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Gromov–Witten invariants of noncompact symplectic manifolds

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Abstract

This is a short survey about our Gromov–Witten invariant theory for noncompact geometrically bounded symplectic manifolds. ***To cite this article: G. Lu, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Invariants de Gromov–Witten des variétés symplectiques non compactes. Nous présentons dans cette Note la théorie des invariants des variétés symplectiques non compactes, géométriquement bornées. ***Pour citer cet article : G. Lu, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

1. Introduction

It has been expected that the Gromov–Witten invariants should also be defined for noncompact symplectic manifolds (see, e.g., the remark on the page 337 of [3] by Kontsevich). We here develop the virtual moduli cycle techniques introduced in [4–7] to generalize work of [8] to arbitrary noncompact geometrically bounded symplectic manifolds. A Riemannian metric μ on a manifold M is said to be *geometrically bounded* if its sectional curvature is bounded above and injectivity radius $i(M,\mu)>0$. Denote by $\mathcal{GR}(M)$ the set of all such Riemannian metrics on M. Let $\mathcal{J}(M,\omega)$ be the space of all ω-compatible almost complex structures on a symplectic manifold (M,ω). A symplectic manifold (M,ω) without boundary is said to be *geometrically bounded* if there exists $J\in\mathcal{J}(M,\omega)$, $\mu\in\mathcal{GR}(M)$ and positive constants α_0 and β_0 such that $\omega(X,JX)\geqslant\alpha_0\|X\|_\mu^2$ and $|\omega(X,Y)|\leqslant\beta_0\|X\|_\mu\|Y\|_\mu$ for all $X,Y\in TM$ (cf. [1,2,10]). We shall also say that such a J is (ω,μ)-*geometrically bounded*. Denote by $\mathcal{J}(M,\omega,\mu)$ the set of all (ω,μ)-geometrically bounded almost complex structures in $\mathcal{J}(M,\omega)$. It is a path-connected subset in $\mathcal{J}(M,\omega)$. Denote by $\mathrm{Symp}_0^S(M,\omega)$ the connected component containing id_M of $\mathrm{Symp}_0(M,\omega)$ with respect to the C^∞-strong topology. For $\mathbb{K}=\mathbb{C},\mathbb{R}$ and $\mathbb{Q}$ we shall consider the $\mathbb{K}$-coefficient deRham

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cohomology $H^*(M,\mathbb{K})$ and deRham cohomology $H^*_c(M,\mathbb{K})$ with compact support; $H^*(M,\mathbb{Q})$ (resp. $H^*_c(M,\mathbb{Q})$) consists of all deRham cohomology classes in $H^*(M,\mathbb{R})$ (resp. $H^*_c(M,\mathbb{R})$) which take rational values over all integral cycles.

2. Gromov–Witten invariants

Let (M,ω,J,μ) be a geometrically bounded symplectic manifold of dimension $2n$, $A\in H_2(M,\mathbb{Z})$ and integers $g\geqslant 0$, $m>0$ with $2g+m\geqslant 3$. Let $\overline{\mathcal{M}}_{g,m}$ be the set of all isomorphism classes of stable curves with m marked points and of genus of g, $\kappa\in H_*(\overline{\mathcal{M}}_{g,m},\mathbb{Q})$ and $\{\alpha_i\}_{1\leqslant i\leqslant m}\subset H^*(M,\mathbb{Q})\cup H^*_c(M,\mathbb{Q})$ satisfy

$$\sum_{i=1}^m \deg\alpha_i+\mathrm{codim}(\kappa)=2c_1(M)(A)+2(3-n)(g-1)+2m. \tag{1}$$

Let $\overline{\mathcal{M}}_{g,m}(M,J,A)$ denote the set of equivalence classes of all m-pointed stable J-maps of genus g and of class $A\in H_2(M,\mathbb{Z})$ in M. It was observed by Gromov in his celebrated paper [2] that the 'size' of the closed J-holomorphic curve can be controlled in this class of symplectic manifolds. So for any compact subset $K\subset M$ the images of all maps in $\overline{\mathcal{M}}_{g,m}(M,J,A;K):=\{[\mathbf{f}]\in\overline{\mathcal{M}}_{g,m}(M,J,A)\mid f(\Sigma)\cap K\neq\emptyset\}$ may be contained in $c(\alpha_0,\beta_0,\mu)\omega(A)$-neighborhood of K in M for some constant $c(\alpha_0,\beta_0,\mu)>0$. It follows that $\overline{\mathcal{M}}_{g,m}(M,J,A;K)$ is compact.

Suppose that $\{\alpha_i\}_{1\leqslant i\leqslant m}\subset H^*_c(M,\mathbb{Q})\cup H^*(M,\mathbb{Q})$ has at least one element, say α_1, belonging to $H^*_c(M,\mathbb{Q})$. We may choose their closed representative forms α_i^*, $i=1,\ldots,m$, and a compact subset K_0 in M such that $\mathrm{supp}(\bigwedge_{i=1}^m\alpha_i^*)\subset K_0$. From $\overline{\mathcal{M}}_{g,m}(M,J,A;K_0)$ we can use the methods developed in [5–7] to construct a family of cobordant virtual moduli cycles

$$\mathcal{C}^{\mathbf{t}}(K_0):=\sum_{I\in\mathcal{N}}\frac{1}{|\Gamma_I|}\bigl\{\hat\pi_I:\mathcal{M}_I^{\mathbf{t}}(K_0)\to\mathcal{W}\bigr\}\quad\forall\mathbf{t}\in\mathbf{B}^{\mathrm{res}}_\varepsilon\bigl(\mathbb{R}^{m_{n_3}}\bigr).$$

Let $\mathrm{ev}_i([f,\Sigma,\bar{\mathbf{z}}])=f(z_i)$, $i=1,\ldots,m$, and $\Pi_{g,m}([f,\Sigma,\bar{\mathbf{z}}])=[\Sigma',\bar{\mathbf{z}}']$ be obtained by collapsing components of $(\Sigma,\bar{\mathbf{z}})$ with genus 0 and at most two special points. Using the map $\mathrm{EV}_{g,m}:=\Pi_{g,m}\times(\prod_{i=1}^m\mathrm{ev}_i):\mathcal{B}^M_{A,g,m}\to\overline{\mathcal{M}}_{g,m}\times M^m$, we define the GW-invariants as

$$\begin{aligned}\mathcal{GW}^{(\omega,\mu,J)}_{A,g,m}(\kappa;\alpha_1,\ldots,\alpha_m)&:=\int_{\mathrm{EV}_{g,m}\circ\mathcal{C}^{\mathbf{t}}(K_0)}\kappa^*\oplus\bigwedge_{i=1}^m\alpha_i^*\\&=\sum_{I\in\mathcal{N}}\frac{1}{|\Gamma_I|}\int_{\mathcal{M}_I^{\mathbf{t}}(K_0)}(\mathrm{EV}_{g,m}\circ\hat\pi_I)^*\Bigl(\kappa^*\oplus\bigwedge_{i=1}^m\alpha_i^*\Bigr),\end{aligned}\tag{2}$$

if (1) is satisfied, and $\mathcal{GW}^{(\omega,\mu,J)}_{A,g,m}(\kappa;\alpha_1,\ldots,\alpha_m)=0$ otherwise. They are well-defined. That is, the left of (2) does not depend on all related choices (see §4.2–§4.6 of [9]). As expected they are multilinear and supersymmetric on $\alpha_1,\ldots,\alpha_m$, and also independent of choices of $J\in\mathcal{J}(M,\omega,\mu)$. Moreover, they only depend on the connected component of μ in $\mathcal{GR}(M)$ with respect to the C^∞ strong topology. (In fact it was proved in [9] that they are invariant under the *weak deformation* of (M,ω,J,μ).) For any $\psi\in\mathrm{Symp}^S_0(M,\omega)$ the following holds

$$\mathcal{GW}^{(\omega,\psi^*\mu,\psi^*J)}_{A,g,m}(\kappa;\alpha_1,\ldots,\alpha_m)=\mathcal{G}W^{(\omega,\mu,J)}_{A,g,m}(\kappa;\alpha_1,\ldots,\alpha_m). \tag{3}$$

Let $\mathcal{F}_m : \overline{\mathcal{M}}_{g,m} \to \overline{\mathcal{M}}_{g,m-1}$ be a map, forgetting the last marked point. It is a Lefschetz fibration and the integration along the fibre induces a map $(\mathcal{F}_m)_\sharp$ from $\Omega^*(\overline{\mathcal{M}}_{g,m})$ to $\Omega^{*-2}(\overline{\mathcal{M}}_{g,m-1})$. It also induces a 'shriek' map $(\mathcal{F}_m)_!$ from $H_*(\overline{\mathcal{M}}_{g,m-1}; \mathbb{Q})$ to $H_{*+2}(\overline{\mathcal{M}}_{g,m}; \mathbb{Q})$.

Theorem 2.1 (Reduction formulas). *If $(g, m) \neq (0, 3), (1, 1)$, then for any $\kappa \in H_*(\overline{\mathcal{M}}_{g,m-1}; \mathbb{Q})$, $\alpha_1 \in H_c^*(M; \mathbb{Q})$, $\alpha_2, \ldots, \alpha_m \in H^*(M; \mathbb{Q})$ with $\deg \alpha_m = 2$ one has*

$$\mathcal{GW}_{A,g,m}^{(\omega,\mu,J)}\big((\mathcal{F}_m)_!(\kappa); \alpha_1, \ldots, \alpha_m\big) = \alpha_m(A) \cdot \mathcal{GW}_{A,g,m-1}^{(\omega,\mu,J)}(\kappa; \alpha_1, \ldots, \alpha_{m-1}),$$
$$\mathcal{GW}_{A,g,m}^{(\omega,\mu,J)}(\kappa; \alpha_1, \ldots, \alpha_{m-1}, \mathbf{1}) = \mathcal{GW}_{A,g,m}^{(\omega,\mu,J)}\big((\mathcal{F}_m)_*(\kappa); \alpha_1, \ldots, \alpha_{m-1}\big).$$

Here $\mathbf{1} \in H^0(M, \mathbb{Q})$ denotes the unit element $H^*(M, \mathbb{Q})$, which is Poincaré dual to the fundamental class $[M]$ in the second singular homology $H_{2n}^{\mathrm{II}}(M, \mathbb{Q})$.

Let integers $g_i \geqslant 0$ and $m_i > 0$ satisfy: $2g_i + m_i \geqslant 3$, $i = 1, 2$. Set $g = g_1 + g_2$ and $m = m_1 + m_2$ and fix a decomposition $Q = Q_1 \cup Q_2$ of $\{1, \ldots, m\}$ with $|Q_i| = m_i$. Then one gets a canonical embedding $\varphi_Q : \overline{\mathcal{M}}_{g_1,m_1+1} \times \overline{\mathcal{M}}_{g_2,m_2+1} \to \overline{\mathcal{M}}_{g,m}$. Let $\psi : \overline{\mathcal{M}}_{g-1,m+2} \to \overline{\mathcal{M}}_{g,m}$ be the natural embedding obtained by gluing together the last two marked points. In the case $\dim H^*(M) < \infty$ we take a basis $\{\beta_i\}$ of $H^*(M)$ and a dual basis $\{\omega_i\}$ of them in $H_c^*(M)$, i.e., $\langle \omega_j, \beta_i \rangle = \int_M \beta_i \wedge \omega_j = \delta_{ij}$. Let $\eta^{ij} = \int_M \omega_i \wedge \omega_j$ and $c_{ij} = (-1)^{\deg \omega_i \cdot \deg \omega_j} \eta^{ij}$.

Theorem 2.2 (Composition laws). *Assume that $\dim H^*(M) < \infty$. Let $\kappa \in H_*(\overline{\mathcal{M}}_{g-1,m+2}, \mathbb{Q})$, and $\alpha_i \in H^*(M, \mathbb{Q})$, $i = 1, \ldots, m$. Suppose that some $\alpha_t \in H_c^*(M, \mathbb{Q})$. Then*

$$\mathcal{GW}_{A,g,m}^{(\omega,\mu,J)}\big(\psi_*(\kappa); \alpha_1, \ldots, \alpha_m\big) = \sum_{i,j} c_{ij} \cdot \mathcal{GW}_{A,g-1,m+2}^{(\omega,\mu,J)}(\kappa; \alpha_1, \ldots, \alpha_m, \beta_i, \beta_j).$$

Moreover, let $\kappa_i \in H_*(\overline{\mathcal{M}}_{g_i,m_i}, \mathbb{Q})$, $i = 1, 2$, and $\alpha_s, \alpha_t \in H_c^*(M, \mathbb{Q})$ for some $s \in Q_1$ and $t \in Q_2$. Then

$$\mathcal{GW}_{A,g,m}^{(\omega,\mu,J)}\big(\varphi_{Q*}(\kappa_1 \times \kappa_2); \alpha_1, \ldots, \alpha_m\big) = \epsilon(Q)(-1)^{\deg \kappa_2 \sum_{i \in Q_1} \deg \alpha_i}$$
$$\times \sum_{A = A_1 + A_2} \sum_{k,l} \eta^{kl} \cdot \mathcal{GW}_{A_1,g_1,m_1+1}^{(\omega,\mu,J)}\big(\kappa_1; \{\alpha_i\}_{i \in Q_1}, \beta_k\big) \cdot \mathcal{GW}_{A_2,g_2,m_2+1}^{(\omega,\mu,J)}(\kappa_2; \beta_l, \{\alpha_i\}_{i \in Q_2}).$$

Here $\epsilon(Q)$ is the sign of the permutation $Q = Q_1 \cup Q_2$ of $\{1, \ldots, m\}$.

For proofs of (3) and Theorems 2.1, 2.2 the readers may refer to [9]. If (M, ω) is a closed symplectic manifold they are reduced to the ordinary ones.

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