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Topology

The loop product for 3-manifolds

Hossein Abbaspour

Centre de mathématiques, École polytechnique, 91128 Palaiseau, France

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Abstract

Let M be a connected, closed, oriented and smooth manifold of dimension d. Let LM be the space of loops in M. Chas and Sullivan introduced the loop product, an associative product of degree $-d$ on the homology of LM. In this Note we aim at identifying 3-manifolds with "non-trivial" loop products. ***To cite this article: H. Abbaspour, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Le produit de Chas–Sullivan pour les variétés de dimension 3. Pour M, une variété connexe, orientée et lisse de dimension d, soit LM l'espace des lacets libres de M. Chas et Sullivan ont défini un produit associatif de degré $-d$ sur l'homologie de LM. Dans cette Note on vise à identifier les variétés de dimension 3 qui ont des produits de Chas–Sullivan « non-triviaux ». ***Pour citer cet article : H. Abbaspour, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Soit M une variété connexe, orientée et lisse de dimension d. Soit $\mathbb{S}^1$ le cercle unité avec un point marqué, disons 1. Un *lacet* est une application continue $f:\mathbb{S}^1 \to M$. L'espace LM de tous les lacets dans M s'appelle *l'espace des lacets libres* de M. Cet espace n'est pas connexe. En fait, il y a une bijection entre les composantes connexes de LM et les classes de conjugaison de $\pi_1(M)$; $H_0(LM)$ est donc le groupe abélien libre engendré par les classes de conjugaison de $\pi_1(M)$. Dans [2], Chas et Sullivan ont défini un produit de degré $-d$ sur $H_*(LM)$, noté $\bullet : H_i(LM) \otimes H_j(LM) \to H_{i+j-d}(LM)$.

Théorème 0.1 (Chas–Sullivan). *Le groupe abélien gradué $H_{*-d}(LM)$, muni du produit de Chas–Sullivan est une algèbre graduée commutative.*

E-mail address: habbaspour@math.polytechnique.fr (H. Abbaspour).

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Considérons l'application $p : H_*(LM) \to H_*(M)$ induite par $f \mapsto f(1)$ et l'application $i : H_*(M) \to H_*(LM)$ induite par l'inclusion des lacets constants. On a $p \circ i = id_{H_*(M)}$. Il existe donc une décomposition canonique $H_*(LM) = H_*(M) \oplus A_M$, où $A_M = \operatorname{Ker} p$.

Définition 0.2. *On dit que M a des produits de Chas–Sullivan non-triviaux si la restriction de $\bullet$ à A_M est non-triviale.*

Nous allons caractériser les variétés de dimension 3 ayant des produits de Chas–Sullivan non-triviaux. Pour cela on introduit la définition suivante :

Définition 0.3. *Soit M une variété fermée de dimension 3. Alors M est algébriquement hyperbolique si le revêtement universel de M est contractible et $\pi_1(M)$ n'a aucun sous-groupe abélien de rang 2.*

Selon la conjecture de géométrisation de Thurston, les variétés algébriquement hyperboliques sont hyperboliques. Le résultat principal de cette note est le théorème suivant :

Théorème 0.4. *Soit M une variété fermée et orientée de dimension 3.*

(i) *Si M est algébriquement hyperbolique alors M et tous ses revêtements finis ont des produits de Chas–Sullivan triviaux.*
(ii) *Si M n'est pas algébriquement hyperbolique alors M ou un revêtement double de M a des produits de Chas–Sullivan non-triviaux.*

La preuve utilise la décomposition d'une variété de dimension trois en variétés premières [5], ainsi que la décomposition JSJ le long de tores [3,4]. Les détails se trouvent dans [1].

1. Introduction

Throughout this article M is a connected oriented smooth manifold and $\mathbb{S}^1$ is the unit circle with a marked point 1. A *loop* in M is a continuous map $f : \mathbb{S}^1 \to M$ and LM, the *free loop space* of M, is the space of all loops in M. Note that LM is not connected and there is a bijection between the connected components of LM and the conjugacy classes of $\pi_1(M)$; hence $H_0(LM)$ is the free Abelian group generated by the conjugacy classes of $\pi_1(M)$. The standard action of the unit circle on LM induces an operator of degree 1, $\Delta : H_*(LM) \to H_{*+1}(LM)$. In [2] Chas and Sullivan introduced a product of degree $-d$ on $H_*(LM)$ called *the loop product* and denoted $\bullet : H_i(LM) \otimes H_j(LM) \to H_{i+j-d}(LM)$, where d is the dimension of M. They proved the following:

Theorem 1.1. *The graded Abelian group $H_{*-d}(LM)$ equipped with the loop product is a graded commutative algebra.*

Let $p : H_*(LM) \to H_*(M)$ be the map induced by $f \mapsto f(1)$ and $i : H_*(M) \to H_*(LM)$ be the map induced by the inclusion of constant loops. We have $p \circ i = id_{H_*(M)}$, and hence $H_*(LM) = H_*(M) \oplus A_M$, where $A_M = \operatorname{Ker} p$.

Definition 1.2. The manifold M has non-trivial loop products if the restriction of $\bullet$ to A_M is non-trivial.

The aim of this Note is to characterize the closed 3-manifolds with non-trivial loop products. For stating our result we need the following definition.

Definition 1.3. A closed 3-manifold M is said to be *algebraically hyperbolic* if its universal cover is contractible and $\pi_1(M)$ has no rank 2 Abelian subgroup.

According to Thurston's geometrization conjecture, algebraically hyperbolic 3-manifolds are actually hyperbolic.

The following is the main result of this Note.

Theorem 1.4. *Let M be a closed* 3*-manifold.*

(i) *If M is algebraically hyperbolic then M and all its finite covers have trivial loop products.*
(ii) *If M is not algebraically hyperbolic then M or some double cover of M has non-trivial loop products.*

The detailed proof can be found in [1]. In this note we try to give a sketch of the proof and some examples of three manifolds with non-trivial loop products.

Notation. The based loop space of M is denoted ΩM. For $\alpha \in \pi_1(M)$, C_α is its centralizer in $\pi_1(M)$ and $[\alpha]$ is its conjugacy class. For a conjugacy class $[\alpha]$, $(LM)_{[\alpha]}$ denotes the corresponding connected component of LM. The projection on A_M is denoted p_{A_M}. For the spaces X and Y where $Y \subset X$, $\overline{Y}$ is the closure of Y in X.

2. Proof of part (i): algebraically hyperbolic 3-manifolds

Let M be an algebraically hyperbolic 3-manifold. Since M has contractible universal cover, it follows from the long exact sequence associated with the fibration $\Omega M \hookrightarrow LM \xrightarrow{p} M$ that each connected component of LM has also a contractible universal cover. Moreover, one can prove that each connected component $(LM)_{[\alpha]}$ is an Eilenberg–Maclane space $K(C_\alpha, 1)$. In [1] we showed that C_α, for $\alpha \neq 1$, has homological dimension 1 by proving that C_α is a subgroup of $\mathbb{Q}$. This proves that $A_M \cong \bigoplus_{[\alpha]\neq[1]} H_*(K(C_\alpha, 1))$ is concentrated in degree at most 1, and therefore $\bullet$ vanishes on A_M.

3. Proof of part (ii): non-algebraically hyperbolic 3-manifolds

The first step is to construct examples of non-trivial loop products for 3-manifolds with finite fundamental group, $S^1 \times S^2$ and Seifert manifolds. Then we use the prime decomposition [5] or the torus decomposition [3,4] to construct homology classes in LM with non-trivial loop products when M has a suitable non-trivial decomposition. Here we give some examples of such constructions. We refer the reader to [1] for more details.

3.1. S^3

Since S^3 is a Lie group, there exits a homeomorphism $j : S^3 \times \Omega S^3 \to LS^3$. This gives rise to an isomorphism of algebras:

$$j_* : \big(H_*\big(S^3\big), \cap\big) \otimes \big(H_*\big(\Omega S^3\big), \times\big) \to \big(H_*\big(LS^3\big), \bullet\big), \tag{1}$$

where $\cap$ denotes the usual intersection product and $\times$ is the Pontrjagin product.

It is known that $(H_*(\Omega S^3), \times) \cong \mathbb{Z}[x]$ where x has degree 2. Let $\mu \in H_3(S^3)$ be the fundamental class of S^3. We set $y_1 = j_*(\mu \otimes x)$ and $y_2 = j_*(\mu \otimes x^2)$. Notice that $p(y_i) = 0$, for $i \in \{1, 2\}$, because the homology of S^3 vanishes in dimension 5 and 7 respectively, thus $y_i \in A_{S^3}$. Under isomorphism (1), $y_1 \bullet y_2$ corresponds to $(\mu \otimes x)(\mu \otimes x^2) = \mu \otimes x^3 \neq 0$ hence $y_1 \bullet y_2 \neq 0$. Therefore S^3 has non-trivial loop products.

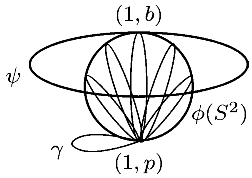

Fig. 1. $S^1 \times S^2$.

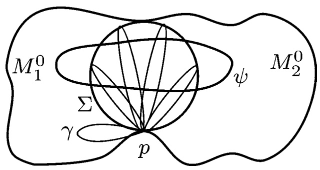

Fig. 2. $M = M_1 \# M_2$.

3.2. $S^1 \times S^2$

Let b and p be two distinct points in S^2. We choose $(1, p)$ as the base point of $S^1 \times S^2$. The map $x \mapsto (x, p)$, $x \in \mathbb{S}^1$, gives rise to an element η of $\pi_1(S^1 \times S^2)$.

Consider the map $\psi : \mathbb{S}^1 \to S^1 \times S^2$ defined by $\psi(x) = (x, b)$. Note that ψ as a loop with the marked point $(1, b)$, represents a homology class $\Psi \in H_0((L(S^1 \times S^2))_{[\eta]})$.

Let $\phi : S^2 \to S^1 \times S^2$ be the map defined by $\phi(y) = (1, y)$. The images of ψ and ϕ intersect exactly at $(1, b)$. We write $\phi(S^2)$ as a union of circles, any two of them having only the point $(1, p)$ in common. This gives rise to a one-dimensional family of loops in $S^1 \times S^2$ (see Fig. 1). Note that the free homotopy type of the loops of this 1-dimensional family is the one of the trivial loop. One can compose the loops of this family with a fixed loop whose marked point is $(1, p)$ and modify their free homotopy type. Suppose that we have done this modification with a fixed loop which does not meet ψ and represents a non-trivial element $\mu \in \pi_1(S^1 \times S^2)$ where $\mu \neq \eta$. This new 1-dimensional family of loops represents a homology class $\Phi \in H_1((L(S^1 \times S^2))_{[\mu]})$.

We prove that $p_{A_{S^1 \times S^2}}(\Delta\Psi) \bullet p_{A_{S^1 \times S^2}}(\Delta\Phi) \neq 0$ which implies that $S^1 \times S^2$ has non-trivial loop products. Since $p_{A_{S^1 \times S^2}}(\Delta\Psi) \bullet p_{A_{S^1 \times S^2}}(\Delta\Phi)$ belongs to $H_0(L(S^1 \times S^2))$, and hence it can be expressed as a sum of conjugacy classes with $+1$ or -1 as the coefficients. Indeed it equals $\pm[\eta\mu] \pm [\eta] \pm [\mu] \pm [1]$. Since 1, η and μ are distinct therefore three terms out of four are distinct and hence there cannot be a complete cancellation.

3.3. Connected sums

Proposition 3.1. *Suppose that $M = M_1 \# M_2$ and $\pi_1(M_i) \neq 1$, $i = 1, 2$. Then M has non-trivial loop products.*

Let $\Sigma \subset M$ be the 2-sphere separating the two components M_1^0 and M_2^0, where M_k^0, for $k \in \{1, 2\}$, is M_k with a ball removed. Just like Section 3.2, the 2-sphere Σ gives rise to a 1-dimensional family of loops which have the same marked point $p \in M$. We set p to be the base point of M (Fig. 2). The loops in this 1-dimensional family have the free homotopy type of the one of the trivial loop. In order to modify their free homotopy type, one can compose the loops of this 1-dimensional family with a fixed loop whose marked point is p. Suppose that we have done this modification using a fixed loop γ (Fig. 2) which represents a non-trivial element $h \in \pi_1(M)$. The new 1-dimensional family of loops represents a homology class $\Phi \in H_1((LM)_{[h]})$.

Now consider a simple smooth curve $\psi : \mathbb{S}^1 \to M$ which intersects Σ exactly at 2 points and has the free homotopy type $[x_1 x_2]$ where $x_i \neq 1 \in \pi_1(M_i)$, $i = 1, 2$ (Fig. 2). Note that $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2)$ and $x_1 x_2$ is regarded as an element of this free product. We choose this curve so that it does not intersect γ. As a loop, ψ represents a homology class $\Psi \in H_0((LM)_{[x_1 x_2]})$. We claim that there exist some choices of x_1, x_2 and h such that $p_{A_M}(\Delta\Psi) \bullet p_{A_M}(\Delta\Phi) \neq 0 \in H_0(LM)$.

In expanding $p_{A_{S^1 \times S^2}}(\Delta\Psi) \bullet p_{A_{S^1 \times S^2}}(\Delta\Phi)$ we get eight terms. By passing to mod 2 only two terms remain, namely $[x_1 x_2 h]$ and $[x_2 x_1 h]$. Now we must show that there exist choices of h such that these two conjugacy classes are different. Indeed $h = x_1 x_2$ is a convenient choice since

$$[x_1 x_2 h] = [x_1 x_2 x_1 x_2] \quad \text{and} \quad [x_2 x_1 h] = [x_2 x_1 x_1 x_2] = \left[x_1^2 x_2^2\right]$$

and the reduced words $x_1 x_2 x_1 x_2$ and $x_1^2 x_2^2$ are cyclically different.

3.4. Manifolds containing non-separating tori

Proposition 3.2. *Suppose that M is a closed oriented 3-manifold which contains a non-separating two sided π_1-injective 2-torus T. Then M has non-trivial loop products.*

Let $\phi : S^1 \times S^1 \to T \subset M$ be a homeomorphism. We set $\phi(1, 1)$ as the base point of M. Consider the one-dimensional family of loops ϕ_t defined by $\phi_t(s) = \phi(t, s)$ (longitudes of T in Fig. 3). This 1-family of loops represents a homology class Φ in $H_1((LM)_{[h]})$, where h is the element of $\pi_1(M)$ represented by ϕ_1.

Now consider a closed simple curve $\psi : \mathbb{S}^1 \to M$ which meets T transversally at exactly one point $\phi(1, 1)$. Note that ψ represents an element $g \in \pi_1(M)$ and also gives rise to a homology class $\Psi \in H_0((LM)_{[g]})$.

We show that $p_{A_M}(\Delta\Psi) \bullet p_{A_M}(\Delta\Phi) \neq 0 \in H_0(LM)$. Similar to $S^1 \times S^2$ we have $p_{A_M}(\Delta\Psi) \bullet p_{A_M}(\Delta\Phi) = \pm[gh] \pm [h] \pm [g] \pm [1]$.

To prove the claim, it is sufficient to show that [1], $[h]$ and $[g]$ are distinct. Since T is π_1-injective then $[h] \neq 1$. Note that the loop ψ intersects T exactly at one point hence the intersection product of the two homology classes (in M) that ψ and T represent are non-trivial and in particular the homology classes are non-trivial, therefore $[g] \neq [1]$. A similar argument shows that $[g] \neq [h]$.

3.5. Manifolds with a hyperbolic factor

Proposition 3.3. *Let M be a 3-manifold which contains a separating two sided π_1-injective torus T. Suppose that $M \setminus T$ has two connected components M_1 and M_2 such that*:

(i) *$\overline{M}_1$ has a hyperbolic interior with finite volume.*
(ii) *Either M_2 has a complete hyperbolic structure of finite volume, or else $\overline{M}_2$ is a Seifert manifold and $\overline{M}_2 \neq S^1 \times S^1 \times [0, 1]$.*

Then M has non-trivial loop products.

Let $\phi : S^1 \times S^1 \to T \subset M$ be a homeomorphism. We choose $\phi(1, 1)$ as the base point. Just like the previous case, ϕ gives rise to a one-dimensional family of loops ϕ_t, $t \in S^1$ (longitudes of T in Fig. 4). This 1-family of loops represents a homology class $\Phi \in H_1((LM)_{[h]})$, where $h \in \pi_1(M)$ is the element represented by ϕ_1.

Now consider a simple smooth curve $\psi : \mathbb{S}^1 \to M$ which intersects T exactly at 2 points and it has the free homotopy type $[x_1x_2]$ where $x_i \in \pi_1(M_i)$, $i = 1, 2$. Note that $\pi_1(M) = \pi_1(\overline{M}_1) *_{\pi_1(T)} \pi_1(\overline{M}_2)$ and x_1x_2 is regarded as an element of this amalgamated free product.

As a loop ψ represents a homology class $\Psi \in H_0((LM)_{[x_1x_2]})$. We claim that there exist choices of x_1, x_2 and h such that $p_{A_M}(\Delta\Psi) \bullet p_{A_M}(\Delta\Phi) \neq 0 \in H_0(LM)$.

In computing $p_{A_{S^1 \times S^2}}(\Delta\Psi) \bullet p_{A_{S^1 \times S^2}}(\Delta\Phi)$ we get eight terms. By passing to mod 2 only two terms survive, namely $[x_1x_2h]$ and $[x_2x_1h]$. Now we must show that there are some choices of x_1, x_2 and h such that these two

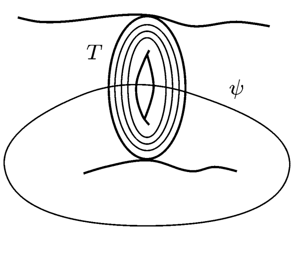

Fig. 3. Non-separating torus T.

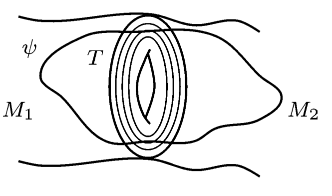

Fig. 4. Separating torus T.

conjugacy classes are different. The following lemma gives some sufficient conditions so that $[x_1x_2h]$ and $[x_2x_1h]$ are distinct. We refer the reader to [1] for the proof of this lemma.

Lemma 3.4. *Suppose that G_1, G_2 and H are three groups and $H = G_1 \cap G_2$. Let $x_1 \in G_1 \setminus H$ and $x_2 \in G_2 \setminus H$ and $h \in H$ such that*:

(a) $x_1^{-1}Hx_1 \cap H = 1$,
(b) $x_2h \neq hx_2$.

Then x_1x_2h and x_2x_1h are not conjugate in $G_1 *_H G_2$.

In our case $G_i = \pi_1(\overline{M}_i)$, $i = 1, 2$, and $H = \pi_1(T)$. Since $\overline{M}_1$ has a hyperbolic interior of finite volume, $\pi_1(T)$ consists of parabolic elements of $PSL(2,\mathbb{C})$ with a common fixed point. Then $x_1^{-1}(\pi_1(T))x_1 \cap \pi_1(T) = 1$ for $x_1 \in \pi_1(\overline{M}_1) \setminus \pi_1(T)$ since conjugation with an element outside of H changes the fixed point. Therefore there exists a choice of x_1.

If $\overline{M}_2$ has a hyperbolic interior with finite volume then it follows from the same reasoning as before that there is a choice of x_2 so that (a) is satisfied. If $\overline{M}_2$ is a Seifert manifold, all we have to do it to modify the embedding ϕ so that h is not in the center of $\pi_1(\overline{M}_2)$ which is generated by a power of the normal fiber. Therefore under the hypothesis above there are choices of x_1, x_2 and y such that the conditions of Lemma 3.4 are satisfied.

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