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COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

Estimates for L^1-vector fields

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Abstract

A simple proof of an integral inequality involving L^1-vector fields is provided. This gives a short proof of estimates of Bourgain and Brezis for elliptic and div–curl systems. ***To cite this article: J. Van Schaftingen, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Estimations pour des champs de vecteurs L^1. Nous donnons une preuve simple d'une inégalité faisant intervenir un champ de vecteurs L^1. Nous en tirons une preuve courte d'estimées de Bourgain et Brezis pour des systèmes elliptiques et des systèmes div–rot. ***Pour citer cet article : J. Van Schaftingen, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Nous donnons une preuve élémentaire de l'inégalité

$$\left| \int_{\mathbb{R}^N} f \cdot u \,\mathrm{d}x \right| \leqslant C_N \|f\|_1 \|\nabla u\|_N + \|\operatorname{div} f\|_1 \|u\|_N,$$

pour tout $f \in L^1(\mathbb{R}^N; \mathbb{R}^N)$ tel que $\operatorname{div} f \in L^1(\mathbb{R}^N)$ et pour tout $u \in (W^{1,N} \cap L^\infty)(\mathbb{R}^N; \mathbb{R}^N)$.

Cette inégalité a été obtenue par Bourgain et Brezis [2] et est une généralisation de l'inégalité

$$\left| \int_\Gamma u \cdot t \right| \leqslant C|\Gamma| \|\nabla u\|_N,$$

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valable pour toute courbe rectifiable fermée Γ. Elle a été obtenue par Bourgain, Brezis et Mironescu [1]. Nous en avons donné une preuve simple [4] avec le même type d'arguments.

Cette inégalité permet de démontrer directement les résultats suivants de Bourgain et Brezis [2]. Si $f \in \mathrm{L}^1_\#(\mathbb{R}^3;\mathbb{R}^3)$, le système $\operatorname{div} Z = 0$, $\operatorname{curl} Z = f$ possède une solution dans $\mathrm{L}^{3/2}(\mathbb{R}^3;\mathbb{R}^3)$. De même, si $f \in \mathrm{L}^1_\#(\mathbb{R}^N;\mathbb{R}^N)$ et $N \geqslant 2$, le système elliptique $\Delta u = f$ possède une solution u telle que $Du \in \mathrm{L}^{N/(N-1)}(\mathbb{R}^N)$.

1. Introduction

In a recent work [2], Bourgain and Brezis considered in $\mathbb{R}^3$ the system

$$\begin{cases} \operatorname{curl} Z = f, \\ \operatorname{div} Z = 0 \end{cases}$$

for a given divergence-free vector field f. The vector field $Z = (-\Delta)^{-1} \operatorname{curl} f$ is a solution of this system. If $f \in \mathrm{L}^p_\#(\mathbb{R}^3;\mathbb{R}^3)$ (here and in the sequel, the subscript # denotes the subspace of vector fields whose divergence vanishes in the sense of distributions), then by the standard Calderón–Zygmund estimates and Sobolev's imbedding, $\|Z\|_{p^*} \leqslant C_p \|f\|_p$, where $p^* = 3p/(3-p)$ and $1 < p < 3$. The Calderón–Zygmund theory does not hold when $p = 1$, but surprisingly, when $p = 1$ one still has

Theorem 1.1 (Bourgain and Brezis [2]). *There exists C, such that for any $f \in \mathrm{L}^1_\#(\mathbb{R}^3;\mathbb{R}^3)$,*

$$\|Z\|_{3/2} \leqslant C\|f\|_1.$$

Bourgain and Brezis gave two proofs of this result. The first one relies on the following

Theorem 1.2. *Given $g \in \mathrm{L}^3_\#(\mathbb{R}^3;\mathbb{R}^3)$, there exists some $Y \in \mathrm{L}^\infty(\mathbb{R}^3;\mathbb{R}^3)$ with $\nabla Y \in \mathrm{L}^3(\mathbb{R}^3;\mathbb{R}^3)$ satisfying*

$$\operatorname{curl} Y = g$$

and the estimate

$$\|Y\|_\infty + \|\nabla Y\|_3 \leqslant C\|g\|_3.$$

The proof of Theorem 1.2 is rather involved and uses a Littlewood–Paley decomposition; no simple proof has been found so far.

Proof of Theorem 1.1 using Theorem 1.2. It suffices to show that

$$\left| \int_{\mathbb{R}^3} Z \cdot h \right| \leqslant C\|f\|_1 \|h\|_3$$

for every $h \in \mathrm{L}^3(\mathbb{R}^3;\mathbb{R}^3)$ with some universal constant C. Given any $h \in \mathrm{L}^3(\mathbb{R}^3;\mathbb{R}^3)$ consider its Hodge decomposition $h = g + \operatorname{grad} p$ with $g \in \mathrm{L}^3_\#(\mathbb{R}^3;\mathbb{R}^3)$ and $\|g\|_3 \leqslant C\|h\|_3$. Then

$$\int_{\mathbb{R}^3} Z \cdot h = \int_{\mathbb{R}^3} Z \cdot (g + \operatorname{grad} p) = \int_{\mathbb{R}^3} Z \cdot g.$$

Next, by Theorem 1.2, we may write $g = \operatorname{curl} Y$ for some Y with $\|Y\|_\infty \leqslant C\|g\|_3$ (here one uses only part of Theorem 1.2) and then

$$\left| \int_{\mathbb{R}^3} Z \cdot h \right| = \left| \int_{\mathbb{R}^3} \operatorname{curl} Z \cdot Y \right| = \left| \int_{\mathbb{R}^3} f \cdot Y \right| \leqslant \|f\|_1 \|Y\|_\infty \leqslant C\|f\|_1 \|g\|_3 \leqslant C\|f\|_1 \|h\|_3. \qquad \square$$

The second proof in [1] uses two ingredients. The first one is

Theorem 1.3 (Bourgain, Brezis and Mironescu [1]). *Let Γ be a closed rectifiable curve in $\mathbb{R}^N$ with unit tangent vector t and let $u \in \mathrm{C}(\mathbb{R}^N;\mathbb{R}^N)$. If $\nabla u \in \mathrm{L}^N(\mathbb{R}^N)$, then*

$$\left|\int_\Gamma u \cdot t\right| \leqslant C|\Gamma|\|\nabla u\|_N,$$

where $|\Gamma|$ denotes the length of Γ. The constant C is independent of the curve Γ and the vector field u.

See [4] for a simple proof of Theorem 1.3.

The next ingredient is Smirnov's theorem which asserts roughly speaking that any divergence free vector field is a limit of convex combinations of the form $\sum_i \alpha_i \mathcal{H}^1_{\llcorner \Gamma_i} t_i$, where $\mathcal{H}^1_{\llcorner \Gamma}$ is Hausdorff's one-dimensional measure restricted to Γ and t_i is the tangent vector to Γ_i.

Combining this with Theorem 1.3, Bourgain and Brezis obtain the following

Corollary 1.4. *There exists a constant C_N such that for each $f \in \mathrm{L}^1_\#(\mathbb{R}^N;\mathbb{R}^N)$ and $u \in \mathrm{L}^\infty(\mathbb{R}^N;\mathbb{R}^N)$ such that $\nabla u \in \mathrm{L}^N(\mathbb{R}^N)$,*

$$\left|\int_{\mathbb{R}^N} f \cdot u \,\mathrm{d}x\right| \leqslant C_N \|f\|_1 \|\nabla u\|_N.$$

Note that Theorem 1.3 is a special case of Corollary 1.4.

Proof of Theorem 1.1 using Corollary 1.4. Indeed we start as in the first proof of Theorem 1.1. Write as above

$$\int_{\mathbb{R}^3} Z \cdot h = \int_{\mathbb{R}^3} Z \cdot g.$$

Next, by standard L^p estimates we may solve $\operatorname{curl} \widetilde{Y} = g$ and $\operatorname{div} \widetilde{Y} = 0$; and then $\|\nabla \widetilde{Y}\|_3 \leqslant \|g\|_3$. (Here we do not use the difficult part in Theorem 1.2 which gives some $Y \in \mathrm{L}^\infty(\mathbb{R}^3;\mathbb{R}^3)$.) Thus

$$\int_{\mathbb{R}^3} Z \cdot h = \int_{\mathbb{R}^3} Z \cdot \operatorname{curl} \widetilde{Y} = -\int_{\mathbb{R}^3} f \cdot \widetilde{Y},$$

so that by Corollary 1.4,

$$\left|\int_{\mathbb{R}^3} Z \cdot h\right| \leqslant C\|f\|_1 \|\nabla \widetilde{Y}\|_3 \leqslant C\|f\|_1\|g\|_3 \leqslant C\|f\|_1\|h\|_3. \qquad \square$$

Remark 1. Note that Corollary 1.4 is an easy consequence of Theorem 1.1. Indeed write, using Theorem 1.1,

$$\left|\int_{\mathbb{R}^3} f \cdot u\right| = \left|\int_{\mathbb{R}^3} (\operatorname{curl} Z) \cdot u\right| = \left|\int_{\mathbb{R}^3} Z \cdot \operatorname{curl} u\right| \leqslant \|Z\|_{3/2}\|\operatorname{curl} u\|_3 \leqslant C\|f\|_1\|\nabla u\|_3.$$

Remark 2. Another consequence of Theorem 1.5 already mentioned in [2] is that $\Delta u = f \in \mathrm{L}^1_\#(\mathbb{R}^N)$ implies $\|\nabla u\|_{3/2} \leqslant C\|f\|_1$. This is proved as follows. Let Z solve $\operatorname{curl} Z = f$ and $\operatorname{div} Z = 0$. Then $\Delta Z = \operatorname{curl} f$, so that, by Theorem 1.1, $\|\Delta^{-1} \operatorname{curl} f\|_{3/2} \leqslant C\|f\|_1$. Therefore $\|\operatorname{curl} u\|_{3/2} = \|\Delta^{-1} \operatorname{curl} \Delta u\|_{3/2} \leqslant C\|f\|_1$. Finally, since $\operatorname{div} u = 0$,

$$\|\nabla u\|_{3/2} \leqslant C\big(\|\operatorname{curl} u\|_{3/2} + \|\operatorname{div} u\|_{3/2}\big) \leqslant C\|f\|_1.$$

The goal of this Note is to give a direct and elementary proof of Corollary 1.4. In fact we present a slightly more general version.

Theorem 1.5. *There exists a constant C_N such that for each $f \in \mathrm{L}^1(\mathbb{R}^N;\mathbb{R}^N)$ such that $\operatorname{div} f \in \mathrm{L}^1$ and $u \in (\mathrm{L}^\infty \cap \mathrm{W}^{1,N})(\mathbb{R}^N;\mathbb{R}^N)$,*

$$\left|\int_{\mathbb{R}^N} f \cdot u \,\mathrm{d}x\right| \leqslant C_N\big(\|f\|_1\|\nabla u\|_N + \|\operatorname{div} f\|_1\|u\|_N\big).$$

In a work in preparation we prove an extension of Corollary 1.4 in which the condition $\operatorname{div} f = 0$ is replaced by a weaker second order condition [5].

2. Proof of Theorem 1.5

First the estimate will be made under the additional assumptions that f and u are in $C^1(\mathbb{R}^N;\mathbb{R}^N)$. The first term in the scalar product is

$$\int_{\mathbb{R}^N} f_1 u_1 \,\mathrm{d}x = \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} f_1 u_1 \,\mathrm{d}y\,\mathrm{d}x_1.$$

The inside integral is estimated as follows. Let $\rho \in \mathrm{L}^1(B(0,1)\cap\mathbb{R}^{N-1})$ be such that $\int_{\mathbb{R}^{N-1}} \rho = 1$. Let $\rho_\varepsilon(\cdot) = \varepsilon^{1-N}\rho(\frac{\cdot}{\varepsilon})$, $u^{x_1}(y) = u(x_1,y)$ and $f^{x_1}(y) = f(x_1,y)$. The integral can be decomposed as

$$\int_{\mathbb{R}^{N-1}} f_1^{x_1} u_1^{x_1}\,\mathrm{d}y = \int_{\mathbb{R}^{N-1}} f_1^{x_1}\big(u_1^{x_1} - \rho_\varepsilon * u_1^{x_1}\big)\,\mathrm{d}y + \int_{\mathbb{R}^{N-1}} f_1^{x_1}\big(\rho_\varepsilon * u_1^{x_1}\big)\,\mathrm{d}y.$$

By the Morrey–Sobolev imbedding in $\mathbb{R}^{N-1}$ (see e.g. [3, Theorem IX.12]),

$$\int_{\mathbb{R}^{N-1}} f_1^{x_1}\big(u_1^{x_1} - \rho_\varepsilon * u_1^{x_1}\big)\,\mathrm{d}y \leqslant C'_N \varepsilon^{1/N}\big\|f^{x_1}\big\|_1\big\|\nabla u_1^{x_1}\big\|_N.$$

On the other hand,

$$\begin{aligned}
\int_{\mathbb{R}^{N-1}} f_1^{x_1}\big(\rho_\varepsilon * u_1^{x_1}\big)\,\mathrm{d}y &= \int_{\mathbb{R}^{N-1}} \int_{-\infty}^{x_1} \frac{\partial}{\partial x_1}\big(f(t,y)\big(\rho_\varepsilon * u_1^{x_1}\big)(y)\big)\,\mathrm{d}t\,\mathrm{d}y \\
&= \int_{(-\infty,x_1)\times\mathbb{R}^{N-1}} \operatorname{div}\big(f(t,y)\big(\rho_\varepsilon * u_1^{x_1}\big)(y)\big)\,\mathrm{d}t\,\mathrm{d}y \\
&= \int_{(-\infty,x_1)\times\mathbb{R}^{N-1}} f(t,y)\cdot\big(0,\nabla\big(\rho_\varepsilon * u_1^{x_1}\big)(y)\big) + \big(\operatorname{div} f(t,y)\big)\big(\rho_\varepsilon * u_1^{x_1}\big)(y)\,\mathrm{d}t\,\mathrm{d}y \\
&\leqslant C''_N \varepsilon^{(1/N)-1}\big(\|f\|_1\big\|\nabla u_1^{x_1}\big\|_N + \|\operatorname{div} f\|_1\big\|u_1^{x_1}\big\|_N\big),
\end{aligned}$$

where C'_N and C''_N are constants which depend only on the dimension N (and of ρ). (The third equality relies on the vector calculus identity $\operatorname{div}(Zf) = (\operatorname{div} f)Z + f\cdot\nabla Z$, and the last inequality comes from Hölder's inequality.)

For each $x_1 \in \mathbb{R}$ such that $\|f^{x_1}\|_1 \|\nabla u_1^{x_1}\|_N \neq 0$, let $\varepsilon = (\|f\|_1 \|\nabla u_1^{x_1}\|_N + \|\operatorname{div} f\|_1 \|u_1^{x_1}\|_N)/(\|f^{x_1}\|_1 \|\nabla u_1^{x_1}\|_N)$, so that

$$\int_{\mathbb{R}^{N-1}} f_1^{x_1} u_1^{x_1}\,\mathrm{d}y \leqslant C_N''' \big(\|f^{x_1}\|_1 \|\nabla u_1^{x_1}\|_N\big)^{(N-1)/N} \big(\|f\|_1 \|\nabla u_1^{x_1}\|_N + \|\operatorname{div} f\|_1 \|u_1^{x_1}\|_N\big)^{1/N}.$$

If $\|f^{x_1}\|_1 \|\nabla u_1^{x_1}\|_N \neq 0$, choosing $\varepsilon \to \infty$ gives the same inequality; hence the inequality is true for any $x_1 \in \mathbb{R}$.

Finally, Hölder's inequality yields

$$\begin{aligned}
\int_{\mathbb{R}^N} f_1 u_1\,\mathrm{d}x &\leqslant \int_{\mathbb{R}} C_N''' \big(\|f^{x_1}\|_1 \|\nabla u_1^{x_1}\|_N\big)^{(N-1)/N} \big(\|f\|_1 \|\nabla u_1^{x_1}\|_N + \|\operatorname{div} f\|_1 \|u_1^{x_1}\|_N\big)^{1/N}\,\mathrm{d}x_1 \\
&\leqslant C_N''' \|f\|_1^{1/N} \bigg(\int_{\mathbb{R}} \|f^{x_1}\|_1\,\mathrm{d}x_1\bigg)^{(N-1)/N} \bigg(\int_{\mathbb{R}} \|\nabla u^{x_1}\|_N^N\,\mathrm{d}x_1\bigg)^{(N-1)/N^2} \\
&\quad \times \bigg(\int_{\mathbb{R}} \big(\|f\|_1 \|\nabla u_1^{x_1}\|_N + \|\operatorname{div} f\|_1 \|u_1^{x_1}\|_N\big)^N\,\mathrm{d}x_1\bigg)^{1/N^2} \\
&\leqslant C_N \big(\|f\|_1 \|\nabla u\|_N\big)^{(N-1)/N} \big(\|f\|_1 \|\nabla u\|_N + \|\operatorname{div} f\|_1 \|u\|_N\big)^{1/N}.
\end{aligned} \tag{1}$$

The same estimate holds for $\int_{\mathbb{R}^N} u_i f_i$, $1 \leqslant i \leqslant N$. By classical approximation arguments, the inequality is true for any $f \in \mathrm{L}^1_\#(\mathbb{R}^N; \mathbb{R}^N)$ and $u \in (\mathrm{L}^\infty \cap \mathrm{W}^{1,N})(\mathbb{R}^N; \mathbb{R}^N)$. □

Remark 3. In fact the proof yields a slightly stronger inequality where $\|\nabla u\|_N$ is replaced by $\sum_{i \neq j} \|\partial_i u_j\|_N$, from which the inequality (1) can be recovered by a scaling argument.

Remark 4. The same arguments show that Theorem 1.5 remains true when f is a measure whose divergence is a measure.

Remark 5. As Bourgain and Brezis pointed out for Theorem 1.3 in [2], the proof works also when $\|\nabla u\|_N$ is replaced by any fractional Sobolev semi-norm $|\cdot|_{s,p}$, with $1 < p < \infty$, $0 < s < 1$, $sp = N$ and

$$|u|_{s,p}^p = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{2N}}\,\mathrm{d}x\,\mathrm{d}y.$$

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