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On the horizontal distribution of zeros of linear combinations of Euler products

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Abstract

We investigate the behavior of the zero counting function of certain natural Dirichlet series with functional equation in the immediate vicinity of the critical line $\{\mathrm{Re}(s) = \frac{1}{2}\}$. ***To cite this article: D.A. Hejhal, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Sur la distribution horizontale des zéros de combinaisons linéaires des produits eulériens. Nous étudions certaines séries de Dirichlet naturelles qui satisfont une équation fonctionelle ainsi qu'une fonction dénombrant leurs zéros dans certains domaines rectangulaires proches de la droite critique $\{\mathrm{Re}(s) = \frac{1}{2}\}$. ***Pour citer cet article : D.A. Hejhal, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

1. Introduction

This is the first of several Notes having to do with the asymptotic distribution of zeros of linear combinations of Euler products over certain types of rectangular regions situated just slightly to the right of the critical line $\{\mathrm{Re}(s) = \frac{1}{2}\}$. The results we obtain are an extension of those in [4] and build in an essential way on the ideas of [5] and [6].

To get started, we need some notation. Let $J \geqslant 2$ and $\{L_1, \ldots, L_J\}$ be any collection of Euler products of degree d satisfying the five basic hypotheses posited in §1 of [5]. Briefly put, the ambient situation will thus be one wherein:

(A) on $\{\mathrm{Re}(s) > 1\}$, one has $\log L_j(s) = \sum c_j(n)(\Lambda(n)/\log n)n^{-s}$ for appropriately chosen $|c_j(n)| \leqslant d$;

(B) each L_j is analytically continuable to all of $\mathbb{C}$ apart from a finite number of poles situated along the line $\{\mathrm{Re}(s) = 1\}$;

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(C) each L_j satisfies a standard functional equation

$$\exp(\mathrm{i}\alpha)G(s)L(s) = \overline{\exp(\mathrm{i}\alpha)G(1-\bar{s})L(1-\bar{s})}$$

for suitably chosen $\alpha \in \mathbb{R}$ and gamma factor $G(s) = Q^s \prod_{\iota=1}^{h} \Gamma(\lambda_\iota s + \mu_\iota)$ (both choices being allowed to depend on j);

(D) the logarithms of the L_j are "formally orthogonal" in the sense that one has

$$\sum_{p \leqslant t} p^{-2\sigma} c_j(p)\overline{c_k(p)} = \aleph_j \delta_{jk} \log\left[\min\left(\log t, \left(\sigma - \frac{1}{2}\right)^{-1}\right)\right] + \mathrm{O}(1),$$

with certain positive constants $\aleph_j$, uniformly for $t \geqslant 2$ and $\frac{1}{2} \leqslant \sigma \leqslant 1$;

(E) the nontrivial zeros of $L_j(s)$ either satisfy GRH for all j, or else a Selberg-type density condition $N_j(\sigma, T, T+H) = \mathrm{O}[H(H/\sqrt{T})^{\beta(1/2-\sigma)} \log T]$ whenever $T^\omega \leqslant H \leqslant T$ and $\frac{1}{2} \leqslant \sigma \leqslant 1$ (the same values of $\frac{1}{2} < \omega \leqslant 1$ and $\beta > 0$ being utilized herein for all j).

This set-up is then completed by choosing any numbers $0 < c_1, \delta < 1$ and $1 < \kappa, c_2 < \infty$, and writing $\psi_0(\sigma, t) \equiv \sum_{p \leqslant t} p^{-2\sigma}$ as in [5, §1]. If GRH holds, ω is taken to be any number in $(0, 1]$.

When the given L_j all have the *same* gamma factor $G(s)$, the linear combination

$$F(s) = \sum_{j=1}^{J} b_j \exp(\mathrm{i}\alpha_j) L_j(s) \tag{1}$$

manifestly satisfies $G(s)F(s) = \overline{G(1-\bar{s})F(1-\bar{s})}$ for any choice of $(b_1, \ldots, b_J) \in S^{J-1}$ (the unit sphere in $\mathbb{R}^J$). Standard techniques show that the total number of zeros of $G(s)F(s)$ inside $\{0 < \mathrm{Im}(s) < T\}$ is $(\Lambda/\pi)T \log T + \mathrm{O}(T)$, where $\Lambda \equiv \sum_{\iota=1}^{h} \lambda_\iota$. At the same time, by virtue of the functional equation, the function GF clearly takes real values along $\{\mathrm{Re}(s) = \frac{1}{2}\}$.

It was proved in [2] that, under GRH plus a modest spacing hypothesis on the zeros of L_j, any linear combination (1) must necessarily have *asymptotically all* of its zeros along the critical line as $T \to \infty$. (Similarly for $T < 0$.) In light of this fact, it is only natural to try to develop bounds for the "horizontal counting function" $N_F(\sigma, T, T+H)$ as σ approaches $\frac{1}{2}$ from the right.[1]

In the case $J = 2$, use of ideas akin to those in [5] and [6, pp. 55 (top), 60 (4.6)] together with the trivial observation that $\max(u, v) = \frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}|u - v|$ make it possible to determine the precise order-of-magnitude of $N_F(\sigma, T, T + \eta T)$ for generic $(b_1, b_2) \in S^1$ and σ lying in the range

$$\frac{1}{2} + \frac{(\log\log T)^\kappa}{\log T} \leqslant \sigma \leqslant \frac{1}{2} + \frac{1}{(\log T)^\delta}. \tag{2}$$

Cf. [4, Theorem A]. (Here η is any positive constant.)

The methods of [4] are readily adapted to apply to $N_F(\sigma, T, T+H)$ in the more general setting of $c_1 T^\omega \leqslant H \leqslant c_2 T$ (ω being as in item (E) if GRH is not assumed). One obtains

$$\frac{H}{\sigma - 1/2} \frac{1}{\sqrt{\log\log T}}$$

as the correct order-of-magnitude.

In this Note, our primary objective will be to announce an extension of this last estimate to a setting in which the variable J is unrestricted.

[1] For any *fixed* $\sigma > 1/2$, one expects that $N_F(\sigma, T, T+H) = \mathrm{O}(T)$, at least under GRH. See [7, §§9.51, 9.623] and [8, §§7.9, 13.2]. Cf. also [2, p. 861 middle].

2. Admissible probability measures

Let $S = S^{J-1}$ and $H(X,Y) = |\langle X,Y\rangle|^{-1}$, where $\langle X,Y\rangle$ is the standard inner product on $\mathbb{R}^J$. A probability measure μ on S will be said to be *admissible* when the integral $\int_S \log H(X,Y)\,\mathrm{d}\mu(X) = \mathrm{O}(1)$ for every $Y \in S$. It is not hard to check that:

(a) for $J = 2$, μ is admissible $\Leftrightarrow$ the logarithmic potential of μ is bounded;

(b) for general J, μ is admissible anytime the Riesz α-potential $\int_S \|X-Y\|^{-\alpha}\,\mathrm{d}\mu(X)$ is bounded for some α strictly bigger than $J-2$.

The proof of (b) uses integration by parts and a covering argument based on solid angles in S^{J-2}. As a consequence of (b), there exist numerous admissible μ supported on any compact subset K of S having Hausdorff dimension $> J-2$. Cf. [3, Chapters 2–4]. (Haar measure on S is, of course, trivially admissible.)

3. Statement of results

Theorem 3.1. *Given Euler products* $\{L_1,\ldots,L_J\}$ *as in Section* 1 *having the* same *gamma factor* $G(s)$*. If* $J \geqslant 3$*, assume that* $\kappa > 2$*. Let* $0 < \varepsilon < 1$ *and* ι *be any probability measure on* S^{J-1} *admissible in the sense of Section* 2 *above. Keep* $\sigma \in [\frac{1}{2}, \frac{1}{2} + (\log T)^{-\delta}]$*,* $H \in [c_1T^{\omega}, c_2T]$*, and* T *bigger than some suitable* $T_0(L_1,\ldots,L_J,c_1,c_2,\delta,\kappa,\omega,\varepsilon,\iota)$*. There will then exist positive constants* C_1 *and* C_2 *depending solely on* $\{L_1,\ldots,L_J,c_1,c_2,\delta,\kappa,\omega,\varepsilon,\iota\}$ (*note the* ε) *such that, subject to* (2),

$$C_1\frac{H}{\sigma-1/2}\frac{1}{\sqrt{\log\log T}} \leqslant N_F(\sigma,T,T+H) \leqslant C_2\frac{H}{\sigma-1/2}\frac{1}{\sqrt{\log\log T}}$$

holds for every $(b_j) \in S^{J-1}$ except *possibly those in an exceptional set* $\mathcal{N}$ *having* ι*-measure* $< \varepsilon$. (*The set* $\mathcal{N}$ *may vary with* σ, T, H.)

To a large extent, once matters are known for $J = 2$ (cf. [4] and the paradigm outlined there), Theorem 3.1 is pretty much a simple corollary of [5] and the following estimate interesting in its own right.

To facilitate the latter's statement, we first introduce independent Gaussians $W_1,\ldots,W_J$ having mean 0 and standard deviation $\sqrt{\aleph_j/2\pi}$. We then write $\Phi_k(x_1,\ldots,x_J) = \prod_{j\neq k}\nu(x_k-x_j)$, where $\nu(x) = \frac{1}{2}[1+\operatorname{sgn}(x)]$ and $\operatorname{sgn}(0) = 0$. For points of $\mathbb{R}^J$ with distinct coordinates, Φ_k has an obvious $\{0,1\}$-interpretation, which leads at once to the identity

$$\max(x_1,\ldots,x_J) = \sum_{k=1}^{J} x_k\Phi_k(x_1,\ldots,x_J). \tag{3}$$

Theorem 3.2. *Given any Euler products* $L_1,\ldots,L_J$ *as in the first part of Section* 1; *i.e., with no particular relation among their gamma factors. Keep* $\sigma \in [\frac{1}{2}, \frac{1}{2} + (\log T)^{-\delta}]$*,* $H \in [c_1T^{\omega}, c_2T]$*, and* T *bigger than some suitable* $T_0(L_1,\ldots,L_J,c_1,c_2,\delta,\kappa,\omega)$*. Let*

$$\psi_0 = \psi_0(\sigma,T), \quad L_j = L_j(\sigma+\mathrm{i}t), \quad \Delta = (12J\kappa/\omega)\psi_0^{\kappa}(\log\psi_0),$$

$$\mathcal{M}(\sigma) = \int_T^{T+H} \max\bigl(\log|L_1|,\ldots,\log|L_J|\bigr)\,\mathrm{d}t, \quad I_k(\sigma) = \int_T^{T+H} \Phi_k\bigl(\log|L_1|,\ldots,\log|L_J|\bigr)\,\mathrm{d}t.$$

We then have

$$\mathcal{M}(\sigma) = H\mathbb{E}\big(\max(W_1, \ldots, W_J)\big)\sqrt{\pi\psi_0} + A\frac{H}{\sqrt{\pi\psi_0}} + \mathrm{O}(H/\psi_0) + \mathrm{O}\big(H\psi_0^{-a}(\log\psi_0)^b\big)$$

$$I_k(\sigma) = H\mathbb{E}\big(\Phi_k(W_1, \ldots, W_J)\big) + \mathrm{O}(H/\psi_0) + \mathrm{O}\big(H\psi_0^{-c}(\log\psi_0)^d\big),$$

where A is some constant depending solely on $\{L_1, \ldots, L_J\}$ and

$$(a,b) = (0, \tfrac{1}{2}), (\tfrac{\kappa}{2} - \tfrac{1}{2}, \tfrac{1}{2}), \quad (c,d) = (\tfrac{1}{4}, \tfrac{5}{4}), (\tfrac{\kappa}{4}, \tfrac{1}{4}) \quad \textit{for } J \geqslant 3;$$

$$(a,b) = (0, \tfrac{1}{2}), (\kappa - \tfrac{1}{2}, 0), \quad (c,d) = (\tfrac{1}{2}, 2), (\tfrac{\kappa}{2}, 0) \quad \textit{for } J = 2.$$

In each instance, the second alternative holds when $\sigma \geqslant \frac{1}{2} + (\Delta/\log T)$ and the respective implied constants will typically depend on $\{c_1, c_2, \delta, \kappa, \omega\}$ in addition to $\{L_1, \ldots, L_J\}$.

Though sufficient for Theorem 3.1, Theorem 3.2 is amenable to improvements of various types. The most important perhaps is that the term $\mathrm{O}(H/\psi_0)$ in $\mathcal{M}(\sigma)$ can be upgraded to an asymptotic development in powers of $1/\sqrt{\pi\psi_0}$ in the spirit of [5].

4. Concerning the proof of Theorem 3.2

The $I_k(\sigma)$ portion of the result is basically obtained by applying [5, Theorem 2.1]. When $J = 2$, one uses a variant specifically adapted to $\log L_1 - \log L_2$ (cf. [4, Eq. (4.14)]). In the general case, one proceeds with the aid of a cruder "over/under" grid-type argument in $\mathbb{R}^J$.

For the $\mathcal{M}(\sigma)$ portion, one uses (3) and an *extension* of the ideas in [5, §3]. One introduces $\Sigma_{yj}(\sigma, t)$ as before and attacks matters first with $\mathrm{Re}\,\Sigma_{yj}$ in place of $\log|L_j|$. Band-limited approximations to $\nu(x)$ are constructed using a "kernel" $\frac{1}{2}\delta_0(v) + \frac{1}{2v}Q(v/\Omega)$, wherein $\delta_0(v)$ is the Dirac delta and Q is an appropriately chosen even function in $C^1(\mathbb{R})$ having support $[-1, 1]$. Cf. [1], [9, Eq. (2.31)], and [4, Eq. (4.7)]. "Morphing" $p^{-\mathrm{i}t}$ into $\exp(2\pi\mathrm{i}\theta_p)$ again plays a decisive role in the subsequent estimates. The terms arising therein due to the x_k multipliers in (3) are best handled *by differentiation*; prototypically,

$$I(v_1, \ldots, v_{2J}) = \int_0^1 \exp\left(\mathrm{i}\sum_{k=1}^{2J} v_k\sigma_k(\theta)\right)\mathrm{d}\theta \Rightarrow \frac{1}{\mathrm{i}}\frac{\partial I}{\partial v_1} = \int_0^1 \sigma_1(\theta)\exp\left(\mathrm{i}\sum_{k=1}^{2J} v_k\sigma_k(\theta)\right)\mathrm{d}\theta.$$

Complexification of the v_k greatly facilitates both the estimates and asymptotics. Those error terms "originating in" the finite bandwidth or morphing process itself are most easily addressed using the $\vec{\Sigma}_y$ counterpart of [5, Theorem 2.1]. One concludes by exploiting [5, Eq. (6)] to get back to $\max_j(\log|L_j|)$.

Full details of this proof will be published elsewhere.

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