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Calculus of Variations

Homogenization of nonlinear integrals via the periodic unfolding method

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Abstract

We consider the periodic homogenization of nonlinear integral energies with polynomial growth. The study is carried out by the periodic unfolding method which reduces the homogenization process to a weak convergence problem in a Lebesgue space. ***To cite this article: D. Cioranescu et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Homogénéisation non linéaire par éclatement périodique. Cette Note présente un résultat d'homogénéisation périodique pour des énergies intégrales à croissance polynômiale. On utilise la méthode d'éclatement périodique qui réduit la démonstration à de la convergence faible dans un espace de Lebesgue. ***Pour citer cet article : D. Cioranescu et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Soit $\mathcal{A}_0$ l'ensemble des ouverts liptschitziens de $\mathbb{R}^n$ et Y la cellule de référence $]0,1[^n$. On considère des densités d'énergies f de Carathéodory satisfaisant (H_0) et (H_1) pour $p \in]1,+\infty[$. Le résultat principal de cette Note est le Théorème 1 qui établit la convergence variationnelle pour $\varepsilon_h \to 0$, pour des énergies intégrales du type $u \mapsto \int_\Omega f(\frac{x}{\varepsilon_h}, \nabla u(x))\,\mathrm{d}x$. Si de plus, f vérifie (H_2), la valeur limite est $\int_\Omega f^p_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x$, où f^p_{hom} est définie par (2).

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Ce résultat avait été démontré par une méthode abstraite de Γ-convergence dans [14] et [7]. Nous donnons ici une démonstration par une méthode directe et élémentaire reposant sur la méthode d'éclatement périodique de [10]. Comme dans le cas linéaire, on est conduit à étudier la semi-continuité inférieure faible d'une fonctionnelle intégrale dans l'espace $L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y))$. La preuve de (1) est contenue dans les Lemmes 3.1 et 3.2. La preuve de la seconde partie du résultat, utilise le Théorème 4 de Castaing [9], et est donné par le Lemme 3.3.

1. Introduction

The homogenization of periodic structures has been carried out in the last thirty years for various kinds of problems involving differential equations as well as integral energies. Starting from the basic works [13,4,15], several methods have been developed to approach the analytic study of the asymptotic behaviour of such structures, which generated a wide bibliography, cf., for example, [3,5,6,11,12] and references therein. In the 1980s, a two-scale convergence technique was introduced in [16] then applied in [1]. In [2] a "dilation" operation was defined for the homogenization of periodic media with double porosity. These two techniques were used in [8] to study perforated domains and thin structures. The periodic unfolding method, presented in [10] for the homogenization of multi-scale periodic problems, combines the dilation technique with ideas from Finite Elements approximations. This approach reduces two-scale convergence to a mere weak convergence in an appropriate space. In particular, when applied to linear problems, this method greatly simplifies the homogenization by reducing it to a weak convergence in a L^2 space.

In the present Note we consider the homogenization problem in the general case of nonlinear integral energies with polynomial growth. Here again, the use of the periodic unfolding method simplifies the homogenization process by reducing it to a weak convergence problem in an L^p space.

Denote by $\mathcal{A}_0$ the class of bounded open subsets of $\mathbb{R}^n$ having a Lipschitz boundary, and by Y the unit reference cell $]0,1[^n$. Consider a Carathéodory energy density f satisfying

$$(\mathrm{H}_0)\qquad \begin{cases} f\colon\ (x,z)\in\mathbb{R}^n\times\mathbb{R}^n\mapsto f(x,z)\in[0,+\infty[,\\ f(\cdot,z)\ \text{Lebesgue measurable},\ Y\text{-periodic}\ \forall z\in\mathbb{R}^n;\ f(x,\cdot)\ \text{convex for a.e. } x\in\mathbb{R}^n.\end{cases}$$

For $p\in[1,+\infty[$, and $M>0$, consider the following two growth conditions:

$$(\mathrm{H}_1)\qquad f(x,z)\leqslant M(1+|z|^p)\ \text{for a.e. } x\in\mathbb{R}^n,\ \text{and every } z\in\mathbb{R}^n,$$
$$(\mathrm{H}_2)\qquad |z|^p\leqslant f(x,z)\ \text{for a.e. } x\in\mathbb{R}^n,\ \text{and every } z\in\mathbb{R}^n.$$

The main result, the proof of which is provided in the next sections, is the following:

Theorem 1.1. *Let f satisfy* (H_0)*, and assume that* (H_1) *holds for some $p\in\,]1,+\infty[$. Let Ω be in $\mathcal{A}_0$, and let $\{\varepsilon_h\}\subseteq\,]0,+\infty[$ converge to* 0*. Then,*

$$\begin{aligned}&\inf\left\{\liminf_{h\to+\infty}\int_\Omega f\left(\frac{x}{\varepsilon_h},\nabla u_h(x)\right)\mathrm{d}x\colon\ \{u_h\}\subseteq W^{1,p}(\Omega),\ u_h\rightharpoonup u\ \text{in}\ W^{1,p}(\Omega)\right\}\\ &=\inf\left\{\limsup_{h\to+\infty}\int_\Omega f\left(\frac{x}{\varepsilon_h},\nabla u_h(x)\right)\mathrm{d}x\colon\ \{u_h\}\subseteq W^{1,p}(\Omega),\ u_h\rightharpoonup u\ \text{in}\ W^{1,p}(\Omega)\right\}\\ &=\inf\left\{\int_{\Omega\times Y} f\big(y,\nabla u(x)+\nabla_y V(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y\colon\ V\in L^p\big(\Omega;W^{1,p}_{\mathrm{per}}(Y)\big)\right\}.\end{aligned}\tag{1}$$

If in addition, (H_2) *holds with the same* p*, this common value equals* $\int_\Omega f^p_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x$*, where* f^p_{hom} *is the homogenized energy density defined by*

$$f^p_{\mathrm{hom}}: z \in \mathbb{R}^n \mapsto \inf\left\{ \int_Y f\big(y, z + \nabla v(y)\big)\,\mathrm{d}y \colon\ v \in W^{1,p}_{\mathrm{per}}(Y) \right\}, \tag{2}$$

and $W^{1,p}_{\mathrm{per}}(Y)$ *is the Banach space of* Y*-periodic functions in* $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ *with the* $W^{1,p}(Y)$*-norm.*

The original proof (which is much more elaborate) of this theorem can be found in [14] and [7], where an abstract Γ-convergence method is used. Our approach here turns out to be elementary.

2. Preliminary results

2.1. *The unfolding operator, and its main properties* (*cf.* [10])

For every $x \in \mathbb{R}^n$, $[x]$ denotes the vector whose coordinates are the integer parts of the corresponding coordinates of x.

Definition 2.1. Let $\Omega \in \mathcal{A}_0$. For $\varepsilon > 0$, the unfolding operator $\mathcal{T}_\varepsilon : L^1(\Omega) \to L^1(\mathbb{R}^n \times Y)$ is defined as

$$\mathcal{T}_\varepsilon(v)(x,y) = \tilde{v}\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) \quad \text{for every } v \in L^1(\Omega), \text{ and a.e. } (x,y) \in \mathbb{R}^n \times Y,$$

where $\tilde{v}$ is the extension of v by zero outside Ω.

Define Ω_ε as the union of all sets $\varepsilon(\xi + \overline{Y})$, as $\xi \in \mathbb{Z}^n$, and $\varepsilon(\xi + \overline{Y}) \cap \overline{\Omega} \neq \emptyset$. Then one has

$$\int_{\Omega_\varepsilon \times Y} \mathcal{T}_\varepsilon(v)(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega v(x)\,\mathrm{d}x \quad \text{for every } \varepsilon > 0, \text{ and } v \in L^1(\Omega). \tag{3}$$

In particular, $\|\mathcal{T}_\varepsilon(v)\|_{L^p(\Omega_\varepsilon \times Y)} = \|v\|_{L^p(\Omega)}$ for every p in $[1, +\infty]$, $\varepsilon > 0$, and v in $L^p(\Omega)$, so that, the unfolding operator restricted to $L^p(\Omega)$ is continuous from $L^p(\Omega)$ to $L^p(\Omega \times Y)$. Moreover,

$$\mathcal{T}_\varepsilon(v) \to \tilde{v} \quad \text{in } L^p\big(\mathbb{R}^n \times Y\big) \text{ as } \varepsilon \to 0 \text{ for every } p \in [1, +\infty[, \text{ and } v \in L^p(\Omega). \tag{4}$$

The main tool in periodic unfolding is

Proposition 2.2. *Let* $\Omega \in \mathcal{A}_0$*,* $p \in]1, +\infty[$*. Let* $\{v_\varepsilon\}$ *be a sequence converging weakly in* $W^{1,p}(\Omega)$ *to some* v *as* $\varepsilon \to 0$*. Then, there exist a subsequence* $\{\varepsilon_k\}$*, and* $V \in L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y))$ *such that,*

$$\mathcal{T}_{\varepsilon_k}(\nabla v_{\varepsilon_k}) \rightharpoonup \nabla v + \nabla_y V \quad \text{in } \big(L^p(\Omega \times Y)\big)^n.$$

2.2. *Castaing's theorem on measurable selections*

Let Ω, X be sets, and Γ a multifunction from Ω to X. A function $\sigma : \Omega \to X$ will be said to be a selection of Γ if $\sigma(x) \in \Gamma(x)$ for every $x \in \Omega$. The measurable selection result below is proved in [9, Theorem III.6 and Proposition III.11].

Theorem 2.3. *Let* X *be a separable metric space,* $(\Omega, \mathcal{M})$ *a measurable space, and* Γ *a multifunction from* Ω *to* X*. Assume that for every* $x \in \Omega$*,* $\Gamma(x)$ *is nonempty and complete in* X*. Assume moreover, that for every closed subset* F *of* X*,* $\{x \in \Omega \colon\ \Gamma(x) \cap F \neq \emptyset\}$ *belongs to* $\mathcal{M}$*. Then* Γ *admits a* $\mathcal{M}$*-measurable selection.*

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is contained in the following three lemmas:

Lemma 3.1. *Assume that f satisfies* (H_0). *Let $\Omega \in \mathcal{A}_0$, $p \in]1,+\infty[$, and $\{\varepsilon_h\} \subseteq]0,+\infty[$ converge to* 0 *as $h \to +\infty$. Then, for every $u \in W^{1,p}(\Omega)$,*

$$\inf\left\{\liminf_{h\to+\infty}\int_\Omega f\left(\frac{x}{\varepsilon_h}, \nabla u_h(x)\right)\mathrm{d}x\colon \{u_h\} \subseteq W^{1,p}(\Omega),\ u_h \rightharpoonup u \text{ in } W^{1,p}(\Omega)\right\}$$
$$\geqslant \inf\left\{\int_{\Omega\times Y} f\big(y, \nabla u(x) + \nabla_y V(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y\colon V \in L^p\big(\Omega; W^{1,p}_{\mathrm{per}}(Y)\big)\right\}.$$

Proof. For $u \in W^{1,p}(\Omega)$, let $\{u_h\}$ converge weakly to u in $W^{1,p}(\Omega)$. Assume for simplicity, that $\lim_{h\to+\infty}\int_\Omega f(\frac{x}{\varepsilon_h}, \nabla u_h(x))\,\mathrm{d}x$ exists and is finite. Then, Proposition 2.2 implies that there exists $\{\varepsilon_{h_k}\} \subseteq \{\varepsilon_h\}$ and $U \in L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y))$ with

$$\mathcal{T}_{\varepsilon_{h_k}}(\nabla u_{h_k}) \rightharpoonup \nabla u + \nabla_y U \quad \text{in } \big(L^p(\Omega\times Y)\big)^n. \tag{5}$$

According to (3), we have $\int_\Omega f(\frac{x}{\varepsilon_{h_k}}, \nabla u_{h_k}(x))\,\mathrm{d}x = \int_{\Omega_\varepsilon\times Y} f(y, \mathcal{T}_{\varepsilon_{h_k}}(\nabla u_{h_k})(x,y))\,\mathrm{d}x\,\mathrm{d}y \geqslant \int_{\Omega\times Y} f(y, \mathcal{T}_{\varepsilon_{h_k}}(\nabla u_{h_k})(x,y))\,\mathrm{d}x\,\mathrm{d}y$, where the last functional is sequentially weakly $(L^p(\Omega))^n$-lower semicontinuous (a well-known consequence of Fatou's Lemma under hypothesis (H_0)). Therefore, using (5), we get the following inequality, from which the lemma follows:

$$\liminf_{h\to+\infty}\int_\Omega f\left(\frac{x}{\varepsilon_h}, \nabla u_h(x)\right)\mathrm{d}x \geqslant \int_{\Omega\times Y} f\big(y, \nabla u(x) + \nabla_y U(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y$$
$$\geqslant \inf\left\{\int_{\Omega\times Y} f\big(y, \nabla u(x) + \nabla_y V(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y\colon V \in L^p\big(\Omega; W^{1,p}_{\mathrm{per}}(Y)\big)\right\}. \quad \square$$

Lemma 3.2. *Assume that f satisfies* (H_0), *and that* (H_1) *holds for some $p \in]1,+\infty[$. Let $\Omega \in \mathcal{A}_0$, and let $\{\varepsilon_h\} \subseteq]0,+\infty[$ converge to* 0 *as $h \to +\infty$. Then, for every $u \in W^{1,p}(\Omega)$,*

$$\inf\left\{\limsup_{h\to+\infty}\int_\Omega f\left(\frac{x}{\varepsilon_h}, \nabla u_h(x)\right)\mathrm{d}x\colon \{u_h\} \subseteq W^{1,p}(\Omega),\ u_h \rightharpoonup u \text{ in } W^{1,p}(\Omega)\right\}$$
$$\leqslant \inf\left\{\int_{\Omega\times Y} f\big(y, \nabla u(x) + \nabla_y V(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y\colon V \in L^p\big(\Omega; W^{1,p}_{\mathrm{per}}(Y)\big)\right\}.$$

Proof. Let $u \in W^{1,p}(\Omega)$ and $U \in C^1(\mathbb{R}^n \times \mathbb{R}^n)$ with $U(x,\cdot)$ Y-periodic for every $x \in \Omega$. For every $h \in \mathbb{N}$ and $x \in \Omega$, we set $u_h(x) = u(x) + \varepsilon_h U(x, \frac{x}{\varepsilon_h})$. Clearly $\nabla u_h(x) = \nabla u(x) + \varepsilon_h \nabla_x U(x, \frac{x}{\varepsilon_h}) + \nabla_y U(x, \frac{x}{\varepsilon_h})$ for every $h \in \mathbb{N}$, $x \in \Omega$.

From the periodicity of U, one has

$$\int_\Omega f\left(\frac{x}{\varepsilon_h}, \nabla u_h(x)\right)\mathrm{d}x = \int_{\Omega_{\varepsilon_h}\times Y} f\big(y, \mathcal{T}_{\varepsilon_h}(\nabla u_h)(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{\Omega_{\varepsilon_h}\times Y} f\left(y, \mathcal{T}_{\varepsilon_h}(\nabla u)(x,y) + \varepsilon_h \nabla_x U\left(\varepsilon_h\left[\frac{x}{\varepsilon_h}\right] + \varepsilon_h y, y\right) + \nabla_y U\left(\varepsilon_h\left[\frac{x}{\varepsilon_h}\right] + \varepsilon_h y, y\right)\right) \mathrm{d}x\,\mathrm{d}y.$$

Due to continuity properties of $\nabla_x U$ and to the periodicity of $\nabla_y U$, we have $\varepsilon_h \nabla_x U(\cdot, \frac{\cdot}{\varepsilon_h}) \to 0$ uniformly in Ω, and $\nabla_y U(\cdot, \frac{\cdot}{\varepsilon_h}) \to \int_Y \nabla_y U(\cdot, y)\,\mathrm{d}y = 0$ weakly* in $(L^\infty(\Omega))^n$. This implies that $u_h \rightharpoonup u$ in $W^{1,p}(\Omega)$. Moreover,

$$\inf\left\{\limsup_{h\to+\infty} \int_\Omega f\left(\frac{x}{\varepsilon_h}, \nabla v_h(x)\right)\mathrm{d}x\colon \{v_h\} \subseteq W^{1,p}(\Omega),\ v_h \rightharpoonup u \text{ in } W^{1,p}(\Omega)\right\}$$
$$\leqslant \limsup_{h\to+\infty} \int_{\Omega_{\varepsilon_h}\times Y} f\left(y, \mathcal{T}_{\varepsilon_h}(\nabla u)(x,y) + \varepsilon_h \nabla_x U\left(\varepsilon_h\left[\frac{x}{\varepsilon_h}\right] + \varepsilon_h y, y\right) + \nabla_y U\left(\varepsilon_h\left[\frac{x}{\varepsilon_h}\right] + \varepsilon_h y, y\right)\right) \mathrm{d}x\,\mathrm{d}y.$$

On the other hand, again by the continuity properties of $\nabla_x U$ and of $\nabla_y U$, one has $\varepsilon_h \nabla_x U(\varepsilon_h[\frac{\cdot}{\varepsilon_h}] + \varepsilon_h \cdot, \cdot) \to 0$ and $\nabla_y U(\varepsilon_h[\frac{\cdot}{\varepsilon_h}] + \varepsilon_h \cdot, \cdot) \to \nabla_y U$ uniformly in Ω. Note that that by (4), $\mathcal{T}_{\varepsilon_h}(\nabla u) \to \widetilde{\nabla u}$ in $(L^p(\mathbb{R}^n \times Y))^n$. This, together with ($H_1$), yields $\lim_{h\to+\infty} \int_{\Omega_{\varepsilon_h}\times Y} f(y, \mathcal{T}_{\varepsilon_h}(\nabla u)(x,y) + \varepsilon_h \nabla_x U(\varepsilon_h[\frac{x}{\varepsilon_h}] + \varepsilon_h y, y) + \nabla_y U(\varepsilon_h[\frac{x}{\varepsilon_h}] + \varepsilon_h y, y))\,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega\times Y} f(y, \nabla u(x) + \nabla_y U(x,y))\,\mathrm{d}x\,\mathrm{d}y$, since $\partial\Omega$ is a null set. Consequently, for every U in $C^1(\mathbb{R}^n \times \mathbb{R}^n)$ with $U(x,\cdot)$ Y-periodic for every $x \in \Omega$, we have

$$\inf\left\{\limsup_{h\to+\infty} \int_\Omega f\left(\frac{x}{\varepsilon_h}, \nabla v_h(x)\right)\mathrm{d}x\colon \{v_h\} \subseteq W^{1,p}(\Omega),\ v_h \rightharpoonup u \text{ in } W^{1,p}(\Omega)\right\}$$
$$\leqslant \int_{\Omega\times Y} f\big(y, \nabla u(x) + \nabla_y U(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y. \tag{6}$$

The set of functions U in $C^1(\mathbb{R}^n \times \mathbb{R}^n)$, with $U(x,\cdot)$ Y-periodic for every $x \in \Omega$, is dense in $L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y))$. We conclude the proof by observing that (H_0) and (H_1) imply the continuity on $L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y))$ of the right-hand side of (6). □

Lemma 3.3. *Assume that f satisfies (H_0), and that (H_1) and (H_2) hold for some $p \in]1, +\infty[$. Let $\Omega \in \mathcal{A}_0$. Then, for every $u \in W^{1,p}(\Omega)$, one has*

$$\inf\left\{\int_{\Omega\times Y} f\big(y, \nabla u(x) + \nabla_y V(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y\colon V \in L^p\big(\Omega; W^{1,p}_{\mathrm{per}}(Y)\big)\right\} = \int_\Omega f^p_{\mathrm{hom}}\big(\nabla u(x)\big)\,\mathrm{d}x.$$

Proof. One inequality is straightforward from definition (2), since for u in $W^{1,p}(\Omega)$ and V in $L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y))$, the following inequality holds for a.e. $x \in \Omega$, $\int_Y f(y, \nabla u(x) + \nabla_y V(x,y))\,\mathrm{d}y \geqslant \inf\{\int_Y f(y, \nabla u(x) + \nabla v(y))\,\mathrm{d}y\colon v \in W^{1,p}_{\mathrm{per}}(Y)\} = f^p_{\mathrm{hom}}(\nabla u(x))$.

The reverse inequality is obvious if $\int_\Omega f^p_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x = +\infty$. To prove it in the case where $\int_\Omega f^p_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x < +\infty$, we make use of Castaing's selection theorem. Note first, that due to (H_0) and (H_1), f^p_{hom} is convex and continuous on $\mathbb{R}^n$. Due to (H_2) and the Poincaré–Wirtinger inequality, the infimum defining $f^p_{\mathrm{hom}}(z)$ in (2), is achieved for every $z \in \mathbb{R}^n$. This and (H_1) imply that for $z \in \mathbb{R}^n$, $\Gamma(z)$ is nonempty, and strongly closed, where Γ is the multifunction defined by

$$\Gamma : z \in \mathbb{R}^n \mapsto \left\{v \in W^{1,p}_{\mathrm{per}}(Y)\colon \int_Y v(y)\,\mathrm{d}y = 0,\ \int_Y f\big(y, z + \nabla v(y)\big)\,\mathrm{d}y = f^p_{\mathrm{hom}}(z)\right\}. \tag{7}$$

We now claim that Γ has a $\mathcal{B}(\mathbb{R}^n)$-measurable selection, where $\mathcal{B}(\mathbb{R}^n)$ denotes the Borel σ-algebra of $\mathbb{R}^n$. By Theorem 2.3, it is enough to show that for every strongly closed subset F of $W^{1,p}_{\mathrm{per}}(Y)$, $\Gamma^-(F) \doteq \{\zeta \in \mathbb{R}^n : \Gamma(\zeta) \cap F \neq \emptyset\}$ belongs to $B(\mathbb{R}^n)$.

Assume first that F is a closed ball in $W^{1,p}_{\mathrm{per}}(Y)$. We prove that $\Gamma^-(F)$ is closed. To do so, let $\{z_h\} \subseteq \Gamma^-(F)$, $z \in \mathbb{R}^n$ with $z_h \to z$. For every $h \in \mathbb{N}$, let $v_h \in \Gamma(z_h) \cap F$. The continuity and the finiteness of f^p_{hom} provide that $\limsup_{h\to+\infty} \int_Y |z_h + \nabla v_h(y)|^p \,\mathrm{d}y \leqslant \lim_{h\to+\infty} \int_Y f(y, z_h + \nabla v_h(y))\,\mathrm{d}y = \lim_{h\to+\infty} f^p_{\mathrm{hom}}(z_h) = f^p_{\mathrm{hom}}(z) < +\infty$. Hence there is a subsequence $\{v_{h_k}\}$ of $\{v_h\}$ and some v_∞ in F such that $v_{h_k} \rightharpoonup v_\infty$ in $W^{1,p}(Y)$. Moreover, $\int_Y v_\infty(y)\,\mathrm{d}y = 0$. From the continuity of f^p_{hom} and the weak $W^{1,p}(Y)$-lower semicontinuity of $w \mapsto \int_Y f(y, \nabla w(y))\,\mathrm{d}y$, we get $f^p_{\mathrm{hom}}(z) \leqslant \int_Y f(y, z + \nabla v_\infty(y))\,\mathrm{d}y \leqslant \liminf_{k\to+\infty} \int_Y f(y, z_{h_k} + \nabla v_{h_k}(y))\,\mathrm{d}y = \lim_{k\to+\infty} f^p_{\mathrm{hom}}(z_{h_k}) = f^p_{\mathrm{hom}}(z)$. Thus v_∞ belongs to $\Gamma(z) \cap F$ and z to $\Gamma^-(F)$.

Suppose now that F is a strongly closed subset of $W^{1,p}_{\mathrm{per}}(Y)$. Since $W^{1,p}_{\mathrm{per}}(Y)$ is separable, F can be written as a countable intersection of countable unions of balls. Consequently, $\Gamma^-(F)$ itself a countable intersection of countable unions of closed sets, belongs to $\mathcal{B}(\mathbb{R}^n)$.

By Theorem 2.3, the multifunction Γ admits a $\mathcal{B}(\mathbb{R}^n)$-measurable selection σ. Fix $u \in W^{1,p}(\Omega)$. For a.e. $x \in \Omega$, set $U(x) = \sigma(\nabla u(x))$. Then U is $\mathcal{L}(\Omega)$-measurable, with values in $W^{1,p}_{\mathrm{per}}(Y)$ and, by (7)

$$f^p_{\mathrm{hom}}\big(\nabla u(x)\big) = \int_Y f\big(y, \nabla u(x) + \nabla_y U(x)(y)\big)\,\mathrm{d}y \quad \text{for a.e. } x \in \Omega. \tag{8}$$

Integrating (8) over Ω yields $\int_{\Omega\times Y} f(y, \nabla u(x) + \nabla_y U(x)(y))\,\mathrm{d}y\,\mathrm{d}x < +\infty$, so that by ($H_2$), $\nabla_y U$ is in $(L^p(\Omega \times Y))^n$. By the Poincaré–Wirtinger inequality, U belongs to $L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y))$, hence $\int_\Omega f^p_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x \geqslant \inf\{\int_{\Omega\times Y} f(y, \nabla u(x) + \nabla_y V(x, y))\,\mathrm{d}x\,\mathrm{d}y \colon V \in L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y))\}$. $\square$

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