Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
C. R. Acad. Sci. Paris, Ser. I 338 (2004) 853–858
Comptes Rendus Mathematique

Functional Analysis

A representation of maximal monotone operators by closed convex functions and its impact on calculus rules

Jean-Paul Penot

Laboratoire de mathématiques appliquées, faculté des sciences, CNRS 2070, BP 1155, 64013 Pau cedex, France

Received 19 January 2004; accepted 16 March 2004

Available online 27 April 2004

Presented by Philippe G. Ciarlet

Abstract

We introduce a new representation for maximal monotone operators. We relate it to previous representations given by Krauss, Fitzpatrick and Martínez-Legaz and Théra. We show its usefulness for the study of compositions and sums of maximal monotone operators. ***To cite this article: J.-P. Penot, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Une représentation des opérateurs maximaux monotones par des fonctions convexes et son impact sur les règles de calcul. Nous introduisons une représentation nouvelle pour les opérateurs maximaux monotones à l'aide de fonctions convexes. Nous la relions à des représentations dues à Krauss, Fitzpatrick, Martínez-Legaz et Théra. Nous montrons son utilité pour obtenir des règles de composition et de somme. ***Pour citer cet article : J.-P. Penot, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Etant donné un espace de Banach réflexif X, de dual X^*, les analogies entre les propriétés des éléments de l'ensemble $\mathcal{M}(X)$ (resp. $\mathcal{M}_{\max}(X)$) des opérateurs monotones (resp. maximaux monotones) de X dans X^* avec les propriétés de l'ensemble $\Gamma(Z)$ des fonctions convexes s.c.i. propres sur un espace de Banach Z ont conduit plusieurs chercheurs à proposer des représentations de $\mathcal{M}_{\max}(X)$ par un sous-ensemble de $\Gamma(Z)$. Le but de cette note est de montrer qu'il existe une telle représentation avec $Z := X \times X^*$, qui est simple et qui permet d'obtenir des résultats cruciaux concernant $\mathcal{M}_{\max}(X)$ à partir de règles classiques de l'analyse convexe.

La représentation et ses propriétés

Etant donné un opérateur $M : X \rightrightarrows X^*$ (identifié avec son graphe $M \subset Z := X \times X^*$), nous lui associons sa fonction indicatrice ι_M, donnée par $\iota_M(z) = 0$ pour $z \in M$, $\iota_M(z) = +\infty$ for $z \in Z\backslash M$ et les fonctions

$$c_M := c + \iota_M, \qquad g_M := \mathrm{co}(c_M), \qquad p_M := c_M^{**},$$

E-mail address: jean-paul.penot@univ-pau.fr (J.-P. Penot).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2004.03.017

où $c : Z \to \mathbb{R}$ est le couplage donné par $c(x,x^*) := \langle x^*, x\rangle$ et g_M est l'enveloppe convexe de c_M. Ici, pour des fonctions $f : Z^* \to \overline{\mathbb{R}}$, $g : Z \to \overline{\mathbb{R}}$, nous définissons les transposées et les conjuguées par

$$f^{\mathsf{T}}(x,x^*) := f(x^*,x), \qquad g^{\mathsf{T}}(x^*,x) = g(x,x^*),$$
$$f^*(x,x^*) := \sup_{(w,w^*)\in X\times X^*} \big\{\langle w^*, x\rangle + \langle x^*, w\rangle - f(w^*,w)\big\},$$
$$g^*(x^*,x) := \sup_{(w,w^*)\in X\times X^*} \big\{\langle w^*, x\rangle + \langle x^*, w\rangle - g(w,w^*)\big\}.$$

La représentation $M \mapsto p_M$ jouit de propriétés plaisantes comme $p_{M^{-1}} = (p_M)^{\mathsf{T}}$, $p_S \geqslant p_T$ pour $S \subset T$, $p_{\lambda M}(x,x^*) = \lambda p_M(x, \lambda^{-1}x^*)$ pour $(x,x^*) \in X \times X^*$. Elle est liée à la représentation de Fitzpatrick $M \mapsto f_M$ par les relations $f_M = p_M^*$, $p_M = f_M^*$. Comme on peut souvent passer de c_M à p_M par une convexification et une fermeture, elle s'avère plus directe. Elle est caractérisée par la propriété suivante.

Lemme 0.1. *Pour toute multiapplication M: $X \rightrightarrows X^*$ et toute fonction convexe s.c.i. propre q majorée par c sur (le graphe de) M on a $q \leqslant p_M$.*

De plus, elle fournit une caractérisation des opérateurs monotones et maximaux monotones.

Proposition 0.2. *Pour toute multiapplication $M : X \rightrightarrows X^*$ les assertions suivantes sont équivalentes* :

(a) *M est monotone : $M \in \mathcal{M}(X)$* ;
(b) *la fonction $p_M := c_M^{**}$ satisfait $p_M \geqslant p_M^*$ (avec égalité sur M)* ;
(c) *la fonction $p_M := c_M^{**}$ prend ses valeurs dans $\mathbb{R} \cup \{+\infty\}$ et vérifie*

$$p_M(x,x^*) + p_M(y,y^*) \geqslant \langle x^*, y\rangle + \langle y^*, x\rangle \quad \forall (x,x^*), (y,y^*) \in X \times X^*; \tag{1}$$

(d) *la fonction h_M sur $X \times X$ donnée par $h_M(x,y) := -(p_M(x,\cdot))^*(y)$ vérifie $h_M \geqslant -h_M^{\mathsf{T}}$* ;
(e) *la fonction $p_M := c_M^{**}$ vérifie $p_M \geqslant c$ et $p_M \mid M = c \mid M$.*

On prolonge ainsi un résultat de Fitzpatrick [6] :

Théorème 0.3. *Si $M \in \mathcal{M}_{\max}(X)$ alors $p_M^{\mathsf{T}} \geqslant p_M^* \geqslant c^{\mathsf{T}}$ et l'on a $M = (p_M - c)^{-1}(0) = ((p_M^*)^{\mathsf{T}} - c)^{-1}(0)$.*

Une réciproque est donnée dans [5] ; nous la complétons quelque peu :

Théorème 0.4. *Soit $g : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ une fonction convexe propre telle que $g \geqslant c$ et soit k donnée par $k(\cdot,y) := h(\cdot,y)^{**}$ avec $h(x,\cdot) := -g(x,\cdot)^*$. Alors les assertions suivantes sont équivalentes* :

(a) $g^* \geqslant c^{\mathsf{T}}$;
(b) $k(x,x) \leqslant 0$ *pour tout $x \in X$* ;
(c) *$M := \{z \in Z\colon g^{**}(z) = c(z)\}$ est un opérateur maximal monotone.*

Liens avec d'autres représentations

Comparons notre représentation avec celle de Krauss $M \mapsto K_M$ donnée par $K_M(\cdot,y) = H_M(\cdot,y)^{**}$, où $H_M(\cdot,y)$ est l'enveloppe convexe de $x \mapsto -c_M(x,\cdot)^*(y)$.

Proposition 0.5. *Soit $M \in \mathcal{M}(X)$ et soient h_M et k_M données par $h_M(x,y) := -(p_M(x,\cdot))^*(y)$, $k_M(x,y) = (h_M(\cdot,y))^{**}(x)$. Alors $H_M \geqslant h_M$, $k_M = K_M$. De plus, $p_M(x,x^*) = (-K_M(x,\cdot))^*(x^*) = (-k_M(x,\cdot))^*(x^*)$.*

Etant donné $M \in \mathcal{M}(X)$ on peut se demander s'il existe $q \in \Gamma(X)$ vérifiant $q^* = q^\mathsf{T}$ et $f_M^\mathsf{T} \leqslant q \leqslant p_M$. Une réponse positive est apportée par le théorème qui suit. L'unicité d'une telle fonction n'est pas assurée.

Théorème 0.6. *Pour tout $M \in \mathcal{M}_{\max}(X)$ il existe $q \in \Gamma(X)$ vérifiant $q^* = q^\mathsf{T}$ et $f_M^\mathsf{T} \leqslant q \leqslant p_M$. De plus, $M = (q - c)^{-1}(0)$.*

Des règles de calcul pour les opérateurs monotones

Les deux règles de calcul pour les opérateurs maximaux monotones qui suivent peuvent être déduites de la généralisation suivante de règles d'analyse convexe qui a un intérêt propre.

Proposition 0.7. *Soient $A : X \to Y$, $B : U \to V$ des applications linéaires continues entre des espaces de Banach et soit $G \in \Gamma(Y \times U)$. Si $\mathbb{R}_+ p_Y(\operatorname{dom} G) - R(A) = Y$, alors pour $F : X \times V \to \mathbb{R} \cup \{+\infty\}$ donné par $F(x, v) := \inf\{G(Ax, u)\colon u \in U, Bu = v\}$ on a*

$$F^*(x^*, v^*) = \min\bigl\{G^*\bigl(y^*, B^\mathrm{T} v^*\bigr)\colon y^* \in Y^*, A^\mathrm{T}(y^*) = x^*\bigr\}.$$

Corollaire 0.8. *Soit $A : X \to Y$ une application linéaire continue entre des espaces de Banach et soit $N : Y \rightrightarrows Y^*$ maximal monotone. Si $\mathbb{R}_+(\operatorname{co}(\operatorname{dom} N) - R(A)) = Y$ alors $M := A^\mathrm{T} N A \in \mathcal{M}_{\max}(X)$.*

Corollaire 0.9. *Soient $S, T \in \mathcal{M}_{\max}(X)$ tels que $\mathbb{R}_+(\operatorname{co}(\operatorname{dom} S) - \operatorname{co}(\operatorname{dom} T)) = X$. Alors $S + T \in \mathcal{M}_{\max}(X)$.*

1. Introduction

The links and analogies between closed proper convex functions and maximal monotone operators are numerous (see [1–3,11,14]) and striking. Among them are: (a) almost convexity of the domains of maximal monotone operators, (b) local boundedness on the interiors of their domains, (c) the Brøndsted–Rockafellar theorem, (d) qualification conditions for calculus rules of sums and compositions, (e) single-valuedness results, (f) regularization processes… They prompted several researchers [4–11,13] to look for a precise connection. Here we give a representation which enables one to deduce results about operations such as sums of operators and composition with a linear map from the classical rules of convex analysis.

The works of Simons and his co-authors (see [11,12]) and the recent book [14] by Zalinescu exploit that vein in a thorough way. However, they use a "big convexification" of the graph $S \subset X \times X^*$ of the operator. Here we remain in the product space $Z := X \times X^*$ which is the natural framework but we avoid the rather sophisticated theory of saddle functions used by Krauss in several papers [7–9].

2. A representation of operators by convex functions

Given a reflexive Banach space X, for $Z := X \times X^*$, $Z^* = X^* \times X$, $f : Z^* \to \overline{\mathbb{R}}$, $g : Z \to \overline{\mathbb{R}}$, we set

$$f^\mathsf{T}(x, x^*) := f(x^*, x), \qquad g^\mathsf{T}(x^*, x) = g(x, x^*),$$
$$f^*(x, x^*) := \sup_{(w, w^*) \in X \times X^*} \bigl\{\langle w^*, x\rangle + \langle x^*, w\rangle - f(w^*, w)\bigr\},$$
$$g^*(x^*, x) := \sup_{(w, w^*) \in X \times X^*} \bigl\{\langle w^*, x\rangle + \langle x^*, w\rangle - g(w, w^*)\bigr\}.$$

Given an operator $M : X \rightrightarrows X^*$ (identified with its graph $M \subset Z := X \times X^*$), we associate to it its indicator function ι_M, given by $\iota_M(z) = 0$ for $z \in M$, $\iota_M(z) = +\infty$ for $z \in Z \backslash M$ and the functions

$$c_M := c + \iota_M, \qquad g_M := \operatorname{co} c_M, \qquad p_M := c_M^{**},$$

where $c : Z \to \mathbb{R}$ is the coupling function and g_M is the convex hull of c_M. We note that the function p_M is directly related to the Fitzpatrick's function f_M associated with M which is given by $f_M := c_M^*$, so that one has $f_M = p_M^*$, $p_M = f_M^*$. Thus p_M is also related to the representation of [10] through conjugacy. The following examples show that it may be simpler to compute p_M than $f_M = p_M^*$.

Example 1. Let M be a linear subspace of $X \times X^*$ such that $\langle x^*, x\rangle \geqslant 0$ for each $(x, x^*) \in M$. Then $p_M = c_{\overline{M}}$ where $\overline{M}$ is the closure of M; when M is the graphe of an invertible symmetric operator the conjugate p_M^* of p_M is given by $p_M^*(y^*, y) = (1/4)\langle M^{-1}(y^* + My), y^* + My\rangle$. In particular, if I is the identity mapping on a Hilbert space X, then $p_I(x, x^*) = \|x\|^2$ if $x = x^*$ and $p_I(x, x^*) = +\infty$ if $x \neq x^*$ while $p_I^*(y^*, y) = (1/4)\|y^* + y\|^2$.

Example 2. Let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation given by $M(x, y) = (-y, x)$. Then $p_M = c_M$.

Example 3. Let $M := \partial s$, where s is a l.s.c. sublinear function. Then $p_M(x, x^*) = s(x) + s^*(x^*)$.

Nice properties follow from the simplicity of the representation: among them we note: $p_{M^{-1}} = (p_M)^{\mathsf{T}}$, $p_S \geqslant p_T$ for $S \subset T$, $p_{\lambda M}(x, x^*) = \lambda p_M(x, \lambda^{-1}x^*)$ for $(x, x^*) \in X \times X^*$.

The predominant role of p_M is illuminated by the following simple statement.

Lemma 2.1. *For any multimapping $M : X \rightrightarrows X^*$ and any closed convex function q majorized by c on the graph of M one has $q \leqslant p_M$.*

In the case M is a monotone operator, the following proposition shows that p is proper and satisfies another domination property. It gathers useful characterizations of monotonicity. The equivalence (a)⇔(h) follows from [6, Proposition 2.2, Theorem 2.4] (see also [10, Theorem 2], [11, Lemma 9.1], [14, Lemma 3.11.1]).

Proposition 2.2. *For a nonempty subset M of $Z := X \times X^*$ the following assertions are equivalent*:

(a) *M is monotone*;
(b) *the function c_M satisfies $c_M(x, x^*) \geqslant c_M^*(x^*, x)$ for any $(x, x^*) \in X \times X^*$*;
(c) *the function $p_M := c_M^{**}$ satisfies $p_M \geqslant p_M^*$ (with equality on M)*;
(d) *the function $p_M := c_M^{**}$ takes its values in $\mathbb{R} \cup \{+\infty\}$ and satisfies*

$$p_M(x, x^*) + p_M(y, y^*) \geqslant \langle x^*, y\rangle + \langle y^*, x\rangle \quad \forall (x, x^*), (y, y^*) \in X \times X^*; \tag{2}$$

(e) *the function h_M on $X \times X$ given by $h_M(x, y) := -(p_M(x, \cdot))^*(y)$ satisfies $h_M \geqslant -h_M^{\mathsf{T}}$*;
(f) *the function $p_M := c_M^{**}$ satisfies $p_M \geqslant c$ and $p_M \mid M = c \mid M$*;
(g) *there exists a closed convex function p on $X \times X^*$ such that $p \geqslant c$ and $p \mid M = c \mid M$*;
(h) *there exists a convex function g on $X \times X^*$ such that $g \geqslant c$ and $g \mid M = c \mid M$.*

Part of the following result is due to Fitzpatrick [6].

Theorem 2.3. *If M is maximal monotone, then $p_M^{\mathsf{T}} \geqslant p_M^* \geqslant c^{\mathsf{T}}$. Moreover one has*

$$M = \{z \in Z:\ p_M(z) = c(z)\} = \{z \in Z:\ (p_M^*)^{\mathsf{T}}(z) = c(z)\}.$$

A converse of Theorem 2.3 is given in [5]; we slightly complete it.

Theorem 2.4. *Let $g : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function such that $g \geqslant c$ and let k be given by $k(\cdot, y) := h(\cdot, y)^{**}$ with $h(x, \cdot) := -g(x, \cdot)^*$. Then the following assertions are equivalent*:

(a) $g^* \geqslant c^\intercal$;
(b) $k(x, x) \leqslant 0$ *for each* $x \in X$;
(c) $M := \{z \in Z\colon\ g^{**}(z) = c(z)\}$ *is a maximal monotone operator.*

Example 4. Let $g(x, x^*) := \varphi(x) + \varphi^*(x^*)$, where φ is a closed proper convex function on X. Then, as easily seen, g is closed proper convex on $X \times X^*$, $g \geqslant c$ and $g^* = g^\intercal \geqslant c^\intercal$. Thus $g^{**} = g$ and $\partial\varphi = \{(x, x^*)\colon\ g(x, x^*) = \langle x^*, x\rangle\}$ is a maximal monotone operator.

3. Other representations

The links with the Krauss' representation are more subtle than the links with the Fitzpatrick's function.

Lemma 3.1. *Given a function $h : X \times X \to \mathbb{R} \cup \{+\infty\}$ such that $h \geqslant -h^\intercal$ and $-h(x, \cdot)$ is closed convex for each $x \in X$, the function k given by $k(\cdot, y) = (h(\cdot, y))^{**}$ satisfies $k^\intercal \geqslant -h$. If moreover the function $g : (x, x^*) \mapsto (-h(x, \cdot))^*(x^*)$ is such that $g = g^{**}$, then one has $h(x, \cdot) = -(-k(x, \cdot))^{**}$.*

In particular, if M is monotone, then, for h_M and k_M given by $h_M(x, y) := -(p_M(x, \cdot))^*(y)$, $k_M(x, y) = (h_M(\cdot, y))^{**}(x)$ one has for any $(x, y) \in X \times X$, $k_M(y, x) \geqslant -h_M(x, y) = (-k_M(x, \cdot))^{**}(y)$.

Let us compare the representation we gave with the one of Krauss. The Krauss' saddle function K_M is given by $K_M(\cdot, y) = H_M(\cdot, y)^{**}$ where $H_M(\cdot, y)$ is the convexification of $x \mapsto -c_M(x, \cdot)^*(y)$.

Proposition 3.2. *Let h_M and k_M be as in the preceding lemma. Then $H_M \geqslant h_M$ and for any $(y, y^*) \in X \times X^*$ one has $f_M(y^*, y) = H_M(\cdot, y)^*(y^*) = K_M(\cdot, y)^*(y^*) = h_M(\cdot, y)^*(y^*) = k_M(\cdot, y)^*(y^*)$, $k_M = K_M$. The function p_M is related to the Krauss' function K_M via $p_M(x, x^*) = (-K_M(x, \cdot))^*(x^*) = (-k_M(x, \cdot))^*(x^*)$.*

Given a monotone operator M, one may wonder whether it is possible to get a closed convex function q such that $q^* = q^\intercal$ and $p_M^\intercal \leqslant q \leqslant p_M$. A positive answer is provided in the next statement.

Theorem 3.3. *For any maximal monotone operator M there exists a closed convex function q such that $q^* = q^\intercal$ and $f_M^\intercal \leqslant q \leqslant p_M$. Moreover $M = \{(x, x^*)\colon\ q(x, x^*) = \langle x^*, x\rangle\}$.*

This result is a consequence of the following proposition inspired by [8], Theorem 4. Let us note that here too uniqueness of q is not ensured. However, as claimed by the second assertion, the coincidence set of q and c is independent of the choice of q when M is maximal monotone; this assertion stems from the fact that this set is also the coincidence set of p_M with c and of $(p_M^*)^\intercal$ with c (see Theorem 2.4).

Proposition 3.4. *Let $p : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ be a closed proper convex function such that $p \geqslant f^\intercal$ for $f := p^*$. Then there exists a closed proper convex function $q : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ such that $q^* = q^\intercal$ and $p \geqslant q \geqslant f^\intercal$.*

In order to point out the links of what precedes with the main result of [8] let us set $h(x, y) := -q(x, \cdot)^*(y)$, $k(x, y) = (h(\cdot, y))^{**}(x)$, so that $(-h(y, \cdot))^* = q(y, \cdot) = q^*(\cdot, y) = k(\cdot, y)^*$, hence

$$k(\cdot, y) = \big(-h(y, \cdot)\big)^{**} = -h(y, \cdot), \tag{3}$$

a useful skew symmetry property. It is simpler than the ones obtained in Lemma 3.1.

Although the preceding result is nonconstructive, there are important examples in which a natural choice of q does exist, as the following examples show. Moreover, we will see in the next section that, under some qualification conditions, calculus rules provide means to find such functions.

Example 5. As noted above, given a closed proper convex function φ on X, the function q defined by $q(x,x^*) := \varphi(x) + \varphi^*(x^*)$ satisfies $q^* = q^\intercal$ and $p_M^\intercal \leqslant q \leqslant p_M$ for $M := \partial\varphi$.

Example 6. Suppose $X := U \times V$ and ℓ is a saddle function on $U \times V$ such that for each $u \in U$ and each $v \in V$ the functions $\ell(\cdot, v)$ and $-\ell(u,\cdot)$ are closed proper convex. Then, the function q on $(U \times V) \times (U^* \times V^*)$ given by $q(u,v,u^*,v^*) := \ell(\cdot,v)^*(u^*) + (-\ell(u,\cdot))^*(v^*)$ satisfies $q^* = q^\intercal$ and $f_M^\intercal \leqslant q \leqslant p_M$ for $M := \partial\ell$, where $(u,v,u^*,v^*) \in \partial\ell$ iff $u^* \in \partial\ell(\cdot,v)(u)$ and $-v^* \in \partial(-\ell(u,\cdot))(v)$.

4. Calculus rules for monotone operators

The following two known calculus rules for monotone operators can be deduced from a generalization of a classical rule of convex analysis.

Proposition 4.1. *Let $A : X \to Y$, $B : U \to V$ be continuous linear maps between Banach spaces and let $G : Y \times U \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. proper convex function. If $\mathbb{R}_+ p_Y(\operatorname{dom} G) - R(A) = Y$, then, for $F : X \times V \to \mathbb{R} \cup \{+\infty\}$ given by $F(x,v) := \inf\{G(Ax,u)\colon u \in U, Bu = v\}$ one has*

$$F^*(x^*,v^*) = \min\bigl\{G^*\bigl(y^*, B^{\mathrm{T}}v^*\bigr)\colon\ y^* \in Y^*, A^{\mathrm{T}}(y^*) = x^*\bigr\}.$$

Corollary 4.2. *Let $A : X \to Y$ be a continuous linear map between two Banach spaces and let $N : Y \rightrightarrows Y^*$ be a maximal monotone operator. Suppose that $\mathbb{R}_+(\operatorname{co}(\operatorname{dom} N) - R(A)) = Y$. Then $M := A^{\mathrm{T}}NA$ is maximal monotone.*

Corollary 4.3. *Let $S, T : X \rightrightarrows X^*$ be maximal monotone operators such that $\mathbb{R}_+(\operatorname{co}(\operatorname{dom} S) - \operatorname{co}(\operatorname{dom} T)) = X$. Then $S + T$ is maximal monotone.*

References

[1] H. Attouch, On the maximality of the sum of two maximal monotone operators, Nonlinear Anal. 5 (2) (1981) 143–147.
[2] H. Attouch, H. Brezis, Duality for the sum of convex functions in general Banach spaces, in: J.A. Barroso (Ed.), Aspects of Mathematics and its Applications, North-Holland, Amsterdam, 1986, pp. 125–133.
[3] H. Brezis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, North-Holland, Amsterdam, 1971.
[4] R.S. Burachik, B.F. Svaiter, ε-enlargements of maximal monotone operators in Banach spaces, Set-Valued Anal. 7 (2) (1999) 117–132.
[5] R.S. Burachik, B.F. Svaiter, Maximal monotone operators, convex functions and a special family of enlargements, Set-Valued Anal. 10 (4) (2002) 297–316.
[6] S. Fitzpatrick, Representing monotone operators by convex functions, in: Functional Analysis and Optimization, Workshop and Miniconference, Canberra, Australia, 1988, in: Proc. Centre Math. Anal. Austral. Nat. Univ., vol. 20, 1988, pp. 59–65.
[7] E. Krauss, A representation of maximal monotone operators by saddle functions, Rev. Roumainl Math. Pures Appl. 30 (1985) 823–836.
[8] E. Krauss, A representation of arbitrary maximal monotone operators via subgradients of skew-symmetric saddle functions, Nonlinear Anal. 9 (1985) 1381–1399.
[9] E. Krauss, Maximal monotone operators and saddle functions. I, Z. Anal. Anwend. 5 (1986) 333–346.
[10] J.E. Martínez-Legaz, M. Théra, A convex representation of maximal monotone operators, J. Nonlinear and Convex Anal. 2 (2) (2001) 243–247.
[11] S. Simons, Minimax and Monotonicity, in: Lecture Notes in Math., vol. 1693, Springer, Berlin, 1998.
[12] S. Simons, Sum theorems for monotone operators and convex functions, Trans. Amer. Math. Soc. 350 (7) (1998) 2953–2972.
[13] B.F. Svaiter, A family of enlargements of maximal monotone operators, Set-Valued Anal. 8 (4) (2000) 311–328.
[14] C. Zalinescu, Convex Analysis in General Vector Spaces, World Scientific, Singapore, 2002.