sqrt{\alpha}} < \infty.$

All the above results are proved in [4]. They rely on formula (5) and on some comparison results for linear eigenvalue problems. Part (a) means that perforations hinder propagation. Note, however, that in [4] we prove that the minimal speed $c^*(e)$ may not be monotone in general with respect to the size of the holes. The monotonicity result in part (b) generalizes the same property for the KPP formula $c^* = 2\sqrt{Bf'(0)}$ in the case of the homogeneous equation $u_t = \Delta u + Bf(u)$ in $\mathbb{R}^N$. Part (c) means that the advection, or stirring, enhances the speed of propagation, whatever the flow is. Furthermore, the influence of advection on the speed of propagation is minimal if and only if the advection is orthogonal to the direction of propagation. The influence of large periodic advection, namely where q is replaced with Cq with large C, is analyzed in [3]. The behaviour of $c^*_{\Omega,A,Cq,f}(e)$ is always at most linear in C for large C, in a general periodic domain Ω. A necessary and sufficient condition for $c^*_{\Omega,A,Cq,f}(e)$ to be at least linear in C is given in [3], shedding light on the role played by the first integrals of the velocity field q. Lastly, part (d) implies that a larger diffusion αA speeds up the propagation, but the rate of growth is controlled by $\sqrt{\alpha}$.

3. Spreading speed in periodic and general domains

Another important notion is that of spreading speed of propagation for the solutions of Cauchy problem like (2) with nonnegative, continuous and compactly supported initial condition $u_0 \not\equiv 0$. A well-known result of Gärtner and Freidlin [9] and Freidlin [8] (see also [12]) gives the existence of $w^*(e)$ (for all unit vector $e \in \mathbb{R}^d$) such that

$$u(t, x + cte, y) \to 1 \quad \text{if } 0 \leqslant c < w^*(e), \quad \text{and} \quad u(t, x + cte, y) \to 0 \quad \text{if } c > w^*(e), \quad \text{as } t \to +\infty,$$

locally with respect to the points (x, y) such that $(x + cte, y) \in \overline{\Omega}$. This value $w^*(e)$ can be viewed as a ray or asymptotic speed in the direction e. Gärtner and Freidlin furthermore establish a formula for $w^*(e)$. Relating this formula with the above (5), it is straightforward to see that

$$w^*(e) = \min_{\xi \in \mathbb{R}^d,\ e\cdot\xi > 0} c^*(\xi)/(e \cdot \xi) > 0.$$

One has $w^*(e) \leqslant c^*(e)$. The equality $w^*(e) = c^*(e)$ $(= 2\sqrt{f'(0)})$ holds for equation $u_t = \Delta u + f(u)$ in $\mathbb{R}^N$, for all e, but it does not hold in general: e.g. for the anisotropic equation $u_t = au_{x_1x_1} + bu_{x_2x_2} + f(u)$ in $\mathbb{R}^2$, where $a \neq b > 0$ are two constants, then $w^*(e) < c^*(e)$ for any direction e which is neither $\pm e_1$ nor $\pm e_2$ (cf. [4]).

The notion of asymptotic speed can be extended, say for the equation

$$\begin{cases} u_t = \Delta u + f(u), \quad u = u(t, z),\ t \geqslant 0,\ z \in \Omega, \\ \nu \cdot \nabla u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{6}$$

in general open connected domains $\Omega \subset \mathbb{R}^N$ which may not be periodic. Then, in general, there are no travelling fronts. Problem (6) comes with a given initial condition $u(0, z) = u_0(z) \not\equiv 0$ which is as above. From now on, $f = f(u)$ satisfies (4) and is extended by $f(s) = f'(1)(s - 1)$ for $s \geqslant 1$. We say that Ω is unbounded in a direction $e \in \mathbb{S}^{N-1}$ if there exists $r > 0$ such that $(B_r + \tau e) \cap \Omega \neq \emptyset$ for all $\tau \geqslant 0$, where B_r is the euclidean ball of $\mathbb{R}^N$ with the origin as centre and radius r (a periodic domain Ω, as in Sections 1 and 2, is unbounded in any direction $e \in \mathbb{R}^d$). In such a direction e, we then define the asymptotic speed of propagation as

$$w^*(e) := \inf\{c > 0,\ u(t, z + cte) \to 0 \text{ as } t \to +\infty \text{ locally uniformly in } z\}. \tag{7}$$

The above convergence $u(t, z + cte) \to 0$ means that, for all $A \geqslant R$,

$$\max_{z \in \overline{B_A},\ z+cte \in \overline{\Omega}} u(t, z + cte) \to 0 \quad \text{as } t \to +\infty.$$

Notice that this definition is coherent with the previous one if Ω is a periodic domain.

In [5], we prove the following results about asymptotic speeds of propagation:

Theorem 3.1. (a) *If Ω is a periodic domain, then $w^*(e) \leqslant 2\sqrt{f'(0)}$ and equality holds if and only if Ω is a cylinder in the direction e.*

(b) *For domains Ω satisfying the 'extension property', $w^*(e) \leqslant 2\sqrt{f'(0)}$.*

(c) *If Ω is an exterior domain ($\mathbb{R}^N \setminus \Omega$ is compact), then $w^*(e) = 2\sqrt{f'(0)}$ for all direction e of $\mathbb{R}^N$.*

The first inequality in part (a) follows from part (a) of Theorem 2.1. The fact that equality $w^*(e) = 2\sqrt{f'(0)}$ holds if Ω is a cylinder in the direction e is based on some independent Liouville type results for elliptic equations of the type

$$\Delta u + b \cdot \nabla u + f(u) = 0 \quad \text{in } \Omega$$

with Neumann boundary conditions on $\partial\Omega$, where b is a constant vector with $|b| < 2\sqrt{f'(0)}$: namely, if $0 \leqslant u \leqslant 1$ is a solution, then $u \equiv 0$ or $u \equiv 1$ (see [4]). For the definition of the 'extension property' in part (b), we refer to Davies [6]: roughly speaking, locally, the connected components of $\partial\Omega$ are not too close to each other (periodic smooth domains Ω satisfy this property). Part (c) extends to the class of exterior domains a classical result in $\mathbb{R}^N$ (cf. [1]; the proof given in [5] is actually simpler than that in [1] and does not rely on the use of some front-like solutions).

Furthermore, we establish the existence of domains in $\mathbb{R}^2$ which are unbounded in every direction e and such that $w^*(e) = 0$ for all e. Lastly, given $N \geqslant 2$ and $e \in \mathbb{S}^{N-1}$, there are domains of $\mathbb{R}^N$ for which $w^*(e) = +\infty$ (this relies on some new heat kernel estimates in domains with very thin infinite cusps).

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