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Number Theory

A new method for lower bounds of L-functions

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Abstract

Let $L(s,\pi,r)$ be an L-function which appears in the Langlands–Shahidi theory. We give a lower bound for $L(s,\pi,r)$ when $\Re(s)=1$ using Eisenstein series. This method is applicable even when $L(s,\pi,r)$ is not known to be absolutely convergent for $\Re(s)>1$. ***To cite this article: S.S. Gelbart et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Une nouvelle méthode pour minorer des fonctions L. Soit $L(s,\pi,r)$ une fonction L présente dans la théorie de Langlands–Shahidi. Nous prouvons une minoration de $L(s,\pi,r)$ quand $\Re(s)=1$, en utilisant les séries d'Eisenstein. Cette méthode s'applique même lorsqu'on ne sait pas que $L(s,\pi,r)$ est absolument convergente pour $\Re(s)>1$. ***Pour citer cet article : S.S. Gelbart et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

1. Introduction

In 1899, de la Vallée Poussin extended his method of proving the Prime Number Theorem to showing that the Riemann zeta function has a zero-free region of the form

$$\left\{\sigma+\mathrm{i}t\colon \sigma>1-\frac{c}{\log(|t|+2)}\right\}$$

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for c an absolute positive constant, or, what is essentially equivalent,

$$\left|\zeta(1+\mathrm{i}t)\right| \geqslant \frac{c}{\log t}, \quad t \geqslant 2. \tag{1}$$

Such lower bounds (and more) are expected to hold for any automorphic L-function.

From a modern point of view, the method of de la Vallée Poussin is based on Rankin–Selberg L-functions and a positivity argument (an effective version of Landau's Lemma – see [3, Appendix]). As pointed out in [12] and [14], it can be applied to any Rankin–Selberg L-function $L(s, \pi_1 \otimes \pi_2)$ *provided that one of the π_i's is self-dual*. Here $\pi_i,\ i = 1, 2$ are cuspidal automorphic representations of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$ (for any number field F) whose central characters are trivial on $\mathbb{R}_+$ imbedded diagonally in the (archimedean) idèles. The zero-free region takes the form

$$\sigma > 1 - \frac{c}{\log(Q_{\pi_1} Q_{\pi_2}(|t|+2))}, \quad |t| \geqslant 1$$

with c an explicit constant depending only on the n_i's and F; here Q_{π_i} is the "analytic conductor" of π_i (see [5]). In particular, we have such a standard zero-free region for $L(s,\pi)$ for any cuspidal representation π of $\mathrm{GL}_n(\mathbb{A}_F)$. Thus, in principle, Langlands' functoriality yields a standard zero-free region for any automorphic L-function.

We note that providing a similar effective zero-free region for $t = 0$ when $L(s) = L(s, \chi)$ with χ a Dirichlet character is a major unsolved problem. See [4] for a discussion of Siegel zeros in this setting.

If the π_i's are not assumed to be self-dual then Brumley [1] recently established a coarse zero-free region

$$\sigma > 1 - \frac{c}{(Q_{\pi_1} Q_{\pi_2}(|t|+2))^N}$$

(for *any* t) where again $c,\ N$ depend (explicitly) only on $n_1,\ n_2$. Brumley's method also uses Rankin–Selberg theory and a positivity argument. Among other things, it has applications to the absolute convergence of the spectral side of Jacquet's relative trace formula [10].

2. The new method

In [14] the third-named author explains how to obtain a slightly weaker form of (1) by quite a different method, using Eisenstein series on SL_2. The argument exploits the Maass–Selberg relations and the computation of Fourier coefficients of Eisenstein series. Comparing the two by Bessel's inequality gives a coarse lower bound for zeta. This can be viewed as an effectuation (for $n = 1$) of the non-vanishing result of Jacquet–Shalika for the standard L-function of cusp forms on $\mathrm{GL}_n(\mathbb{A}_F)$ at $\Re(s) = 1$ [6]. (To obtain a better bound in the spirit of (1), a deeper analysis using an elementary sieve method is required.)

In this note we explain the generalization of the method above to the higher rank case, yielding lower bounds for any of the L-functions appearing in Langlands' formula for the constant term of Eisenstein series. Details will appear in a forthcoming paper of the two first-named authors.

3. Eisenstein series and L-functions

Let G be a reductive group over a number field F and let P be a maximal parabolic subgroup over F with Levi decomposition $P = MU$. Let ϖ be the fundamental weight corresponding to P. For any cuspidal $\varphi \in \mathcal{A}(M(F)U(\mathbb{A}_F)\backslash G(\mathbb{A}_F))$ (suitably normalized under A_M) we consider the Eisenstein series $E(g, \varphi, s\varpi),\ g \in G(\mathbb{A}_F),\ s \in \mathbb{C}$. (See [11] for unexplained notation.) Let π be a cuspidal automorphic representation of $M(\mathbb{A}_F)$ and suppose that $m \to \varphi(mk)$ belongs to the space of π for all $k \in \mathbf{K}$. Then the constant term of $E(\cdot, \varphi, s\varpi)$ along $\overline{P}$ is expressed in terms of the intertwining operator which is in turn given, up to a finite number of local factors, by

$$\prod_{j=1}^{m} \frac{L^S(js, \tilde{\pi}, r_j)}{L^S(1+js, \tilde{\pi}, r_j)}$$

for sufficiently large S [9]. Here $\bigoplus_{j=1}^{m} r_j$ is the decomposition of the adjoint representation of ${}^L M$ on the Lie algebra of ${}^L U$ into irreducible constituents, indexed by the terms in the lower central series of ${}^L U$, and $L^S(s, \pi, r_j)$ denotes the corresponding (partial) L-function.

4. Finiteness of order

By a result of Müller [13, Theorem 0.2] there exists an entire function $q(s)$ of finite order such that $q(s)E(g, \varphi, s)$ is entire and of finite order for all $g \in G(\mathbb{A}_F)$. This fact and Langlands' formula imply, using induction and a little complex analysis, the following Theorem.

Theorem 4.1. *Each* $L(s, \pi, r_j)$ *is of finite order* (*as a meromorphic function*).

We note that a similar result holds for other L-functions which admit an integral representation.

5. Generic representations

In order to generalize [14] it is necessary that π be *generic*. Suppose from now on that G (or equivalently, M) is quasi-split and let ψ be a non-degenerate character of $U_0(F)\backslash U_0(\mathbb{A}_F)$. *We will henceforth assume that* π, r_j *and* S *are all fixed, that* S *contains the archimedean places and that* π *is generic with respect to the restriction of* ψ *to* $U_0 \cap M$. Then, by the "Langlands–Shahidi method" the ψ-th Fourier coefficient of $E(\cdot, \varphi, s\varpi)$ is given, up to a global constant and local Jacquet integrals, by

$$\left[\prod_{j=1}^{m} L^S(1+js, \tilde{\pi}, r_j)\right]^{-1}.$$

Moreover, Shahidi has obtained an exact functional equation for $L(s, \pi, r_j)$ and proved finiteness of poles for the partial L-functions [16]. Invoking a standard argument using the Phragmen–Lindelof principle we deduce the following proposition from Theorem 4.1.

Proposition 5.1. *There exists a polynomial* $P(s)$ *such that for every* $s_0 < s_1$ *there exist constants* $c, n > 0$ *such that*

$$\left|P(s)L^S(s, \pi, r_j)\right| \leqslant c\bigl(1+|s|\bigr)^n \tag{2}$$

in the strip $R = \{s \in \mathbb{C}\colon\ s_0 < \Re(s) < s_1\}$ *and similarly for the derivative of* L^S.

This proposition sharpens and simplifies the main result of [2]. We point out that a similar result ought to hold for other automorphic L-functions (even for non-generic representations) which admit integral presentations (of Rankin–Selberg type). This would have an important consequence in the application of the converse theorem.

6. The main result

Theorem 6.1. *There exist constants* $c,\ n > 0$ *such that*

$$\left|L^S(1+\mathrm{i}t, \pi, r_j)\right| \geqslant c\bigl(1+|t|\bigr)^{-n}, \quad |t| \geqslant 1,\ j = 1, \ldots, m.$$

This theorem answers in a strong form a conjecture posed in [2]. It implies a similar zero-free region for $L(s,\pi,r_j)$. The proof of Theorem 6.1 is based, as in the SL_2 case, on estimating (for $s\in \mathrm{i}\mathbb{R}$) $\|\Lambda^T E(\cdot,\varphi,s\varpi)\|_2^2$ from above by the Maass–Selberg relations, and from below by squares of Fourier coefficients. The crux of the matter is the non-homogeneity of the ensuing inequality. We remark that the non-vanishing of $L^S(1+\mathrm{i}t,\pi,r_j)$, $t\neq 0$ was proved by Shahidi [15].

Let now π be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Applying Theorem 6.1 for the exceptional group $G=E_8$ and using the third and fourth symmetric power liftings of π [8,7] we obtain:

Corollary 6.2. *There exist constants $c,n>0$ such that for all $t\in\mathbb{R}$ with $|t|\geqslant 1$*

$$L^S\bigl(1+\mathrm{i}t,\pi,\mathrm{sym}^9\bigr)\geqslant \frac{c}{(1+|t|)^n}.$$

Interestingly enough, it is not known whether $L^S(s,\pi,\mathrm{sym}^9)$ converges absolutely for $\Re(s)>1$ or if it has zeros or poles in $[1,\infty)$. Thus, clearly, Corollary 6.2 lies beyond the scope of the method of de la Vallée Poussin and Theorem 6.1 contains all the cases of such non-vanishing proved by the last method as special cases. Another consequence of Theorem 6.1 is to uniform upper bounds of Eisenstein series.

Theorem 6.3. *There exist constants c, n such that for all $g\in G(\mathbb{A}_F)$ and $s\in \mathrm{i}\mathbb{R}$*

$$\bigl|E(g,\varphi,s)\bigr|\leqslant c\cdot\bigl(1+\|g\|\bigr)^n\cdot\bigl(1+|s|\bigr)^n.$$

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