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Partial Differential Equations

Compactness of solutions to the Yamabe problem

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Abstract

We establish compactness of solutions to the Yamabe problem on any smooth compact connected Riemannian manifold (not conformally diffeomorphic to standard spheres) of dimension $n \leqslant 7$ as well as on any manifold of dimension $n \geqslant 8$ under some additional hypothesis. ***To cite this article: Y.Y. Li, L. Zhang, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Compacité des solutions du problème de Yamabe. On établit la compacité des solutions du problème de Yamabe sur toute variété riemannienne, régulière compacte connexe (non conformément équivalente à la sphère standard) de dimension $n \leqslant 7$. Le même résultat est valable en dimension $n \geqslant 8$ sous une hypothèse supplémentaire. ***Pour citer cet article : Y.Y. Li, L. Zhang, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Soit (M, g) une variété riemannienne régulière, compacte et connexe sans bord de dimension n. On considère l'équation de Yamabe

$$-\Delta_g u + \frac{n-2}{4(n-1)} R_g u = u^{(n+2)/(n-2)}, \quad u > 0, \quad \text{sur } M, \tag{1}$$

où Δ_g désigne l'opérateur de Laplace–Beltrami.

Soit

$$\mathcal{M} = \left\{ u \mid u \in C^2(M),\ u \text{ vérifie } (1) \right\}.$$

On considère les deux cas suivant :

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1°. Dimension de $M \leqslant 7$.
2°. Dimension de $M \geqslant 8$ et $|W_g| > 0$ sur M, où W_g désigne le tenseur de Weyl de g.

Théorème 0.1. *On suppose que (M, g) n'est pas conformément équivalent à la sphère standard. On fait l'hypothèse* 1° *ou* 2°. *Alors il existe une constante C dependant seulement de (M, g) telle que*

$$\|u\|_{L^\infty(M)} \leqslant C \quad \forall u \in \mathcal{M}.$$

1. The Yamabe conjecture

Let (M, g) be an n-dimensional smooth compact Riemannian manifold without boundary. For $n \geqslant 3$, the Yamabe conjecture states that there exist metrics which are pointwise conformal to g and have constant scalar curvature. The Yamabe conjecture was proved through the works of Yamabe [11], Trudinger [10], Aubin [1] and Schoen [8]. Different proofs in the case $n \leqslant 5$ and in the case (M, g) is locally conformally flat were given by Bahri and Brezis [3] and Bahri [2].

Consider the Yamabe equation

$$-\Delta_g u + \frac{n-2}{4(n-1)} R_g u = u^{(n+2)/(n-2)}, \quad u > 0, \quad \text{on } M, \tag{2}$$

where Δ_g denotes the Laplace–Beltrami operator.

Let

$$\mathcal{M} = \{u \mid u \in C^2(M),\ u \text{ satisfies (2)}\}.$$

For $n \geqslant 3$, under the assumption that (M, g) is locally conformally flat and is not conformally diffeomorphic to standard spheres, Schoen proved in 1991, see [9], that for any non-negative integer k,

$$\|u\|_{C^k(M,g)} \leqslant C, \quad \forall u \in \mathcal{M}, \tag{3}$$

where C is some constant depending only on (M, g) and k. He also announced in the same paper the same result for general Riemannian manifolds, without the locally conformally flatness assumption. The proof of this claim has not been made available. For general Riemannian manifolds of dimension $n = 3$, a proof was given by Li and Zhu in [7]; while for dimension $n = 4$, the combination of results of Li and Zhang [5] and Druet [4] yields a proof.

2. New results

We consider the following two cases:

1°. Dimension $n \leqslant 7$.
2°. Dimension $n \geqslant 8$ and $|W_g| > 0$ on M, where W_g denotes the Weyl tensor of g.

Theorem 2.1. *Let (M, g) be an n-dimensional smooth compact connected Riemannian manifold without boundary which is not conformally diffeomorphic to standard spheres. Assume either* 1° *or* 2°. *Then, for any non-negative integer k, there exists some constant C depending only on (M, g) and k such that* (3) *holds.*

Remark 1. Theorem 2.1 was announced in November 2003 by the first author in his talk at the Joint Analysis Seminar in Princeton University.

Remark 2. In fact our proof yields a stronger result: replacing $u^{(n+2)/(n-2)}$ in (2) by u^p with $1 < 1+\varepsilon \leqslant p \leqslant (n+2)/(n-2)$, then positive solutions u satisfy (3) with C depending also on ε. As a result, the total Leray–Schauder degree of solutions is equal to -1.

It is well known that (3) does not hold when (M, g) is conformally diffeomorphic to standard spheres.

For $Q \in M$ and $\lambda > 0$, let

$$\xi_{Q,\lambda}(P) = \big(n(n-2)\big)^{(n-2)/4}\left(\frac{\lambda}{1+\lambda^2 \operatorname{dist}_g(P,Q)^2}\right)^{(n-2)/2}, \quad P \in M,$$

where $\operatorname{dist}_g(P, Q)$ denotes the geodesic distance between P and Q.

Let (M, g) be an n-dimensional smooth compact connected Riemannian manifold without boundary, we first prove that for any $u \in \mathcal{M}$, there exist local maximum points $P_1, \ldots, P_m$ of u such that

$$\operatorname{dist}_g(P_i, P_j) \geqslant \frac{1}{C}, \quad \forall i \neq j,$$
$$\frac{1}{C}u(P_i) \leqslant u(P_j) \leqslant Cu(P_i), \quad \forall i, j,$$

and

$$\frac{1}{C}\sum_{l=1}^{m} \xi_{P_l, u(P_l)^{2/(n-2)}}(P) \leqslant u(P) \leqslant C\sum_{l=1}^{m} \xi_{P_l, u(P_l)^{2/(n-2)}}(P), \quad \forall P \in M,$$

where C is some positive constant depending only on (M, g).

We also establish sharp pointwise estimates on $|\nabla^k(u - \xi_{P_l, u(P_l)^{2/(n-2)}})(P)|$ for $\operatorname{dist}_g(P, P_l) < C^{-1}$ for each $1 \leqslant l \leqslant m$, where C depends only on (M, g).

To prove Theorem 2.1, we only need to prove

$$\|u\|_{L^\infty(M)} \leqslant C, \quad \forall u \in \mathcal{M}, \tag{4}$$

for some C depending only on (M, g), since the rest follows from standard elliptic theories. With the above estimates, we prove (4) by contradiction argument based on the Pohozaev identity and, in the case of $n \leqslant 7$, the positive mass theorem of Schoen and Yau. The detail of the proof of Theorem 2.1 is given in [6].

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