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Stability and locally exact differentials on a curve

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Abstract

We show that the locally free sheaf $B_1 \subset F_*(\Omega^1_X)$ of locally exact differentials on a smooth projective curve of genus $g \geqslant 2$ over an algebraically closed field k of characteristic p is a stable bundle. This answers a question of Raynaud. ***To cite this article: K. Joshi, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Stabilité et des différentielles localement exactes sur une courbe. Soit X une courbe propre, lisse, connexe, de genre g, définie sur un corps k algébriquement clos de caractéristique $p > 0$. Soit $F : X \to X$ le Frobenius absolu et $B_1 \subset F_*(\Omega^1_X)$, le faisceau des formes différentielles localement exactes sur X. C'est un fibré vectoriel sur X de rang $p - 1$. Nous montrons qu'il est stable pour $g \geqslant 2$. ***Pour citer cet article : K. Joshi, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

dh̲īre dh̲īre re manā, dh̲īre sab kuch̲ hoye
māli siṇch̲e sau gh̲arā, ritū āye ph̲al hoye
Kabīr

1. Introduction

Let k be an algebraically closed field of characteristic $p > 0$. Let X/k be a smooth, projective curve of genus $g \geqslant 2$ over k. Let $F : X \to X$ be the absolute Frobenius morphism of X. If V is a vector bundle we will write $\mu(V) = \deg(V)/\operatorname{rk}(V)$ for the slope of V. We will say that V is stable (resp. semi-stable) if for all subbundles $W \subset V$ we have $\mu(W) < \mu(V)$ (resp. $\mu(W) \leqslant \mu(V)$). We will write $\Omega^1_X = K_X$ for the canonical sheaf of the curve.

Let B_1 be the locally free sheaf of locally exact differential forms on X. This may also be defined by the exact sequence of locally free sheaves

$$0 \to \mathcal{O}_X \to F_*(\mathcal{O}_X) \to B_1 \to 0.$$

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As X is a curve, by definition, B_1 is a sub-bundle of $F_*(\Omega_X^1)$ of rank $p-1$. In [5] Raynaud showed that B_1 is a semi-stable bundle of degree $(p-1)(g-1)$ and slope $g-1$. Raynaud has asked if B_1 is in fact stable (see [6]). In this Note we answer Raynaud's question. We prove:

Theorem 1.1. *Let k be an algebraically closed field of characteristic $p>0$. Let X be a smooth, projective curve over k of genus g and let B_1 be the locally free sheaf of locally exact differentials on X. Then B_1 is a stable vector bundle of slope $g-1$ and rank $p-1$.*

Observe that when $p=2$, B_1 is a line bundle of degree $g-1$ and so the result is immediate in this case. Our proof also gives a new proof of Raynaud's theorem that B_1 is semi-stable. Theorem 1.1 will follow from the corresponding assertion for curves of sufficiently large genus. More precisely:

Theorem 1.2. *Let k be an algebraically closed field of characteristic $p \neq 2$. Let X be a smooth, projective curve over k of genus g and assume that the genus $g > (1/2)(p-1)(p-2)$. Then B_1 is stable.*

2. The proofs

We will first explain the reduction of Theorem 1.1 to the Theorem 1.2 (I owe this argument to A. Tamagawa).

Proof (Theorem 1.2 $\Rightarrow$ Theorem 1.1). As the genus g_X of X is at least two, we know that there exists a connected finite étale covering of X of arbitrarily large genus. Choose a finite étale covering $f: Y \to X$ such that the genus g_Y of Y is sufficiently large genus (more precisely with $g_Y > (p-1)(p-2)/2$). Observe that the formation of B_1 commutes with any finite étale base change, that is, $B_{1,Y} = f^*(B_{1,X})$. Hence the stability of $B_{1,X}$ follows from that of $B_{1,Y}$ and the latter assertion is the content of Theorem 1.2. □

The rest of this Note will be devoted to the proof of Theorem 1.2. The idea of the proof is to get an upper bound on the slope of the destabilizing subsheaf (if it exists!). We recall some facts which we need. The first two lemmas are from [3] which is not yet published so we provide proofs.

Lemma 2.1. *We have* $\deg(F_*(\Omega_X^1)) = (p+1)(g-1)$.

Proof. By Riemann–Roch theorem $\chi(\Omega_X^1) = \chi(F_*(\Omega_X^1))$. Which gives $\deg(F_*(\Omega_X^1)) + p(1-g) = \deg(\Omega_X^1) + (1-g)$. Simplifying this gives $\deg(F_*(\Omega_X^1)) = 2(g-1) + p(g-1) + (1-g)$, which leads to the claimed result. □

Lemma 2.2. *Let $M \subset F_*(\Omega_X^1)$ be any line subbundle. Then*

$$\mu(M) \leqslant \mu\big(F_*\big(\Omega_X^1\big)\big) - \frac{(p-1)(g-1)}{p}.$$

Proof. This was proved in [3]. By adjunction we get a map

$$F^*(M) \to \Omega_X^1.$$

Hence by degree considerations we see that

$$\mu(M) \leqslant \frac{\deg(\Omega_X^1)}{p} = \mu\big(F_*\big(\Omega_X^1\big)\big) - \frac{(p-1)(g-1)}{p},$$

where we use Lemma 2.1 for the last equality. □

We will also need to identify the dual B_1^* of B_1. This was done by Raynaud in [5].

Lemma 2.3. *The dual* $B_1^* = B_1 \otimes K_X^{-1}$.

Now we recall a theorem of Mukai–Sakai (see [4, p. 251]). This will be used to give upper bounds on destabilizing subbundles along with Lemma 2.2. For more on this see Remark 2.

Theorem 2.4. *Let* X/K *be a smooth, projective curve over an algebraically closed field* K *of arbitrary characteristic. Let* W *be a vector bundle. Fix an integer* $1 \leqslant k \leqslant r = \mathrm{rk}(W)$. *Then there exists a subbundle* $U \subset W$ *of rank* k *such that*

$$\mu(W) \leqslant \mu(U) + g(1 - k/r).$$

Proof of Theorem 1.2. As remarked in the introduction, when $p = 2$, B_1 is a line bundle and so there is nothing to prove in this case. In what follows we will assume that $p \neq 2$. Assume, if possible, that B_1 is not stable. Then there exists some $W \subset B_1$ which is locally free of rank r with $\mu(W) \geqslant \mu(B_1) = g - 1$.

We first claim that we can assume with out loss of generality that $r \leqslant (p-1)/2$. Indeed, if not then the dual of B_1 surjects on W^* the dual of W and the kernel has rank $\leqslant (p-1)/2$. Moreover writing W_1 for the kernel we get an exact sequence

$$0 \to W_1 \to B_1^* = B_1 \otimes K_X^{-1} \to W^* \to 0$$

and writing $W_2 = W_1 \otimes K_X$, we get

$$0 \to W_2 \to B_1 \to W^* \otimes K_X \to 0.$$

Now $\deg(W_1) + \deg(W^*) = \deg(B_1^*) = (p-1)(1-g)$ and a simple calculation using $-\mu(W^*) = \mu(W) \geqslant (g-1)$ shows that $\deg(W_1) \geqslant \mathrm{rk}(W_1)(1-g)$. So that we have

$$\deg(W_2) = \deg(W_1 \otimes K_X) \geqslant \mathrm{rk}(W_1)(1-g) + \mathrm{rk}(W_1)(2g-2),$$

which simplifies to

$$\deg(W_2) \geqslant \mathrm{rk}(W_2)(g-1)$$

or equivalently $\mu(W_2) \geqslant (g-1)$.

Thus we may assume without loss of generality that $\mathrm{rk}(W) \leqslant (p-1)/2$.

We apply Theorem 2.4 to the following situation. We take W as above of slope $\geqslant (g-1)$ with $\mathrm{rk}(W)$ at most $(p-1)/2$ and we take $k = 1$ and let U to be the line bundle given by Theorem 2.4. Then we get

$$\mu(W) \leqslant \mu(U) + g\bigl(1 - 1/\mathrm{rk}(W)\bigr).$$

Now as $U \subset W \subset B^1 \subset F_*(\Omega_X^1)$, we know by Lemma 2.2 that

$$\mu(U) \leqslant \mu\bigl(F_*\bigl(\Omega_X^1\bigr)\bigr) - (p-1)(g-1)/p.$$

Putting these two inequalities together and using $\mu(W) \geqslant g - 1$ we get

$$(g-1) \leqslant \mu(W) \leqslant \mu\bigl(F_*\bigl(\Omega_X^1\bigr)\bigr) - (p-1)(g-1)/p + g\bigl(1 - 1/\mathrm{rk}(W)\bigr). \tag{1}$$

Writing $r = \mathrm{rk}(W)$ this simplifies to

$$(g-1) \leqslant \mu(W) \leqslant \frac{2(g-1)}{p} + g\left(1 - \frac{1}{r}\right) \tag{2}$$

$$\leqslant \left(\frac{2}{p} + 1 - \frac{1}{r}\right)g - \frac{2}{p} \tag{3}$$

and so in particular we get

$$g-1 \leqslant \left(\frac{2}{p}+1-\frac{1}{r}\right)g-\frac{2}{p},$$

which easily simplifies to

$$g \leqslant \frac{r(p-2)}{p-2r},$$

and as r can be as large as $(p-1)/2$, this says that $g \leqslant \frac{1}{2}(p-1)(p-2)$. Now we have assumed that $g > \frac{1}{2}(p-1)(p-2)$ so that our assumption that $W \subset B_1$ has slope $\geqslant g-1$ leads to a contradiction. Thus B_1 is stable. □

Remark 1. In fact, as was pointed out to us by A. Tamagawa, Theorem 1.2 can be slightly sharpened as follows. We claim that the following conditions are equivalent (i) the bundle B_1 is stable; (ii) either $p=2$ or $g>1$.

After Theorem 1.1, the assertion (ii) $\Rightarrow$ (i) is immediate. Now to prove that (i) $\Rightarrow$ (ii) observe that the case $p=2$ is trivial so assume, if possible, that $g \leqslant 1$. By Theorem 2.4 for line bundles applied to B_1, we see that B_1 has a line subbundle $M \hookrightarrow B_1$ with

$$\mu(M) \geqslant \mu(B_1) - g\left(1-\frac{1}{p-1}\right).$$

Then the right-hand side is greater than $(g-1)-1$. As $\mu(M)$ is an integer, this says that $\mu(M) \geqslant (g-1)$ and this contradicts the stability of B_1.

Remark 2. The method of using coverings of large degree can also be used to improve the results of [2] (resp. [3] for small p) where we showed that $F_*(L)$ is stable for any line bundle L on X provided that the genus of X is sufficiently large.

Remark 3. The method of proof given above can also be used to show that the bundles B_n defined in [1] using iterated Cartier operations are stable. These are bundles of rank p^n-1 and slope $g-1$.

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