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Numerical Analysis

External approximation of $H^3(\Omega)$ in a bounded domain of $\mathbb{R}^3$ with piecewise cubics of weak C^2-class

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Abstract

A nonconforming finite element method is introduced to approximate triharmonic boundary value problems in $\mathbb{R}^3$, among other applications. It is constructed upon tetrahedra and piecewise cubic representations. The finite element can be viewed as the primitive of a quadratic one proposed by the first author to solve biharmonic problems, which can be considered in turn as the three-dimensional analogue of the well-known Morley triangle. The new method is proven to be first order convergent in the natural discrete H^3-norm for the problem under consideration. ***To cite this article: V. Ruas et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Approximation externe de $H^3(\Omega)$ dans un domaine borné de $\mathbb{R}^3$ par des fonctions cubiques par morceaux faiblement de classe C^2. On introduit une méthode d'éléments finis non conformes pour résoudre des problèmes aux limites triharmoniques tridimensionnels, entre autres applications. L'élément fini est basé sur des maillages en tétraèdres et des fonctions cubiques par morceaux. Il est une sorte de primitive d'un autre élément fini quadratique proposé par le premier auteur, pour résoudre des problèmes biharmoniques, ce dernier étant à son tour la version tridimensionnelle du très classique triangle de Morley. On établit pour la nouvelle méthode des résultats de convergence au premier ordre dans la norme H^3 discrète, naturelle pour le problème considéré. ***Pour citer cet article : V. Ruas et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***
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Version française abrégée

Soit Ω un domaine borné de $\mathbb{R}^n$ de frontière Γ. L'approximation interne d'espaces de Sobolev $H^m(\Omega)$ pour $m \geqslant 2$ par des fonctions polynomiales par morceaux, lorsque $n > 2$ pose des problèmes de construction algébrique de grande complexité. Ceci conduit naturellement à l'emploi d'approximations externes employant des polynômes d'ordre plus bas dont la qualité est assurée, pourvu que les traces des polynômes aux interfaces des éléments

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de la partition du domaine utilisée possèdent des propriétés de raccord minimales. Or même dans ce cas peu de constructions sont connues pour $n > 2$. Ici on introduit une approximation externe de type éléments finis pour $n = m = 3$, basée sur des maillages en tétraèdres.

Pour ce faire on emploie les notations suivantes associées à un tétraèdre T. Les indices $i,\ j,\ k,\ l$ étant distincts et parcourant l'ensemble $\{1, 2, 3, 4\}$, S_i désigne un sommet de T ; F_i représente la face de T opposée à S_i ; G_i désigne le barycentre de la face F_i ; $\partial^2(\cdot)/\partial n_i^2$ exprime la dérivée seconde normale extérieure à F_i ; s_{ij} représente la longueur de l'arête S_iS_j ; $\vec{\tau}_i^j$ est le vecteur unitaire de F_i orthogonal à l'arête S_kS_l orientée de celle-ci vers S_j ; $\partial(\cdot)/\partial\tau_i^j$ exprime la dérivée première dans la direction de $\vec{\tau}_i^j$.

Maintenant, on définit sur $C^2(T)$ les trois types de fonctionnelles (degrés de liberté) suivants, où p représente une fonction générique de cet espace :

$$\mathcal{F}_i(p) := p(S_i)\ ; \quad \mathcal{F}_i^j(p) := \frac{1}{s_{kl}} \int_{S_k}^{S_l} \frac{\partial p}{\partial \tau_i^j}\, \mathrm{d}s, \quad i, j, k, l \text{ distinct}\ ; \quad \mathcal{F}^j(p) := \frac{\partial^2 p}{\partial n_j^2}(G_j). \tag{1}$$

Désormais on suppose que le domaine Ω est un polyèdre. Ce faisant on considère un maillage de Ω en tetraèdres noté $\mathcal{T}_h$, satisfaisant les conditions de compatibilité habituelles relativement à la méthode des éléments finis. Comme à l'accoûtumé, on suppose aussi que $\mathcal{T}_h$ appartient à une famille régulière $\mathcal{P}$ de maillages en tétraèdres, h désignant la plus grande arête de tous les tétraèdres du maillage.

Les trois types de fonctionnelles définis par (1) engendrent les vingt degrés de liberté du nouvel élément fini, lequel à son tour sert à définir l'espace de fonctions V_h associé à $\mathcal{T}_h$ de la façon suivante.

Soit $W_h = \{v \mid v_{|T} \in P_3,\ \forall T \in \mathcal{T}_h\}$, P_m étant l'espace des polynômes de degré inférieur ou égal à m. On définit l'espace V_h en tant que le sousespace de W_h des fonctions v continues aux sommets des éléments de $\mathcal{T}_h$, telles que les dérivées normales secondes au barycentre de toute face commune à deux tétraèdres de $\mathcal{T}_h$ des restrictions de v à ces éléments sont identiques, et telles que les moyennes simples le long de toute arête commune à des tétraèdres de $\mathcal{T}_h$ des projections du gradient sur le plan orthogonal à cette arête des restrictions de v aux éléments contenant celle-ci coïncident. La possibilité de procéder à une telle construction est démontrée dans la Section 2. En fait elle découle du fait qu'on peut exhiber vingt fonctions de base de l'espace P_3 dans chaque tétraèdre, associées aux fonctionnelles définies plus haut.

Maintenant on introduit la semi-norme $|\cdot|_{3,h}$ de V_h donnée par (6), où $\overrightarrow{\mathrm{grad}}\,\mathcal{H}(f)$ est le tenseur du 3-ème ordre sur $\mathbb{R}^3$ composé des dérivées première du hessien $\mathcal{H}(f)$ d'une fonction f, et l'expression $|\mathcal{D}|$ représente la norme euclidienne standard d'un tenseur $\mathcal{D}$ d'ordre quelconque. $|\cdot|_{3,h}$ s'étend de façon évidente aux fonctions v de $H^3(\Omega)$, auquel cas elle n'est autre que la semi-norme standard de cet espace.

Entre autres applications (cf. [6]) de l'élément fini qu'on vient de définir, on va considérer la résolution du problème triharmonique suivant où, pour $f \in L^2(\Omega)$ donnée, $u \in H_0^3(\Omega)$:

$$-\Delta^3 u = f. \tag{2}$$

Afin d'approcher cette équation, on définit le sous-espace de V_h^0 de V_h des fonctions v telles que les fonctionnelles $\mathcal{F}_i(v_{|T})$, $\mathcal{F}_i^j(v_{|T})$ et $\mathcal{F}^j(v_{|T})$ pour $T \in \mathcal{T}_h$, s'annulent respectivement, en tout sommet appartenant à Γ, sur toute arête située sur Γ et sur toute face contenue dans Γ. Notons que sur V_h^0 $|\cdot|_{3,h}$ est bien une norme, comme on peut démontrer aisément. Ce faisant, pour $a_h(\cdot,\cdot)$ et $L(\cdot)$ définis dans (4), on pose le problème approché de trouver $u_h \in V_h^0$ vérifiant :

$$a_h(u_h, v) = L(v) \quad \forall v \in V_h^0, \tag{3}$$

$$a_h(u, v) := \sum_{T \in \mathcal{T}_h} \int_T \overrightarrow{\mathrm{grad}}\,\mathcal{H}(u_{|T}) \cdot \overrightarrow{\mathrm{grad}}\,\mathcal{H}(v_{|T})\, \mathrm{d}x, \quad \forall u, v \in V_h + H^3(\Omega)\ ; \quad L(v) := \int_\Omega f v\, \mathrm{d}x, \tag{4}$$

où ci-dessus $\mathcal{D} \cdot \mathcal{E}$ désigne le produit scalaire euclidien standard de deux tenseurs $\mathcal{D}$ et $\mathcal{E}$ d'ordre quelconque. Nous avons alors le résultat de convergence suivant : si $u \in H^4(\Omega)$, $\Delta u \in H^3(\Omega)$ et $\Delta^2 u \in H^2(\Omega)$, alors il existe une constante C indépendante de h telle qu'on ait :

$$|u - u_h|_{3,h} \leqslant C\big[h\|u\|_4 + h\|\Delta u\|_3 + h^2\big\|\Delta^2 u\big\|_2\big]. \tag{5}$$

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ and Γ be its boundary. The internal approximation of Sobolev spaces $H^m(\Omega)$ for $m \geqslant 2$ by piecewise polynomial functions whenever $n > 2$ is a matter of great complexity in terms of algebraic constructions. Even in the case where $n = 2$ and $m = 2$ the known constructions are rather elaborate (cf. [2]), let alone the three-dimensional case, where the use of such approximation methods becomes unreasonable. This fact naturally leads to external approximations, that is, to the so-called nonconforming methods when dealing with finite element approximations. In this case the use of lower order polynomials is admissible, as long as some conditions are fulfilled in order to ensure the quality of the approximations. More specifically the traces of the polynomials at element interfaces should have suitable continuity properties. In the three-dimensional case, a nonconforming tetrahedron based quadratic finite element method to approximate biharmonic problems was studied in [5]. Actually it is a nontrivial extension of the classical Morley triangle (cf. [4]). In this Note we establish the existence of a cubic finite element, such that every first order derivative determined elementwise of any function in the underlying space, belongs to the one corresponding to that quadratic finite element.

2. Finite element definition

Let us first introduce some notation associated with a given nondegenerated tetrahedron T. In the following the letters $i,\ j,\ k$ and l denote an integer belonging to the set $\{1, 2, 3, 4\}$. Also whenever i and j appear together in the same expression or notation as either a subscript or a superscript, then $i \neq j$.

- The S_is denote the vertices of T;
- F_i represents the face of T opposite to S_i;
- G_i denotes the barycenter of face F_i;
- $\vec{n}_i$ is the unit outer normal vector to face F_i;
- λ_i is the barycentric coordinate of T corresponding to vertex S_i;
- β_i is the first order derivative of λ_i in the direction of $\vec{n}_i$;
- $\partial^2(\cdot)/\partial n_i^2$ denotes the second order derivative in the direction of $\vec{n}_i$;
- h_i^j represents the height of F_i corresponding to S_j, or yet its length;
- s_{ij} represents the length of edge S_iS_j;
- $\vec{\tau}_i^j$ is the unit vector along h_i^j directed from the edge opposite to S_j towards this vertex;
- $\partial(\cdot)/\partial\tau_i^j$ denotes the first order derivative in the direction of $\vec{\tau}_i^j$;
- $\beta_i^j(k)$ denotes the first order derivative of λ_k in the direction of $\vec{\tau}_i^j$.

Now we define by (1) three types of functionals applied to a generic function $p \in C^2(T)$ yielding the twenty degrees of freedom of the new finite element. Henceforth we assume that Ω is a polyhedron, and we consider a partition $\mathcal{T}_h$ of Ω into tetrahedra, satisfying the usual compatibility conditions for the finite element method. Moreover, we assume that $\mathcal{T}_h$ belongs to a quasiuniform family of partitions $\mathcal{P}$. h denotes the largest edge of all the tetrahedra of $\mathcal{T}_h$.

The new finite element generates a subspace V_h of $W_h = \{v \mid v_{|T} \in P_3,\ \forall T \in \mathcal{T}_h\}$, where P_m is the space of polynomials of degree less than or equal to m. Its definition is as follows:

Definition 2.1. V_h is the subspace of W_h of those functions v, which are continuous at the vertices of the elements of $\mathcal{T}_h$, and such that the second order normal derivative at the barycenter of every common face to two tetrahedra of the partition of the restrictions of v to them are identical, and such that the mean values along every edge of the projection of the gradient onto the plane orthogonal to it of the restrictions of v to the tetrahedra of $\mathcal{T}_h$ containing this edge coincide.

The actual possibility of constructing such space V_h is a consequence of

Proposition 2.2. *Given a nondegenerated tetrahedron T and any set of twenty scalars α_i, α_i^j, α^j, $i, j \in \{1,2,3,4\}$, there exists a unique function $p \in P_3$, such that: $\mathcal{F}_i(p) = \alpha_i$; $\mathcal{F}_i^j(p) = \alpha_i^j$; $\mathcal{F}^j(p) = \alpha^j$.*

Proof. Noticing that in three-dimension space the dimension of P_3 equals twenty, in order to prove this proposition, it suffices to exhibit a set of canonical basis functions f_i, f_i^j and f^j of P_3, for $i, j \in \{1,2,3,4\}$, respectively associated with the functionals $\mathcal{F}_i$, $\mathcal{F}_i^j$ and $\mathcal{F}^j$, for tetrahedron T. Let us then define $f^j := (\lambda_j^2 - \lambda_j^3)/(2\beta_j^2)$, $j = 1,2,3,4$; $f_i^j := \tilde{f}_i^j - \sum_{m=1}^4 \partial^2(\tilde{f}_i^j)/\partial n_m^2(G_m) f^m$, $i, j \in \{1,2,3,4\}$ with $\tilde{f}_i^j := h_i^j[\lambda_i^2 - \lambda_i^3 + \lambda_j - 2\lambda_i\lambda_j + \lambda_j^2(3\lambda_i + 2\lambda_j - 3)]$; $f_i := \lambda_i - \sum_{k=1}^3 \sum_{l=k+1}^4 [\beta_k^l(i) f_k^l + \beta_l^k(i) f_l^k]$, $i = 1,2,3,4$. By straightforward calculations one can check that the following function p fulfills the required condition: $p = \sum_{m=1}^4 [\alpha_m f_m + \alpha^m f^m] + \sum_{m=1}^3 \sum_{n=m+1}^4 [\alpha_m^n f_m^n + \alpha_n^m f_n^m]$. □

Corollary 2.3. *For every $\mathcal{T}_h$ there exists a N_h-dimensional space V_h defined in accordance with Definition* 2.1, *where N_h equals the sum of the number of vertices, the number of faces and twice the number of edges, generated by this partition.*

3. Properties of the finite element space

Lemma 3.1. *Let e be an edge of a given tetrahedron of $\mathcal{T}_h$, and $\tau(e)$ be the subset of this partition consisting of the elements that contain e. For every function $v \in V_h$ the mean value along e of the trace of $\overrightarrow{\mathrm{grad}}(v_{|T})$ on this edge is the same $\forall T \in \tau(e)$.*

Proof. Let T_0 be an element of $\tau(e)$. The vertices of T_0 belonging to e are S_k and S_l for a certain pair of subscripts k and l. Let $\vec{\sigma}_{kl}$ be the unit vector along e, oriented from S_k towards S_l. $(\vec{\sigma}_{kl}, \vec{\tau}_i^j, \vec{\tau}_j^i)$, with i, j, k, l distinct, form a basis of $\mathbb{R}^3$, where $\vec{\tau}_i^j$ and $\vec{\tau}_j^i$ are linked to T_0. Now for an arbitrary $T \in \tau(e)$, let us express $\overrightarrow{\mathrm{grad}}(v_{|T})$ in this basis. Since $\vec{\tau}_i^j$ and $\vec{\tau}_j^i$ span a plane π orthogonal to e, the mean value along e of the projection of $\overrightarrow{\mathrm{grad}}(v_{|T})$ onto π is the same for all $T \in \tau(e)$ by the construction of V_h.

On the other hand we have $\int_{S_k}^{S_l} \overrightarrow{\mathrm{grad}}(v_{|T}) \cdot \vec{\sigma}_{kl}\, \mathrm{d}s = v_{|T}(S_l) - v_{|T}(S_k)$. Since v is continuous at the vertices of $\mathcal{T}_h$, the mean value along e of the projection of $\overrightarrow{\mathrm{grad}}(v_{|T})$ onto e also coincide $\forall T \in \tau(e)$. □

Lemma 3.2. *Let v be an arbitrary function of V_h. For every face F common to two elements of $\mathcal{T}_h$ the Hessian of the restrictions of v to both tetrahedra coincide at the barycenter of F.*

Proof. Let T_1 and T_2 be the tetrahedra of $\mathcal{T}_h$ having F as a common face, and G be the barycenter of F. Denoting by $\vec{n}_F$ the unit vector normal to F directed in a given sense, let $\vec{\tau}_F^1$ and $\vec{\tau}_F^2$ be two unit vectors parallel to the plane of F, such that $(\vec{n}_F, \vec{\tau}_F^1, \vec{\tau}_F^2)$ form a direct orthonormal basis of $\mathbb{R}^3$. In the expressions that follow the notation employed for the partial derivatives in connection with these three directions are self-explanatory.

By construction we have $\partial^2(v_{/T_1})/\partial n_F^2(G) = \partial^2(v_{/T_2})/\partial n_F^2(G)$. Moreover the gradient of v restricted to every tetrahedron of the partition is a quadratic field that possesses the property specified in Lemma 3.1. Therefore we may apply Lemma 1 of [5] to derive $\partial\, \overrightarrow{\mathrm{grad}}(v_{/T_1})/\partial \tau_F^m(G) = \partial\, \overrightarrow{\mathrm{grad}}(v_{/T_2})/\partial \tau_F^m(G)$, $m \in \{1,2\}$. Projecting both sides of this expression successively onto $\vec{n}_F$, $\vec{\tau}_F^1$, $\vec{\tau}_F^2$ the result follows. □

Now we introduce the seminorm $|\cdot|_{3,h}$ of V_h given by,

$$|v|_{3,h} = \sum_{T\in\mathcal{T}_h}\int_T \left|\overrightarrow{\mathrm{grad}}\,\mathcal{H}(v_{|T})\right|^2 \mathrm{d}x, \tag{6}$$

where $\overrightarrow{\mathrm{grad}}\,\mathcal{H}(f)$ is the third order tensor on $\mathbb{R}^3$ whose components are the first order derivatives of the Hessian $\mathcal{H}(f)$ of a function f, and $|\mathcal{D}|$ denotes the standard Euclidean norm of a tensor $\mathcal{D}$ of any order.

Since each term of the summation (6) is the square of the standard seminorm of $H^3(T)$, $|\cdot|_{3,h}$ trivially extends to arbitrary functions v in $H^3(\Omega)$.

Definition 3.3. Subspace V_h^0 of V_h is that consisting of those functions v for which the functionals $\mathcal{F}_i$, $\mathcal{F}_i^j$, $\mathcal{F}^j$ applied to the restriction of v to every element of $\mathcal{T}_h$ intersecting Γ vanish respectively, at every vertex belonging to Γ, for every edge located on Γ, and for every face contained in Γ.

Proposition 3.4. *Seminorm $|v|_{3,h}$ is a norm over V_h^0.*

4. Application to a model problem

In the aim of illustrating a use among others (see, e.g., [6]), of the above finite element space, we apply it to solving problem (2), where $u \in H_0^3(\Omega)$ and $f \in L^2(\Omega)$ is given.

In order to approximate this problem we search for $u_h \in V_h^0$ satisfying (3), where a_h and L are given by (4), $\mathcal{D}\cdot\mathcal{E}$ being the standard Euclidean inner product of two tensors $\mathcal{D}$ and $\mathcal{E}$ of arbitrary order.

Since a_h is obviously a coercive bilinear form over $V_h^0 \times V_h^0$ for the norm $|\cdot|_{3,h}$, the approximate problem (3), (4) has a unique solution $u_h \in V_h^0$. Moreover, according to the improvement due to Dupire [3] of the celebrated Strang's inequality, the following error bound applies:

$$|u-u_h|_{3,h} \leqslant \inf_{v\in V_h^0}|u-v|_{3,h} + \sup_{v\in V_h^0,\ v\neq 0}\frac{a_h(u,v)-L(v)}{|v|_{3,h}}. \tag{7}$$

Let us denote by $|\cdot|_m$ and $\|\cdot\|_m$ the standard seminorm and norm of $H^m(\Omega)$, for $m\in\mathbb{N}$ (see, e.g., [1]).

The two terms on the right-hand side of (7) may be estimated as follows:

Proposition 4.1 (cf. [2]). *If $u\in H^4(\Omega)$ $\exists C_1$ independent of h such that* $\inf_{v\in V_h^0}|u-v|_{3,h}\leqslant C_1 h|u|_4$.

Proposition 4.2. *If $u\in H^4(\Omega)$, $\Delta u\in H^3(\Omega)$ and $\Delta^2 u\in H^2(\Omega)$, $\exists C_2$ independent of h such that,*

$$\sup_{v\in V_h^0,\ v\neq 0}\frac{a_h(u,v)-L(v)}{|v|_{3,h}} \leqslant C_2\left[h\|u\|_4 + h\|\Delta u\|_3 + h^2\left\|\Delta^2 u\right\|_2\right]. \tag{8}$$

Proof. First we expand the expression in the numerator of the fraction on the left-hand side of (8) using reiterately integration by parts. In order to do so we denote by ∂T the boundary of $T\in\mathcal{T}_h$ and by $\vec{n}_T$ the unit outer normal vector to ∂T, and by $\partial_{n_T}(\cdot)$ the first order partial derivative in the direction of $\vec{n}_T$. Then under our assumptions on u that validate the trace properties required in the expressions below (see, e.g., [1]), taking into account (4), we obtain after straightforward calclulations:

$$a_h(u,v)-L(v) = b_h(u,v) - c_h(u,v) + d_h(u,v), \tag{9}$$

where $b_h(u,v) := \sum_{T\in\mathcal{T}_h}[\int_{\partial T}\partial_{n_T}(\mathcal{H}(u))\cdot\mathcal{H}(v_{|T})\,\mathrm{d}S]$; $c_h(u,v) := \sum_{T\in\mathcal{T}_h}[\int_{\partial T}\partial_{n_T}(\overrightarrow{\mathrm{grad}}\,\Delta u)\cdot\overrightarrow{\mathrm{grad}}(v_{|T})\,\mathrm{d}S]$; $d_h(u,v) := \sum_{T\in\mathcal{T}_h}[\int_{\partial T}\partial_{n_T}(\Delta^2 u)v_{|T}\,\mathrm{d}S]$.

From Lemmas 3.1 and 3.2 we may apply the arguments in [5]. Thus $\exists C_3, C_4$ independent of h such that

$$\left|b_h(u,v)\right| \leqslant C_3 h\|u\|_4|v|_{3,h} \quad \text{and} \quad \left|c_h(u,v)\right| \leqslant C_4 h\|\Delta u\|_3|v|_{3,h}. \tag{10}$$

Now in order to properly estimate $d_h(u,v)$, given a tetrahedron $T \in \mathcal{T}_h$, for every face F of ∂T, we introduce a quadratic interpolant of the trace of $v_{|T}$ over F denoted by $\Pi_F^T(v_{|T})$ defined as follows: $\Pi_F^T(v_{|T})(S) = v(S)$ $\forall s$ vertex of F and $\partial \Pi_F^T(v_{|T})/\partial \tau_F^e(M_e) = (1/\text{length}(e)) \int_e \partial(v_{|T})/\partial \tau_F^e \,\mathrm{d}e$ $\forall e$ edge of F, where M_e is the mid-point of e, and the first order partial derivative means the one in the direction of the unit vector of the plane of F orthogonal to e and directed from e outwards F. Notice that the interpolating function $\Pi_F^T(v_{|T})$ is uniquely defined $\forall v \in C^1(T)$, and $\forall F \subset \partial T$ for any nondegenerated tetrahedron T, since the degrees of freedom used to define it are those of the Morley triangle based on F.

Due to the coincidence of the mean values along the edges of F of the first order derivative normal to them on the plane of this face, together with the continuity of v at the vertices of F, for every pair of tetrahedra T and T' of $\mathcal{T}_h$ having F as a common face, we have $\Pi_F^T(v_{|T}) = \Pi_F^{T'}(v_{|T'})$. Furthermore by a similar argument $\Pi_F^T(v_{|T})$ vanishes identically for every F contained in Γ from the construction of V_h^0.

On the other hand the assumption $\Delta^2 u \in H^2(\Omega)$ implies that $\partial_{n_T}(\Delta^2 u) + \partial_{n_{T'}}(\Delta^2 u) = 0$ on every face F of the partition common to two tetrahedra T and T'.

Taking into account both arguments above, we may rewrite $d_h(u,v)$ in the following manner:

$$d_h(u,v) = \sum_{T \in \mathcal{T}_h} \sum_{F \subset \partial T} \int_F \partial_{n_T}\big(\Delta^2 u\big)\big[v_{|T} - \Pi_F^T(v_{|T})\big]\,\mathrm{d}S. \tag{11}$$

Since $\Pi_F^T(v_{|T}) = v_{|T}$ whenever $v_{|T} \in P_2$, we may apply again standard estimates for nonconforming finite elements (cf. [2] and [5]) to $d_h(u,v)$. Thus there exists a constant C_5 independent of h such that

$$\big|d_h(u,v)\big| \leqslant C_5 h^2 \big\|\Delta^2 u\big\|_2 |v|_{3,h}. \tag{12}$$

Finally (8) results from the combination of (9), (10) and (12). □

As a direct consequence of Propositions 4.1 and 4.2, we have the following convergence result:

Theorem 4.3. *If $u \in H^4(\Omega)$, $\Delta u \in H^3(\Omega)$ and $\Delta^2 u \in H^2(\Omega)$, then there exists a constant C independent of h such that* (5) *holds.*

5. Concluding remark

An amazing thing about the cubic tetrahedral finite element studied in this Note, is the fact that it has no two-dimensional analogue. Indeed if this happened to be the case, such triangular element would have no more than one degree of freedom applying to the function itself, as one may easily check. However in this case the trace continuity property necessary to derive estimate (12) becomes out of reach.

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