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Domains of type 1, 1 operators: a case for Triebel–Lizorkin spaces

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Abstract

Pseudo-differential operators of type 1, 1 are proved continuous from the Triebel–Lizorkin space $F^d_{p,1}$ to L_p, $1 \leqslant p < \infty$, when of order d, and this is, in general, the largest possible domain among the Besov and Triebel–Lizorkin spaces. Hörmander's condition on the twisted diagonal is extended to this framework, using a general support rule for Fourier transformed pseudo-differential operators. ***To cite this article: J. Johnsen, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Domaines des opérateurs de type 1, 1 et des espaces de Triebel–Lizorkin. On démontre que les opérateurs pseudo-différentiels de type 1, 1 et d'ordre d sont continus de l'espace $F^d_{p,1}$ de Triebel–Lizorkin dans L_p, $1 \leqslant p < \infty$, et que parmi les espaces de Besov et Triebel–Lizorkin, ces domaines sont, en général, les plus grand possible. La condition de Hörmander sur la diagonale–miroir est établie pour ce cadre, en utilisant un résultat général sur le support de la transformation de Fourier d'un opérateur pseudo-différentiel. ***Pour citer cet article : J. Johnsen, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

1. Introduction

Recall that for symbols $a \in S^d_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$, i.e. $|D^\alpha_\xi D^\beta_x a(x,\xi)| \leqslant C_{\alpha\beta}(1+|\xi|)^{d-\rho|\alpha|+\delta|\beta|}$,

$$a(x,D) = \mathrm{OP}(a) = (2\pi)^{-n}\int e^{\mathrm{i}x\cdot\xi} a(x,\xi)\hat{u}(\xi)\,\mathrm{d}\xi \tag{1}$$

map the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ continuously into itself, say for $0 \leqslant \delta \leqslant \rho \leqslant 1$. For $(\rho,\delta) \neq (1,1)$ these operators extend to continuous, 'globally' defined maps

$$a(x,D) : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n). \tag{2}$$

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However, for $\rho = \delta = 1$ Ching [2] proved existence of $a \in S^0_{1,1}$ such that $a(x,D) \notin \mathbb{B}(L_2(\mathbb{R}^n))$. That every $A \in \mathrm{OP}(S^0_{1,1})$ is bounded on C^s and H^s for $s > 0$ was first proved by Stein (unpublished); Meyer [6] proved continuity from H^{s+d}_p to H^s_p for $s > 0$, $1 < p < \infty$.

For $s \leqslant 0$, Hörmander [4] gave a condition on the *twisted diagonal* $\{(\xi,\eta) \mid \xi + \eta = 0\}$: $a(x,D)$ is bounded $H^{s+d} \to H^s$ for *all* $s \in \mathbb{R}$ if $\hat{a}(\xi,\eta) := \mathcal{F}_{x\to\xi} a(x,\eta)$ fulfils

$$\hat{a}(\xi,\eta) = 0 \quad \text{for} \quad C\big(|\xi+\eta|+1\big) \leqslant |\eta|, \quad \text{for some} \quad C \geqslant 1. \tag{3}$$

For $s \geqslant 0$ and $1 \leqslant p \leqslant \infty$, the next result gives a maximal domain by means of the Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ (albeit with a Besov space for $p = \infty$).

Theorem 1.1. *Every $a \in S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$, $d \in \mathbb{R}$, gives a bounded operator*

$$a(x,D) : F^d_{p,1}\big(\mathbb{R}^n\big) \to L_p\big(\mathbb{R}^n\big) \quad \textit{for } p \in [1,\infty[, \tag{4}$$

$$a(x,D) : B^d_{\infty,1}\big(\mathbb{R}^n\big) \to L_\infty\big(\mathbb{R}^n\big). \tag{5}$$

$\mathrm{OP}(S^d_{1,1})$ *contains $a(x,D) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$, that are discontinuous when $\mathcal{S}(\mathbb{R}^n)$ is given the induced topology from any $F^d_{p,q}(\mathbb{R}^n)$ or $B^d_{p,q}(\mathbb{R}^n)$ with $p \in [1,\infty]$ and $q \in]1,\infty]$.*

So for fixed $p \in [1,\infty[$, every $A \in \mathrm{OP}(S^0_{1,1})$ is bounded $F^0_{p,1} \to L_p$ and everywhere defined, but not so on any larger $B^s_{p,q}$- or $F^s_{p,q}$-space (regardless of the codomain).

In comparison with Besov spaces, arguments in favour of *Triebel–Lizorkin* spaces have, perhaps, been less convincing. Indeed, $F^s_{p,2} = H^s_p$ for $1 < p < \infty$, cf. [9], but this does not necessarily make the $F^s_{p,q}$ a *useful* extension of the H^s_p-scale. However, Theorem 1.1 shows that also $F^s_{p,q}$-spaces with $q \neq 2$ are indispensable for a natural L_p-theory.

The next result extends Hörmander's condition in (3) to $F^s_{p,q}$.

Theorem 1.2. *Any $a(x,D) \in \mathrm{OP}(S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n))$ is continuous, for $s > 0$, p, $q \in [1,\infty]$,*

$$a(x,D) : F^{s+d}_{p,q}\big(\mathbb{R}^n\big) \to F^s_{p,q}\big(\mathbb{R}^n\big), \quad \textit{for } p < \infty. \tag{6}$$

If (3) *holds,* (6) *does so for $s \in \mathbb{R}$.* (*The result extends to $B^s_{p,q}$ and $p,q \in]0,\infty]$.*)

The proofs of Theorems 1.1 and 1.2 treat the symbols directly without approximation by elementary symbols, so it is crucial to control the spectra of the terms appearing in the paradifferential splitting of $a(x,D)$, and for this purpose the following was established.

Proposition 1.3 (The support rule). *If $b \in S^d_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ and $v \in \mathcal{F}^{-1}\mathcal{E}'(\mathbb{R}^n)$, then*

$$\operatorname{supp} \mathcal{F}\big(b(x,D)v\big) \subset \big\{\xi + \eta \mid (\xi,\eta) \in \operatorname{supp} \hat{b}(\cdot,\cdot),\ \eta \in \operatorname{supp} \hat{v}\big\}. \tag{7}$$

Proposition 1.4. *Any A in $\mathrm{OP}(S^\infty_{1,1})$ extends to a map $\mathcal{F}^{-1}\mathcal{E}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, that coincides with the usual one for $A \in \mathrm{OP}(S^\infty_{1,0})$.*

The support rule generalises to $b \in S^\infty_{1,1}$, for all $v \in \mathcal{F}^{-1}\mathcal{E}'$, using Proposition 1.4.

2. On the proofs

With $1=\sum_{j=0}^{\infty}\Phi_j$ so that $\Phi_j(\xi)=1 \Leftrightarrow |\xi|\sim 2^j$ $(j>0)$, set $\tilde{\Phi}_j=\Phi_{j-1}+\Phi_j+\Phi_{j+1}$, $a_{j,k}(x,\eta)=\mathcal{F}^{-1}_{\xi\to x}(\Phi_j\hat{a}(\cdot,\eta))\tilde{\Phi}_k(\eta)$ and $u_j=\Phi_j(D)u$. One can then make the ansatz

$$a(x,D)u(x)=a^{(1)}(x,D)u(x)+a^{(2)}(x,D)u(x)+a^{(3)}(x,D)u(x), \tag{8}$$

when the pair (a,u) is such that the following series converge in $\mathcal{D}'(\mathbb{R}^n)$:

$$a^{(1)}(x,D)u=\sum_{k=2}^{\infty}\sum_{j=0}^{k-2}a_{j,k}(x,D)u_k,\qquad a^{(3)}(x,D)u=\sum_{j=2}^{\infty}\sum_{k=0}^{j-2}a_{j,k}(x,D)u_k, \tag{9}$$

$$a^{(2)}(x,D)u=\sum_{k=0}^{\infty}\sum_{j,l=0,1,\ j+l\leqslant 1}a_{k-j,k-l}(x,D)u_{k-l}. \tag{10}$$

Here $a\in S^{\infty}_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$ implies $a_{j,k}\in S^{-\infty}$, and if $K_{j,k}$ denotes the distribution kernel,

$$a_{j,k}(x,D)u_k=\int_{\mathbb{R}^n}K_{j,k}(x,y)u_k(y)\,\mathrm{d}y,\quad \text{for } u\in\mathcal{S}'\big(\mathbb{R}^n\big). \tag{11}$$

This definition of $a(x,D)$ extends other ones, e.g. (1). Proposition 1.4 follows, for if $\hat{u}\in\mathcal{E}'$ both $a^{(1)}(x,D)u$, $a^{(2)}(x,D)u$ exist as finite sums; with $K_k(x,y):=\mathcal{F}^{-1}_{\xi\to y}(a\tilde{\Phi}_k)(x,x-y)$ one can sum over $j\leqslant N$ in (11) and majorise to show $\mathcal{S}'$-convergence to $\int K_k(x,\cdot)u_k\,\mathrm{d}y$.

To exploit the ansatz further, the 'pointwise' estimate in the next lemma is useful.

Lemma 2.1. *Let $v\in\mathcal{S}'(\mathbb{R}^n)$ and $b\in S^{\infty}_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$ such that* $\operatorname{supp}\mathcal{F}v\cup\bigcup_{x\in\mathbb{R}^n}\operatorname{supp}b(x,\cdot)$ *is contained in a ball $B(0,2^k)$, $k\in\mathbb{N}$. Then there exists a $c>0$ such that*

$$\big|b(x,D)v(x)\big|\leqslant c\big\|b(x,2^k\cdot)\mid \dot{B}^{n/t}_{1,t}\big(\mathbb{R}^n\big)\big\|M_tv(x). \tag{12}$$

Here $M_tf(x)=\sup_{r>0}(\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|^t\,\mathrm{d}y)^{1/t}$ is the maximal function; $0<t\leqslant 1$.

Lemma 2.1 is similar to [5, Proposition 5(a)], except that $b\in S^{\infty}_{1,1}$ replaces the vague assumption made there of being a 'symbol $\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$' ([5, Proposition 5(a)] itself is not easy to read, as it is extracted from an earlier proof with another set-up. However, $b\in S^{\infty}_{1,1}$ implies that $b(x,D)v$ is given by an integral like (11), and estimates in [5, Proposition 4] apply to this.)

The proof of Theorem 1.1 combines (12) with $L_p(\ell_1)$-boundedness of M_t for $t<1$, so that $\frac{n}{t}<n+1$. Further estimates of a follow from the embeddings $W^{n+1}_1\hookrightarrow B^{n+1}_{1,\infty}\hookrightarrow\dot{B}^{n/t}_{1,t}$: since $\frac{1}{4}\leqslant|\eta|\leqslant 4$ on $\operatorname{supp}\tilde{\Phi}$, so eg $2^{kd}\sim(1+|2^k\eta|)^d$, then if $\Psi_k=\Phi_0+\cdots+\Phi_k$,

$$\Bigg\|\sum_{j=0}^{k-2}a_{j,k}\big(x,2^k\cdot\big)\mid\dot{B}^{n/t}_{1,t}\Bigg\|\leqslant\sum_{|\alpha|\leqslant n+1}\big\|D^{\alpha}_{\xi}\big(\Psi_{k-2}(D_x)a\big(x,2^k\cdot\big)\tilde{\Phi}\big)\mid L_{1,\xi}\big\|\leqslant c2^{kd}, \tag{13}$$

where $c=c'\|\tilde{\Phi}\mid W^{n+1}_1\|\|\check{\Psi}\|_1\sup_{x,\xi;|\alpha|\leqslant n+1}(1+|\xi|)^{-(d-|\alpha|)}|D^{\alpha}_{\xi}a(x,\xi)|$. Using (12),

$$\begin{aligned}\Bigg\|\sum_k\sum_{j=0}^{k-2}a_{j,k}(x,D)u_k\Bigg\|_p^p&\leqslant\int\Bigg|\sum_k 2^{kd}M_tu_k(x)\Bigg|^p\mathrm{d}x\Bigg(\sup_{x,k}2^{-kd}\Bigg\|\sum_{j=0}^{k-2}a_{j,k}\big(x,2^k\cdot\big)\mid\dot{B}^{n/t}_{1,t}\Bigg\|\Bigg)^p\\&\leqslant c\int\Bigg(\sum_k 2^{kd}\big|u_k(x)\big|\Bigg)^p\mathrm{d}x.\end{aligned} \tag{14}$$

For k in finite sets, it now follows that the $a^{(1)}(x, D)u$-series is fundamental in L_p when $u \in F^d_{p,1}(\mathbb{R}^n)$ for $1 \leqslant p < \infty$, and (14) gives that $a^{(1)}(x, D)$ is bounded. The sum $\sum_{j=0}^{k-2}$ may then be replaced by the one pertinent for $a^{(2)}$, with a similar argument. To handle $a^{(3)}$, one may further invoke Taylor's formula and [10, Lemma 3.8]. The case $B^d_{\infty,1}(\mathbb{R}^n)$ is analogous, and the counterexamples of [2] adapts easily to give the sharpness.

In the proof of Theorem 1.2, the key point is to obtain (with Φ_j as in [10])

$$\operatorname{supp}\mathcal{F}\left(\sum_{j=0}^{k-2} a_{j,k}(x, D)u_k\right) \cup \operatorname{supp}\mathcal{F}\left(\sum_{j=0}^{k-2} a_{k,j}(x, D)u_j\right) \subset \left\{\frac{1}{5}2^k \leqslant |\xi| \leqslant 5 \cdot 2^k\right\}, \tag{15}$$

$$\operatorname{supp}\mathcal{F}\left(\sum_{j,l=0,1,\ j+l\leqslant 1} a_{k-j,k-l}(x, D)u_{k-l}\right) \subset \left\{|\xi| \leqslant 4 \cdot 2^k\right\}. \tag{16}$$

If (3) holds, then (16) may be supplemented by the property that, for k large enough,

$$\operatorname{supp}\mathcal{F}\left(\sum_{j,l=0,1,\ j+l\leqslant 1} a_{k-j,k-l}(x, D)u_{k-l}\right) \subset \left\{\xi \,\middle|\, \frac{1}{4C}2^k \leqslant |\xi| \leqslant 4 \cdot 2^k\right\}. \tag{17}$$

By Proposition 1.3, (15) and (16) are easy. (17) is seen thus: given (3), Proposition 1.3 implies that any $\xi + \eta$ in $\operatorname{supp}\mathcal{F}(a_{k-j,k-l}(x, D)u_{k-l})$ for large k fulfils

$$|\xi + \eta| \geqslant \frac{1}{C}|\eta| - 1 \geqslant \frac{11}{20C}2^{k-l} - 1 \geqslant \left(\frac{11}{40C} - 2^{-k}\right)2^k > \frac{1}{4C}2^k. \tag{18}$$

To complete the proof of Theorem 1.2 one can modify the estimates (14) ff. into $L_p(\ell^s_q)$ estimates; then convergence criteria for series of distributions, e.g. Theorems 3.6 and 3.7 of [10], apply by (15) and (16) (similar to arguments used in [6,10,5] etc.). The ball on the r.h.s. of (16) only yields estimates of $\|a^{(2)}(x, D)u \mid F^s_{p,q}\|$ for $s > 0$, as is well known. But if (3) holds, one can, by (17), use the criteria for series with spectra in dyadic annuli, like for $a^{(1)}$ and $a^{(3)}$ (the finitely many other terms of $a^{(2)}$ are in $\bigcap_{s>0} F^s_{p,q}$).

Remark 2.2. The class $\mathrm{OP}(S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n))$ was first treated in $F^s_{p,q}$-spaces by Runst [7], but unfortunately the proofs are somewhat flawed, since in Lemma 1 there the spectral estimates require a support rule under rather weak assumptions, like in Proposition 1.3 above. This was seemingly overlooked in [7] and by Marschall [5]. Using the φ-decomposition of Frazier and Jawerth [3], Torres [8] extended the H^s_p-continuity of [6] to the $F^s_{p,q}$-scale. The borderline $s = 0$ was treated by Bourdaud [1, Theorem 1]; his result on $B^0_{p,1}$ is improved by Theorem 1.1 above. Theorem 1.2 is a novelty concerning (3).

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