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Random Euclidean embeddings in spaces of bounded volume ratio

Alexander Litvak [a], Alain Pajor [b], Mark Rudelson [c,1], Nicole Tomczak-Jaegermann [d,2], Roman Vershynin [e,3]

[a] *Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1*
[b] *Équipe d'analyse et mathématiques appliquées, université de Marne-la-Vallée, 5, boulevard Descartes, Champs sur Marne, 77454 Marne-la-Vallée cedex 2, France*
[c] *Department of Mathematics, University of Missouri, Columbia, MO 65211, USA*
[d] *Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1*
[e] *Department of Mathematics, University of California, Davis, CA 95616, USA*

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Abstract

Let $(\mathbb{R}^N, \|\cdot\|)$ be the space $\mathbb{R}^N$ equipped with a norm $\|\cdot\|$ whose unit ball has a bounded volume ratio with respect to the Euclidean unit ball. Let Γ be any random $N \times n$ matrix with $N > n$, whose entries are independent random variables satisfying some moment assumptions. We show that with high probability Γ is a good isomorphism from the n-dimensional Euclidean space $(\mathbb{R}^n, |\cdot|)$ onto its image in $(\mathbb{R}^N, \|\cdot\|)$: there exist $\alpha, \beta > 0$ such that for all $x \in \mathbb{R}^n$, $\alpha\sqrt{N}|x| \leqslant \|\Gamma x\| \leqslant \beta\sqrt{N}\,|x|$. This solves a conjecture of Schechtman on random embeddings of ℓ_2^n into ℓ_1^N. ***To cite this article: A. Litvak et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Plongements aléatoires de l'espace euclidien dans un espace à volume ratio borné. Soit $(\mathbb{R}^N, \|\cdot\|)$ l'espace $\mathbb{R}^N$ muni d'une norme $\|\cdot\|$ dont la boule unité est à volume ratio borné par rapport à la boule unité euclidienne. On montre qu'une matrice aléatoire Γ, de taille $N \times n$ $(N > n)$, dont les coefficients sont des variables aléatoires indépendantes, vérifiant certaines hypothèses de moments, réalise avec une grande probabilité, un bon isomorphisme de l'espace euclidien de dimension n, de norme $|\cdot|$, sur son image dans $(\mathbb{R}^N, \|\cdot\|)$: il existe $\alpha, \beta > 0$ tels que pour tout $x \in \mathbb{R}^n$, $\alpha\sqrt{N}|x| \leqslant \|\Gamma x\| \leqslant \beta\sqrt{N}\,|x|$; ce qui démontre une conjecture de Schechtman sur les plongements aléatoires de ℓ_2^n dans ℓ_1^N. ***Pour citer cet article : A. Litvak et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

E-mail addresses: alexandr@math.ualberta.ca (A. Litvak), alain.pajor@univ-mlv.fr (A. Pajor), rudelson@math.missouri.edu (M. Rudelson), nicole@ellpspace.math.ualberta.ca (N. Tomczak-Jaegermann), vershynin@math.ucdavis.edu (R. Vershynin).

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Soit $N \geqslant n$. Dans cette Note, nous nous intéressons à des sections « aléatoires » de dimension n de corps convexes de $\mathbb{R}^N$, dont l'espace est engendré par les n colonnes de matrices Γ de taille $N \times n$, dont les coefficients sont des variables aléatoires réelles sur un espace probabilisé $(\Omega, \mathcal{A}, \mathbb{P})$. On considère ces matrices comme des opérateurs entre les espaces euclidiens ℓ_2^n et ℓ_2^N et l'on note $\|\Gamma\|_{2\to 2}$ leur norme dans $\mathrm{L}(\ell_2^n, \ell_2^N)$.

Soient $\mu \geqslant 1$ et $a_1, a_2 > 0$. On considère l'ensemble $\mathrm{M}(N, n, \mu, a_1, a_2)$ des matrices $N \times n$ dont les coefficients sont des variables réelles symétriques indépendantes $(\xi_{ij})_{1\leqslant i\leqslant N, 1\leqslant j\leqslant n}$ sur $(\Omega, \mathcal{A}, \mathbb{P})$, vérifiant :

$$1 \leqslant \|\xi_{ij}\|_{L^2} \leqslant \|\xi_{ij}\|_{L^3} \leqslant \mu \quad \text{pour tout } 1 \leqslant i \leqslant N,\ 1 \leqslant j \leqslant n \quad \text{et} \quad \mathbb{P}\big(\|\Gamma\|_{2\to 2} \geqslant a_1\sqrt{N}\big) \leqslant \mathrm{e}^{-a_2 N}.$$

Parmi les exemples importants de matrices de $\mathrm{M}(N, n, \mu, a_1, a_2)$, figurent les matrices aléatoires dont les entrées sont des gaussiennes standard ou des variables de Bernoulli ± 1.

On note $|\cdot|$ la norme euclidienne de $\mathbb{R}^N$, B_2^N sa boule unité et S^{N-1} sa sphère unité. On note aussi $|L|$ le volume d'une partie mesurable $L \subset \mathbb{R}^N$. Soit K un corps convexe symétrique (par rapport à l'origine), on pose $V_K := (|K|/|B_2^N|)^{1/N}$.

Lorsque $B_2^N \subset K$ et que V_K est majoré par une constante indépendante de la dimension, on dit que K est à volume ratio borné par rapport à la boule unité euclidienne. Soit ℓ_1^N, l'espace $\mathbb{R}^N$ muni de la norme $\sum_{i\geqslant 1} |x_i|$ pour $x = (x_i) \in \mathbb{R}^N$ et B_1^N sa boule unité. La boule $K = \sqrt{N} B_1^N$ est à volume ratio borné par rapport à la boule unité euclidienne.

Soit $\delta > 0$, Kashin [4] (voir aussi Szarek [9]) a montré que pour des sections E de dimension $n \leqslant N/(1+\delta)$ de $\mathbb{R}^N$ qui sont aléatoires au sens de la mesure de Haar de la grassmanienne, on a

$$B_2^N \cap E \subset \sqrt{N} B_1^N \cap E \subset a(\delta) B_2^N \cap E \tag{1}$$

pour une certaine fonction $a(\delta)$; ce qui conduit, quand $N = 2n$ à un résultat bien connu de Kashin sur la décomposition orthogonale de l'espace ℓ_1^N. En utilisant la méthode de ([9]), ces résultats ont été généralisé dans [10] aux boules à volume ratio borné.

Notons que la mesure sur la grassmanienne est induite par une matrice Γ de taille $N \times n$ dont les coefficients sont des variables gaussiennes. Schechtman ([8]) a étudié une question similaire à la propriété (1) ci-dessus, pour des sous-espaces E de ℓ_1^N qui sont engendrés par les colonnes d'une matrice de variables de Bernoulli et pour $\delta > 0$ quelconque. Plus précisément, il montre ([8], Proposition 3) l'existence d'une section (non-aléatoire) vérifiant (1) et conjecture que le résultat reste vrai pour des matrices aléatoires de ± 1. Sur un autre plan, dans [6] les auteurs ont montré que les noyaux d'une matrice aléatoire de ± 1 de taille $n \times N$ vérifiaient aussi les inclusions 1 (avec une grande probabilité).

Dans cette Note on répond à la question de Schechtman, pour tout $\delta > 0$ et d'une part, on montre ce résultat pour des espaces à volume ratio borné, d'autre part, pour des sections obtenues à partir des colonnes de matrices qui appartiennent à une très large classe, contenant le cas gaussien et le cas de variables de Bernoulli. Le principal résultat de cette Note est le suivant :

Théorème 0.1. *Soient $\delta > 0$, $n > 1$ et $N = (1+\delta)n$. Soit Γ une matrice $N \times n$ de $\mathrm{M}(N, n, \mu, a_1, a_2)$, avec $\mu \geqslant 1$, $a_1, a_2 > 0$. Soit $K \subset \mathbb{R}^N$ un corps convexe symétrique tel que $B_2^N \subset K$. Il existe $\alpha = (2V_K)^{-c_3'(1+1/\delta)}$ et $\tilde{c}_1, \gamma > 0$ tels que pour tout $n \geqslant \tilde{c}_1^{1+1/\delta}$, on a*

$$\mathbb{P}\big(\|\Gamma x\|_K \geqslant \sqrt{N}\alpha\, |x| \text{ for all } x \in \mathbb{R}^n\big) \geqslant 1 - \mathrm{e}^{-\gamma N},$$

où c_3' dépend de μ, a_1, a_2 et $\tilde{c}_1$ dépend de a_1, μ et enfin γ dépend de μ, a_2.

L'estimation de α est optimale quand δ tend vers 0 (aux constantes numériques près). Dans cette Note, on donne une esquisse de démonstration avec une estimation en $1 + 1/\delta^2$ au lieu de $1 + 1/\delta$. Les démonstrations complètes ainsi que d'autres applications seront développées dans un article en préparation.

On en déduit le corollaire suivant :

Corollaire 0.2. *Sous les hypothèses du Théorème* 0.1*, l'espace E engendré par les colonnes de la matrice Γ, vérifie, avec une probabilité* $\geqslant 1 - e^{-\gamma N}$,

$$\left(1/a_1\sqrt{N}\right)\Gamma\left(B_2^n\right) \subset B_2^N \cap E \subset K \cap E \subset \left(1/\alpha\sqrt{N}\right)\Gamma\left(B_2^n\right) \subset (a_1/\alpha)B_2^N \cap E$$

et

$$\alpha\sqrt{N}|x| \leqslant \|\Gamma x\|_K \leqslant a_1\sqrt{N}\,|x| \quad \textit{pour tout } x \in \mathbb{R}^n.$$

Dans le cas gaussien, on retrouve le résultat de Szarek [9,10]. Dans le cas de variables de Bernoulli ± 1, cela démontre le résultat conjecturé par Schechtman [8] pour ℓ_1^N.

La démonstration du Théorème 0.1 s'appuie sur la proposition suivante, qui est une modification du Théorème 3.1 de [5] sur la plus petite valeur singulière des matrices de $\mathrm{M}(N, n, \mu, a_1, a_2)$ ainsi que sur un lemme d'entropie métrique (Lemma 2.4 de la version anglaise).

Proposition 0.3. *Soient $\delta > 0$, $n > 1$ et $N = (1+\delta)n$. Soit Γ une matrice $N \times n$ de* $\mathrm{M}(N, n, \mu, a_1, a_2)$*, avec $\mu \geqslant 1$, $a_1, a_2 > 0$. Il existe $c_1 > 0$ de la forme $c_1 = c_3^{1+1/\delta}$ et $\tilde{c}_1$, c_2 tels que pour tout $n \geqslant \tilde{c}_1^{1+1/\delta}$ et tout $z \in \mathbb{R}^N$, on a*

$$\mathbb{P}\left(\exists\, x \in S^{n-1} \textit{ t.q. } \Gamma x \in z + c_1\sqrt{N}B_2^N\right) \leqslant \exp(-c_2 N)$$

où $0 < c_3 < 1$ et $\tilde{c}_1$ dépendent de μ, a_1 et $c_2 > 0$ dépend de μ, a_2.

1. Introduction

Let $N \geqslant n$. In this paper we are interested in "random" sections of convex bodies in $\mathbb{R}^N$ given by n-dimensional subspaces of $\mathbb{R}^N$, spanned by the columns of rectangular $N \times n$ matrices Γ, whose entries are real-valued random variables on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We consider these matrices as operators acting from the Euclidean space ℓ_2^n to ℓ_2^N and we denote by $\|\Gamma\|_{2\to 2}$ the norm of Γ in $\mathrm{L}(\ell_2^n, \ell_2^N)$.

Let $\mu \geqslant 1$ and $a_1, a_2 > 0$. We define the set of $N \times n$ matrices $\mathrm{M}(N, n, \mu, a_1, a_2)$ to consist of matrices with real-valued independent symmetric random variable entries $(\xi_{ij})_{1\leqslant i\leqslant N, 1\leqslant j\leqslant n}$ on $(\Omega, \mathcal{A}, \mathbb{P})$, satisfying:

$$1 \leqslant \|\xi_{ij}\|_{L^2} \leqslant \|\xi_{ij}\|_{L^3} \leqslant \mu \quad \text{for all } 1 \leqslant i \leqslant N,\ 1 \leqslant j \leqslant n \quad \text{and} \quad \mathbb{P}\left(\|\Gamma\|_{2\to 2} \geqslant a_1\sqrt{N}\right) \leqslant e^{-a_2 N}.$$

Basic examples of matrices from $\mathrm{M}(N, n, \mu, a_1, a_2)$ are random matrices with standard Gaussian or Bernoulli ± 1 entries.

By $|\cdot|$ and $\langle\cdot,\cdot\rangle$ we denote the canonical Euclidean norm and the canonical inner product on $\mathbb{R}^N$; the corresponding unit ball and the unit sphere are denoted by B_2^N and S^{N-1}, respectively. For any Lebesgue measurable set $L \subset \mathbb{R}^N$, by $|L|$ we denote the volume of L. By a symmetric convex body $K \subset \mathbb{R}^N$ we mean a centrally symmetric convex compact set with the non-empty interior. For such a K we set $V_K := (|K|/|B_2^N|)^{1/N}$.

Whenever $B_2^N \subset K$ and V_K is bounded by a constant independent on the dimension, we say that K has bounded volume ratio with respect to the Euclidean unit ball. For example, denoting by ℓ_1^N the space $\mathbb{R}^N$ with the norm $\sum_{i\geqslant 1}|x_i|$, for $x = (x_i) \in \mathbb{R}^N$, and by B_1^N its unit ball, the body $K = \sqrt{N}B_1^N$ has bounded volume ratio with respect to the Euclidean unit ball.

Let $\delta > 0$. It was shown by Kashin in [4] (see also Szarek [9] for a different argument) that "random" n-dimensional subspaces $E \subset \mathbb{R}^N$ (in sense of the Haar measure on the Grassman manifold) with $n \leqslant N/(1+\delta)$, satisfy condition (1), namely $B_2^N \cap E \subset \sqrt{N}\, B_1^N \cap E \subset a(\delta) B_2^N \cap E$ for a certain function $a(\delta)$. When $N = 2n$, this implies a well-known result of Kashin on the orthogonal decomposition of ℓ_1^N. Szarek's proof worked in a more general case of spaces with bounded volume ratio ([10], see also [7]).

Observe that the Haar measure on the Grassman manifold is induced by an $N \times n$ matrix Γ whose entries are independent Gaussian random variables. Recently Schechtman studied in [8] an analogue of (1) for subspaces of ℓ_1^N spanned by the columns of matrices build from Bernoulli ± 1 variables. More precisely, he has shown that for a $2n \times n$ matrix A such that $A^* = [\sqrt{n} I_n \quad B]$, where I_n is the identity $n \times n$ matrix, and B is an $n \times n$ matrix whose entries are independent Bernoulli ± 1 variables, the subspace spanned by the columns of A satisfies (1) with probability exponentially close to 1. He further showed [8, Proposition 3] that for any $\delta > 0$ and $N \geqslant (1+\delta)n$, there exists an $N \times n$ (non-random) matrix consisting of ± 1 entries only, whose columns span a subspace satisfying 1, and he conjectured that the result remains valid for "random" ± 1 matrices as well. On a related direction, it has been recently shown in [6] that the kernels of a random ± 1 matrix of size $n \times N$ also satisfy 1 with probability exponentially close to 1.

2. Main result

In this Note we answer the question of Schechtman, for all δ. On one hand, we show the result for spaces with bounded volume ratio, and on the other hand, for sections spanned by the columns of matrices belonging to $\mathrm{M}(N, n, \mu, a_1, a_2)$, which, as mentioned earlier, contains matrices with standard Gaussian and Bernoulli entries. The main result of this paper states

Theorem 2.1. *Let $\delta > 0$, let $n > 1$ and $N = (1+\delta)n$. Let Γ be an $N \times n$ matrix from $\mathrm{M}(N, n, \mu, a_1, a_2)$, for some $\mu \geqslant 1$, $a_1, a_2 > 0$. Let $K \subset \mathbb{R}^N$ be a symmetric convex body such that $B_2^N \subset K$. There exist $\alpha > 0$ of the form $\alpha = (2V_K)^{-c_3' f(\delta)}$ and $\tilde{c}_1, \gamma > 0$, such that whenever $n \geqslant \tilde{c}_1^{1+1/\delta}$ then*

$$\mathbb{P}\big(\|\Gamma x\|_K \geqslant \sqrt{N}\alpha\, |x| \text{ for all } x \in \mathbb{R}^n\big) \geqslant 1 - \mathrm{e}^{-\gamma N}.$$

Here $f(\delta)$ is a function of δ only, $c_3' > 0$ depends on μ, a_1, a_2, while $\tilde{c}_1$ depends on μ, a_1, and finally γ depends on μ, a_2 only.

Theorem 2.1 holds with $f(\delta) = 1 + 1/\delta$, which is an optimal order as $\delta \to 0$. In this Note we shall outline a simplified proof which only gives $f(\delta) \leqslant 1 + 1/\delta^2$. The complete proof of the optimal estimate, related results for non-symmetric and shifted bodies, as well as some further applications, will appear elsewhere.

We have an immediate corollary

Corollary 2.2. *Under the assumptions of Theorem 2.1, the subspace E spanned by the n columns of the matrix Γ satisfies, with probability $\geqslant 1 - \mathrm{e}^{-\gamma N}$,*

$$\big(1/a_1\sqrt{N}\big)\Gamma\big(B_2^n\big) \subset B_2^N \cap E \subset K \cap E \subset \big(1/\alpha\sqrt{N}\big)\Gamma\big(B_2^n\big) \subset (a_1/\alpha) B_2^N \cap E \quad \text{and}$$

$$\sqrt{N}\alpha |x| \leqslant \|\Gamma x\|_K \leqslant a_1\sqrt{N}\,|x| \quad \text{for all } x \in \mathbb{R}^n.$$

The proof of Theorem 2.1 relies on the following result which is a modification of Theorem 3.1 from [5] on the smallest singular value of matrices from $\mathrm{M}(N, n, \mu, a_1, a_2)$. We have

Proposition 2.3. *Let $\delta > 0$, let $n > 1$ and $N = (1+\delta)n$. Let Γ be an $N \times n$ matrix from $\mathrm{M}(N, n, \mu, a_1, a_2)$, for some $\mu \geqslant 1$, $a_1, a_2 > 0$. There exist $c_1 > 0$ of the form $c_1 = c_3^{1+1/\delta}$, and $\tilde{c}_1, c_2 > 0$ such that whenever $n \geqslant \tilde{c}_1^{1+1/\delta}$ then, for every fixed $z \in \mathbb{R}^N$, we have $\mathbb{P}(\exists x \in S^{n-1}$ s.t. $\Gamma x \in z + c_1\sqrt{N} B_2^N) \leqslant \exp(-c_2 N)$.*

Here $0 < c_3 < 1$ and $\tilde{c}_1$ depend on μ, a_1, while $c_2 > 0$ depends on μ, a_2 only.

Another key fact provides a control, for a symmetric convex body $K \subset \mathbb{R}^N$, of the covering number of a certain body L of the form $L = \alpha K \cap \bar{a} B_2^N$ by $\bar{c} B_2^N$, where $\bar{a}$ and $\bar{c}$ are fixed and α depends on V_K. Recall that for any subsets L and L' of $\mathbb{R}^N$, the covering number $N(L, L')$ is the smallest number of translates of L' needed to cover L.

Lemma 2.4. *There exists an absolute constant $c > 0$ such that for every $\bar{a} \geqslant \mathrm{e}\bar{c} > 0$, every symmetric convex body $K \subset \mathbb{R}^N$ satisfying $B_2^N \subset K$, and every $0 < \eta \leqslant \ln(4\pi V_K)/\ln(\bar{a}/\bar{c})$ one has*

$$N\big(\alpha K \cap \bar{a} B_2^N, \bar{c} B_2^N\big) \leqslant 2^{\eta N} \quad \text{for } \alpha = \bar{a}(4\pi V_K)^{-(c/\eta)\ln(\bar{a}/\bar{c})}.$$

Sketch of the proof. Set $L := \alpha K \cap \bar{a} B_2^N$, and $A = 4\pi V_K$. By Szarek's volume ratio theorem, for every $1 \leqslant k \leqslant n$ there exists a subspace $E \subset \mathbb{R}^N$ with $\operatorname{codim} E = k$ such that $L \cap E \subset \min(\bar{a}, \alpha A^{N/k}) B_2^N \cap E$. It is now convenient to use some terminology of so-called s-numbers of operators. For an operator $u : (\mathbb{R}^N, K) \to \ell_2^N$ and any j, the j'th Gelfand number is defined by $c_j(u) = \inf\{\|u|_E\|:\ E \subset \mathbb{R}^N, \operatorname{codim} E < j\}$, and the j'th entropy number is defined by $e_j(u) = \inf\{\varepsilon:\ N(u(K), \varepsilon B_2^N) \leqslant 2^{j-1}\}$. In particular, letting u to be the formal identity operator from $(\mathbb{R}^N, K)$ to ℓ_2^N, we have $c_{k+1}(u) \leqslant \min(\bar{a}, \alpha A^{n/k})$.

Set $\bar{r} = \bar{a}/\bar{c}$, $\beta = \ln \bar{r} \geqslant 1$, and $m = [\eta N]$. By Carl's theorem ([1], cf., [7, Theorem 5.2]) we get $m^\beta e_m(u) \leqslant \rho_\beta \sup_{k \leqslant m} k^\beta c_{k+1}(u) \leqslant (c\beta)^\beta \sup_{0<t\leqslant m}(t^\beta \min(\bar{a}, \alpha A^{N/t}))$, where $c > 0$ is an absolute constant. Since the function $f(t) = t^\beta A^{N/t}$ is decreasing on the interval $(0, N(\ln A)/\beta]$ and $m \leqslant N(\ln A)/\beta$, the supremum above is attained for $t = N(\ln A)/\ln(\bar{a}/\alpha)$. Thus $e_m(u) \leqslant (m^{-1} c\beta N(\ln A)/\ln(\bar{a}/\alpha))^\beta \bar{a} \leqslant (2c\,(\ln \bar{r})\,(\ln A)/(\eta \ln(\bar{a}/\alpha)))^{\ln \bar{r}} \bar{a} \leqslant \bar{c}$ for $\alpha \leqslant \bar{a} A^{-(2\mathrm{e}c/\eta)\ln \bar{r}}$. That proves the result. □

Assuming Proposition 2.3, Theorem 2.1 now easily follows.

Proof of Theorem 2.1. Note that by the definition of $\mathrm{M}(N, n, \mu, a_1, a_2)$, for any $\alpha > 0$, we have

$$\mathbb{P}\big(\exists x \in S^{n-1} \text{ s.t. } \Gamma x \in \alpha\sqrt{N} K\big) \leqslant \mathrm{e}^{-a_2 N} + \mathbb{P}\big(\exists x \in S^{n-1} \text{ s.t. } \Gamma x \in \sqrt{N} L\big), \tag{2}$$

where we let $L := \alpha K \cap a_1 B_2^N$.

Let c_1, c_2 be as in Proposition 2.3. Apply Lemma 2.4 for $\eta = \min(c_2/2, \ln(4\pi V_K)/\ln(a_1/c_1))$, $\bar{a} = a_1$ and $\bar{c} = c_1$. Let α be an appropriate function of V_K, which can be taken of the form $\alpha = (2V_K)^{-c_3' f(\delta)}$ with $f(\delta) \leqslant 1 + 1/\delta^2$. Then $M := N(L, c_1 B_2^N) = N(\alpha K \cap a_1 B_2^N, c_1 B_2^N) \leqslant \mathrm{e}^{c_2 N/2}$. Pick $\mathcal{M}$ in $\mathbb{R}^N$ with $|\mathcal{M}| = M$ such that $L \subset \bigcup_{z \in \mathcal{M}} (z + c_1 B_2^N)$. Then the latter probability in (2) is less than or equal to $M\,\mathbb{P}(\exists x \in S^{n-1} \text{ s.t. } \Gamma x \in z + c_1\sqrt{N} B_2^N) \leqslant M\mathrm{e}^{-c_2 N} \leqslant \mathrm{e}^{-c_2 N/2}$. By (2), this completes the proof. □

We shall now comment on the proof of Proposition 2.3. It is based on two key estimates, the proofs of which are based on similar ideas as Proposition 3.2 and 3.4 in [5].

Lemma 2.5. *Let $(\xi_i)_{i=1}^n$ be a sequence of independent symmetric random variables with $1 \leqslant \|\xi_i\|_{L_2} \leqslant \|\xi_i\|_{L_3} \leqslant \mu$ for all $i = 1, \ldots, n$. For any subset $\sigma \subset \{1, \ldots, n\}$ let P_σ denote the coordinate projection in $\mathbb{R}^n$. Then for any $x = (x_i) \in \mathbb{R}^n$, $\sigma \subset \{1, \ldots, n\}$, we have, for all $s \in \mathbb{R}$ and $t > 0$,*

$$\mathbb{P}\left(\left|\sum_{i=1}^n \xi_i x_i - s\right| < t\right) \leqslant \sqrt{2/\pi}\,\frac{t}{|P_\sigma x|} + c\left(\frac{\|P_\sigma x\|_3}{|P_\sigma x|}\mu\right)^3,$$

where $c > 0$ is a universal constant.

The proof of this lemma depends on the Berry–Esséen theorem (see [5]). The second lemma is a general estimate for the norm $|\Gamma x - z|$ of an arbitrary shift (by $z \in \mathbb{R}^N$) of Γx, for a fixed $x \in \mathbb{R}^n$.

Lemma 2.6. *Let $1 \leqslant n < N$. Let Γ be an $N \times n$ random matrix from $\mathrm{M}(N, n, \mu, a_1, a_2)$, for some $\mu \geqslant 1$ and $a_1, a_2 > 0$. Then for every $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^N$ we have $\mathbb{P}(|\Gamma x - z| \leqslant c'\mu^{-3}\sqrt{N}|x|) \leqslant \exp(-c''N/\mu^6)$, where $0 < c', c'' < 1$ are absolute constants.*

Let $K \subset \mathbb{R}^N$ be a symmetric convex body. Recall the important definition of the M^*-functional, $M^*(K) := \int_{S^{N-1}} \sup_{y \in K} \langle x, y\rangle \, \mathrm{d}x$. Now, consider the function $M_K^*(\cdot) : (0, \infty) \to [0, 1]$ defined by $M_K^*(r) = M^*(L)$, where $L := (K/r) \cap B_2^N$. In [2,3] many properties of K were investigated using the function $M_K^*(r)$. The following proposition provides estimates for this function in terms of V_K.

Proposition 2.7. *Let $K \subset \mathbb{R}^N$ be a symmetric convex body such that $B_2^N \subset K$. There exists an absolute constant C such that $M_K^*(r) \leqslant C\sqrt{\frac{\ln(2V_K)}{\ln(r^2 \ln(2V_K))}}$ for every $r > \frac{1}{\sqrt{\ln(2V_K)}}$. In particular, if $r \geqslant 2(2V_K)^{1/\eta}$ then $M_K^*(r) \leqslant C\sqrt{\eta}$.*

Remark 1. By Sudakov's inequality this proposition implies Lemma 2.4 with $\alpha = \bar{a}(2V_K)^{-c(\bar{a}/\bar{c})^2/\eta}$.

Sketch of the proof. Denote $M_K^*(r) = M^*(L)$ by M^*. Since $L \subset B_2^N$, by the dual version of Dvoretzky theorem, there exist an absolute constant $0 < c' < 1/4$ and a subspace $E \subset \mathbb{R}^N$ of dimension $k \geqslant c'(M^*)^2 N$ such that $P_E K \supset r P_E L \supset (rM^*/2) P_E B_2^N$. Here P_E denotes the orthogonal projection onto E. Since $K \supset B_2^N$, Rogers–Shephard inequality (see [7, Lemma 8.8]) implies

$$V_K = \left(\frac{|K|}{|B_2^N|}\right)^{1/N} \geqslant \binom{N}{k}^{-1/N} \left(\frac{|(rM^*/2)B_2^k||B_2^{N-k}|}{|B_2^N|}\right)^{1/N} \geqslant \frac{1}{2}\left(\frac{rM^*}{2}\right)^{k/N}.$$

Thus if $M^* \geqslant 2/r$ then $2V_K \geqslant (rM^*/2)^{c'M^{*2}}$, which implies $M^{*2} \leqslant 4\ln(2V_K)/(c'\ln(r^2\ln(2V_K)))$. Finally, if $M^* \leqslant 2/r$, then the conclusion follows from the fact that $r > 1/\sqrt{\ln(2V_k)}$. □

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