Available online at www.sciencedirect.com
SCIENCE DIRECT®
C. R. Acad. Sci. Paris, Ser. I 338 (2004) 661–666
COMPTES RENDUS MATHEMATIQUE

Mathematical Problems in Mechanics

Travelling breathers in Klein–Gordon chains

Yannick Sire, Guillaume James

Laboratoire MIP (UMR 5640), INSA Toulouse, département GMM, 135, av. Rangueil, 31077 Toulouse cedex 4, France

Received 11 September 2003; accepted after revision 7 January 2004

Presented by Gérard Iooss

Abstract

We study the existence of travelling breathers in Klein–Gordon chains, which describe nonlinear oscillators linearly coupled in a local anharmonic potential. In this work, we consider a case when the period of the breather and the inverse of the velocity are commensurable. In a neighborhood of critical values of velocity and coupling, we show by a center manifold reduction that the infinite-dimensional problem can be locally reduced to a eight-dimensional reversible differential equation. ***To cite this article: Y. Sire, G. James, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Travelling breathers dans les chaînes de Klein–Gordon. Dans cette Note, on étudie l'existence de travelling breathers pour des chaînes de Klein–Gordon, qui décrivent des oscillateurs non linéaires linéairement couplés entre eux et plongés dans un potentiel anharmonique. Dans ce travail, on considère un cas où la période du breather et l'inverse de sa vitesse sont commensurables. Dans un voisinage de valeurs critiques pour le couplage et la vitesse, on montre, par réduction à une variété centrale, que le problème en dimension infinie se réduit localement à une équation différentielle réversible en dimension huit. ***Pour citer cet article : Y. Sire, G. James, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Les chaînes de Klein–Gordon, décrites par (1), sont des réseaux d'oscillateurs non linéaires linéairement couplés à leurs premiers voisins. On suppose que le potentiel anharmonique V est C^k ($k \geqslant 5$) et admet le développement de Taylor (2) en $x = 0$. Différentes preuves ont été données pour l'existence d'ondes progressives (solutions satisfaisant $x_n(\tau) = x_{n-1}(\tau - T)$ où T est une constante) dans des réseaux non linéaires (voir [7,5,4,3] et leurs références). Dans cette Note, on étudie l'existence de solutions plus générales (ondes progressives pulsatoires) vérifiant la condition (3). En particulier, les solutions qui vérifient (3) et $\lim_{\tau\to\pm\infty} x_n(\tau) = 0$ sont appelées *travelling breathers* [2]. On utilise la méthode développée par Iooss et Kirchgässner [7], qui est basée sur une réduction à une variété centrale pour un problème d'évolution avec avance et retard en temps. Pour faciliter la résolution du problème de régularité optimale [7], on choisit comme variables du système $(u_1(t), u_2(t)) = (x_1(\tau), x_2(\tau + T/2))$, avec $\tau = T\,t$. Ce changement de variables conduit au problème d'évolution (5) qui est réversible (il existe une symétrie $R \neq \mathrm{Id}$ qui anticommute avec l'opérateur d'évolution). Pour pouvoir réduire

E-mail addresses: sire@gmm.insa-tlse.fr (Y. Sire), james@gmm.insa-tlse.fr (G. James).

1631-073X/$ – see front matter © 2004 Académie des sciences. Published by Elsevier SAS. All rights reserved.
doi:10.1016/j.crma.2004.01.031

localement le problème à une variété centrale au voisinage de valeurs critiques des paramètres T et γ, on démontre d'abord que la partie centrale du spectre de l'opérateur linéaire L est isolée de la partie hyperbolique et de dimension finie (les valeurs propres $\pm \mathrm{i}q$ vérifient la relation de dispersion (7)). La Fig. 1 donne l'évolution du spectre central dans l'espace des paramètres (γ, T). On définit alors Δ_0 comme l'ensemble des paramètres (γ, T) tels que la partie centrale du spectre soit constituée d'une paire de valeurs propres doubles non semi-simples $\pm \mathrm{i}q_0$ et de deux paires de valeurs propres simples, en éliminant les voisinages de résonances fortes (voir Fig. 1, spectres Σ_1, Σ_8 et Σ_{10}). Pour la partie hyperbolique du spectre, on montre ensuite que le problème d'évolution affine a la propriété de régularité optimale (voir [14], hypothèse (ii), p. 127). On utilise pour cela des estimations obtenues dans [7] (Lemme 3, p. 448). On peut alors appliquer la réduction à une variété centrale [14], qui ramène localement le problème de dimension infinie à un système d'équations différentielles réversible en dimension 8 pour $(\gamma, T) \approx \Delta_0$. En mettant le système sous forme normale (Éq. (9)) et en le tronquant à l'ordre 4, on obtient un système intégrable qui donne des solutions approchées de (1). A l'extérieur de Δ_0 on choisit $(\gamma, T) \approx (\gamma_0, T_0) \in \Delta_0$, tel que L admette 2 paires de valeurs propres simples voisines de $\pm \mathrm{i}q_0$, de parties réelles non nulles. Lorsque le coefficient s_2 défini par (11) est négatif, il existe des solutions de la forme normale tronquée homoclines à des tores de dimension 2. La persistence de telles orbites pour le système complet est pour l'instant un problème ouvert. Pour le système (1), ces solutions devraient constituer la partie principale de travelling breathers superposés à l'infini à une queue oscillante (voir Éq. (10)). La forme normale tronquée possède également des solutions réversibles homoclines à 0, mais nous conjecturons (par analogie avec les résultats de Lombardi [10]) que ces solutions ne persistent pas génériquement pour le système complet. Nos résultats sont résumés dans le théorème suivant.

Théorème 0.1. *On note* $\mathbb{D}$ *l'espace fonctionnel* (6)*. Pour* (γ, T) *dans un voisinage de* Δ_0*, les solutions* U *de* (5) *voisines de* 0 *dans* $\mathbb{D}$ *pour tout* $t \in \mathbb{R}$ *se trouvent sur une variété centrale locale* $\mathcal{M}$ *de dimension* 8*, invariante par le flot de* (5)*. De plus, lorsque* $s_2 < 0$ *et* L *admet* 2 *paires de valeurs propres simples de parties réelles non nulles voisines de* $\pm \mathrm{i}q_0$*, l'équation différentielle réversible sur* $\mathcal{M}$*, mise sous forme normale et tronquée à l'ordre* 4*, admet des orbites homoclines à des* 2*-tores. Pour le système* (1)*, ces solutions devraient constituer la partie principale de travelling breathers superposés à une queue oscillante.*

1. Introduction and statement of the problem

We consider the following system (Klein–Gordon chains)

$$\frac{\mathrm{d}^2 x_n}{\mathrm{d}\tau^2} + V'(x_n) = \gamma(x_{n+1} + x_{n-1} - 2x_n), \quad n \in \mathbb{Z}, \tag{1}$$

where x_n is the displacement of the nth particle from an equilibrium position, V an on-site potential and $\gamma > 0$ a coupling constant. This system describes a chain of particles linearly coupled with their first neighbors, in a local anharmonic potential V. We assume V to be C^k ($k \geqslant 5$) with the following Taylor expansion at $x = 0$

$$V(x) = \frac{1}{2}x^2 - \frac{a}{3}x^3 - \frac{b}{4}x^4 + \text{h.o.t.} \tag{2}$$

Proofs have been given for the existence of spatially localized time-periodic solutions, known as discrete breathers, for small values of γ [11]. Travelling waves form an other class of solutions which satisfy $x_n(\tau) = x_{n-1}(\tau - T)$ for a constant T. Various existence proofs for such solutions (in particular solitary waves) have been obtained in the last few years for Klein–Gordon and Fermi–Pasta–Ulam (FPU) chains (see, e.g., [7,5,4,3] and references therein). The work of Iooss and Kirchgässner [7,5] has shown that center manifold theory is well adapted to the study of small amplitude travelling waves in lattices. A discrete version of this method (applied to the existence of breathers in FPU lattices) has been developped in [9]. Travelling wave solutions of (1) have also been studied numerically (see [1,13] and references therein).

This paper deals with solutions satisfying

$$x_n(\tau) = x_{n-2}(\tau - T), \tag{3}$$

where T is a given number. These solutions consist in travelling pulses, which are exactly translated of 2 sites after a fixed propagation time T. Solutions satisfying (3) and $\lim_{\tau\to\pm\infty} x_n(\tau)=0$ are called *travelling breathers* [2]. Note that one can look for travelling breathers in the form $x_n(\tau)=x(n-c\tau,\tau)$ where $\lim_{\xi\to\pm\infty} x(\xi,\tau)=0$ and x is T-periodic in its second argument. Condition (3) implies the commensurability condition $c=2/T$. Eqs. (1)–(3) reduce to find $(x_1(\tau),x_2(\tau))$ satisfying

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}(x_1,x_2)\\ &= \bigl(-V'(x_1)+\gamma\bigl(x_2(\tau)-2x_1(\tau)+x_2(\tau+T)\bigr),\ -V'(x_2)+\gamma\bigl(x_1(\tau)-2x_2(\tau)+x_1(\tau-T)\bigr)\bigr). \qquad (4)\end{aligned}$$

Note that travelling wave solutions of (1) satisfying $x_n(\tau)=x_{n-1}(\tau-T/2)$ are particular solutions of (3). Consequently, the solutions considered in our case include those found by Iooss and Kirchgässner [7]. We shall analyze small amplitude solutions of (4) using the center manifold reduction method introduced by Iooss and Kirchgässner in the context of nonlinear lattices. For this purpose, one has to make a convenient choice of variables which allows us to recover an essential optimal regularity result in the reduction process. We rescale (4) using $t=\tau/T$ and consider the new variable $(u_1(t),u_2(t))=(x_1(\tau),x_2(\tau+\frac{T}{2}))$. As in [7] we set $U=(u_1,u_2,\dot u_1,\dot u_2,X_1(t,v),X_2(t,v))$ where $v\in[-1/2,1/2]$ and $X_i(t,v)=u_i(t+v)$, $i=1,2$. We define the operator $\delta_a X_i(t,\cdot)=X_i(t,a)$. We then write the system (4) as an evolution problem

$$\frac{\mathrm{d}U}{\mathrm{d}t}=LU+F(U), \qquad (5)$$

where the linear operator L is given by $LU=(\dot u_1,\dot u_2,\alpha_1u_1+\alpha_2(\delta_{1/2}+\delta_{-1/2})X_2,\alpha_1u_2+\alpha_2(\delta_{1/2}+\delta_{-1/2})X_1,$ $\partial_v X_1,\partial_v X_2)$, with $\alpha_1=-T^2(1+2\gamma)$, $\alpha_2=T^2\gamma$. The nonlinear operator F is $F(U)=T^2(0,0,f(u_1),f(u_2),0,0)$ with $f(u)=au^2+bu^3+\text{h.o.t.}$ Solutions of (5) satisfy $U\in C^0(\mathbb{R},\mathbb{D})\cap C^1(\mathbb{R},\mathbb{H})$, where

$$\mathbb{H}=\mathbb{R}^4\times\bigl(C^0[-1/2,1/2]\bigr)^2,\quad \mathbb{D}=\bigl\{U\in\mathbb{R}^4\times\bigl(C^1[-1/2,1/2]\bigr)^2/X_1(0)=u_1,X_2(0)=u_2\bigr\}. \qquad (6)$$

The operator L maps $\mathbb{D}$ into $\mathbb{H}$ continuously and $F:\mathbb{D}\to\mathbb{D}$ is C^{k-1} with $F(U)=\mathrm{O}(\|U\|^2_{\mathbb{D}})$. We observe that the symmetry R on $\mathbb{H}$ defined by $R(u_1,u_2,\xi_1,\xi_2,X_1(v),X_2(v))=(u_1,u_2,-\xi_1,-\xi_2,X_1(-v),X_2(-v))$ satisfies $(L+F)R=-R(L+F)$. Therefore, system (5) is reversible under R. In addition, note that the permutational symmetry $S(u_1,u_2,\xi_1,\xi_2,X_1,X_2)=(u_2,u_1,\xi_2,\xi_1,X_2,X_1)$ commutes with $L+F$ (travelling wave solutions are fixed points of S).

The problem (5) is ill-posed as an initial value problem in $\mathbb{D}$. Nevertheless, it is possible to construct bounded solutions for all $t\in\mathbb{R}$. Using center manifold theory as in [7], we are able to reduce (5) locally to a finite-dimensional system of ordinary differential equations (eight-dimensional in our parameter range).

2. Analytical results

We first study the spectrum of L, which consists in isolated eigenvalues with finite multiplicities. Solving $L\,U=\sigma\,U$ yields the dispersion relation $N(\sigma,T,\gamma):=(\sigma^2+T^2(1+2\gamma))^2-4(\gamma T^2)^2\cosh^2(\sigma/2)=0$. Since L has real coefficients and due to reversibility, its spectrum is invariant under the reflections on the real and the imaginary axis. Therefore, we can restrict our analysis to the case $\sigma=p+\mathrm{i}q$ with $(p,q)\in\mathbb{R}^2_+$. Due to the invariance $\tau\to-\tau$ of (1), we also restrict ourselves to the case $T>0$. For the central part of the spectrum ($\sigma=\mathrm{i}q$), the dispersion relation reads

$$\bigl(-q^2+T^2(1+2\gamma)\bigr)^2=4\bigl(\gamma T^2\bigr)^2\cos^2(q/2). \qquad (7)$$

In what follows we study the solutions of (7). For $\gamma\,T^2<4$, the spectrum of L on the imaginary axis consists in two pairs of simple eigenvalues (or one pair of double semi-simple eigenvalues for particular parameter values, see curves Σ_{11} in Fig.1). In the following lemma we consider the occurence of double eigenvalues.

Lemma 2.1. *Consider the curve Γ parametrized by $(\gamma(q), T(q))$ with $q \in \mathbb{R}^+$ and T, γ defined by the system $N(\mathrm{i}q, \gamma, T) = 0$ and $\mathrm{d}N(\mathrm{i}q, \gamma, T)/\mathrm{d}q = 0$. This curve (which we call bifurcation curve) is given by*

$$T^2 = q^2 - 4q \tan\left(\frac{q}{4}\right), \quad \gamma = \frac{2q}{T^2 \sin(q/2)}, \quad \textit{for } q \in \big(4k\pi, (2k+1)2\pi\big),$$
$$T^2 = q^2 + \frac{4q}{\tan(q/4)}, \quad \gamma = -\frac{2q}{T^2 \sin(q/2)}, \quad \textit{for } q \in \big((2k-1)2\pi, 4k\pi\big),$$

where $k \geqslant 1$ is an integer. The range of q is determined by the condition $T^2 > 0$.

For $(\gamma, T) \in \Gamma$, except on a countable set of points (e.g., cusps, double points), the spectrum of L on the imaginary axis consists in a pair of double nonsemi-simple eigenvalues $\pm\mathrm{i}q$ and at least two distinct pairs of simple eigenvalues.

We denote by Γ_k the restriction of Γ to the interval $q \in (2k\pi, 2(k+1)\pi)$. The spectrum of L on the imaginary axis as a function of (γ, T) is sketched in Fig. 1 (Γ lies above the curve $\gamma T^2 = 4$). We shall see that small amplitude solutions which bifurcate in the neighborhood of Γ_{2k} include travelling wave solutions of [7]. These solutions can be combined with an additional mode corresponding to an extra pair of simple eigenvalues on the imaginary axis. On the contrary, small amplitude solutions which bifurcate in the neighborhood of Γ_{2k+1} mainly consist (apart from spatially periodic travelling waves) in travelling pulse solutions not described in [7]. In what follows, we define Δ as the subset of Γ such that the central part of the spectrum is $\Sigma = \{\pm\mathrm{i}q_1, \pm\mathrm{i}q_2, \pm\mathrm{i}q_0\}$, where $\pm\mathrm{i}q_0$ is a pair of double nonsemi-simple eigenvalues (eigenvector $V_0 = (\varepsilon, 1, \varepsilon\mathrm{i}q_0, \mathrm{i}q_0, \varepsilon\,\mathrm{e}^{\mathrm{i}q_0 v}, \mathrm{e}^{\mathrm{i}q_0 v})$, generalized eigenvector $\widehat{V_0} = (0, 0, \varepsilon, 1, \varepsilon v\,\mathrm{e}^{\mathrm{i}q_0 v}, v\,\mathrm{e}^{\mathrm{i}q_0 v})$, $\varepsilon = (-1)^k$ for $(\gamma, T) \in \Gamma_k$) and $\pm\mathrm{i}q_1, \pm\mathrm{i}q_2$ two pairs of simple ones (eigenvectors $V_1 = (-1, 1, -\mathrm{i}q_1, iq_1, -\mathrm{e}^{\mathrm{i}q_1 v}, \mathrm{e}^{\mathrm{i}q_1 v})$, $V_2 = (1, 1, \mathrm{i}q_2, \mathrm{i}q_2, \mathrm{e}^{\mathrm{i}q_2 v}, \mathrm{e}^{\mathrm{i}q_2 v})$). This parameter set corresponds to regions Σ_1, Σ_8, Σ_{10} in Fig. 1. In the following we fix $(\gamma, T) \in \Delta$.

As in [7], L is not sectorial and the central part of its spectrum is isolated from the hyperbolic part. We define P as the spectral projection on the 8-dimensional L-invariant central subspace associated to Σ and use the notations $\mathbb{D}_h = (\mathbb{I} - P)\mathbb{D}$, $\mathbb{H}_h = (\mathbb{I} - P)\mathbb{H}$, $\mathbb{D}_c = P\mathbb{D}$, $U_h = (\mathbb{I} - P)U$, $U_c = PU$. The property of optimal regularity (see [14], hypothesis (ii) p. 127) is fulfilled by the affine linearized system $\frac{\mathrm{d}U_h}{\mathrm{d}t} = LU_h + (\mathbb{I} - P)g(t)$ on $\mathbb{H}_h$, where $g(t) = (0, 0, g_1(t), g_2(t), 0, 0)$ lies in the range of the nonlinear term F in Eq. (5) (this part of the analysis is similar to [7], Lemma 3, p. 448). Thanks to this property one has the following center manifold reduction result [14].

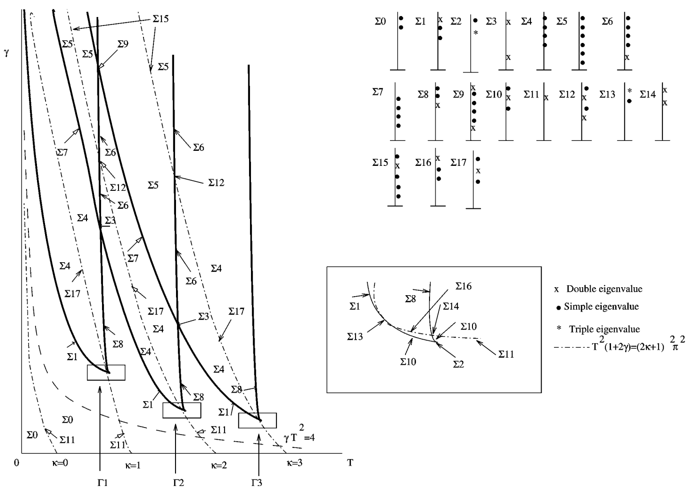

Fig. 1. Bifurcation curves and purely imaginary eigenvalues of L (upper half complex plane).

Fig. 1. Courbes de bifurcation et valeurs propres imaginaires pures de L (demi-plan supérieur).

Theorem 2.2. *Fix $(\gamma_0, T_0) \in \Delta$. There exists a neighborhood $\mathcal{U} \times \mathcal{V}$ of $(0, \gamma_0, T_0)$ in $\mathbb{D} \times \mathbb{R}^2$ and a map $\psi \in C_b^{k-1}(\mathbb{D}_c \times \mathbb{R}^2, \mathbb{D}_h)$ such that the following properties hold for all $(\gamma, T) \in \mathcal{V}$ (with $\psi(0, \gamma, T) = 0$, $D\psi(0, \gamma_0, T_0) = 0$).*

- *If $U : \mathbb{R} \to \mathbb{D}$ solves* (5) *and $U(t) \in \mathcal{U}$ $\forall t \in \mathbb{R}$, then $U_h(t) = \psi(U_c(t), \gamma, T)$ for all $t \in \mathbb{R}$ and $U_c : \mathbb{R} \to \mathbb{D}_c$ is a solution of*

$$\frac{\mathrm{d}U_c}{\mathrm{d}t} = LU_c + PF\big(U_c + \psi(U_c, \gamma, T)\big). \tag{8}$$

- *If U_c is a solution of* (8) *with $U_c(t) \in \mathcal{U}$ $\forall t \in \mathbb{R}$, then $U = U_c + \psi(U_c, \gamma, T)$ is a solution of* (5).
- *The map $\psi(\cdot, \gamma, T)$ commutes with R and S, and* (8) *is reversible under R and S-equivariant.*

In the following we study the reduced equation (8). According to normal form theory (see, e.g., [6]), one can perform a polynomial change of variables $U_c = \widetilde{U}_c + \widetilde{P}_{\gamma,T}(\widetilde{U}_c)$ close to the identity which simplifies the reduced equation (8) and preserves its symmetries. In the sequel we set $\widetilde{U}_c = AV_0 + B\widehat{V}_0 + CV_1 + DV_2 + \text{c.c.}$ To compute the normal form, we exclude points of Δ which are close to points where $sq_0 + rq_1 + r'q_2 = 0$ for $s, r, r' \in \mathbb{Z}$ and $0 < |s| + |r| + |r'| \leqslant 4$ (such values correspond to strong resonances), and denote this new set as Δ_0. The normal form computation is very similar to [7] (Section 6 and Appendix 2), to which we refer for details. The normal form of (8) at order 3 is given in the following lemma.

Lemma 2.3. *The normal form of* (8) *at order* 3 *reads*

$$\begin{aligned}
&\frac{\mathrm{d}A}{\mathrm{d}t} = \mathrm{i}q_0 A + B + iA\mathcal{P}(u_1, u_2, u_3, u_4) + h.o.t., \qquad \frac{\mathrm{d}C}{\mathrm{d}t} = \mathrm{i}q_1 C + \mathrm{i}C\mathcal{Q}(u_1, u_2, u_3, u_4) + h.o.t.,\\
&\frac{\mathrm{d}B}{\mathrm{d}t} = \mathrm{i}q_0 B + \mathrm{i}B\mathcal{P}(u_1, u_2, u_3, u_4) + A\mathcal{S}(u_1, u_2, u_3, u_4) + h.o.t.,\\
&\frac{\mathrm{d}D}{\mathrm{d}t} = \mathrm{i}q_2 D + \mathrm{i}D\mathcal{T}(u_1, u_2, u_3, u_4) + h.o.t.,
\end{aligned} \tag{9}$$

where $u_1 = A\bar{A}$, $u_2 = C\overline{C}$, $u_3 = D\overline{D}$, $u_4 = \mathrm{i}(A\overline{B} - \bar{A}B)$ and $\mathcal{P}$, $\mathcal{S}$, $\mathcal{Q}$, $\mathcal{T}$ are affine functions of their arguments, with smoothly parameters dependent real coefficients for (γ, T) in the neighborhood of Δ_0. Higher order terms are $\mathrm{O}((|A| + |B| + |C| + |D|)^4)$.

In the sequel we set $\mathcal{S}(u_1, u_2, u_3, u_4) = s_1 + s_2 u_1 + s_3 u_2 + s_4 u_3 + s_5 u_4$ (note that s_1 vanishes on Δ_0). The study of the normal form truncated at order 4 directly follows from the classical 1 : 1 resonance case [8], since $|C|^2$ and $|D|^2$ are first integrals of the truncated system. The solutions of system (9) truncated at order 4 yield approximate (leading order) solutions of the Klein–Gordon system

$$x_n(\tau) \approx \big[(-1)^{nm} A + (-1)^n C + D\big]\left(\frac{\tau}{T} - \frac{n-1}{2}\right) + \text{c.c.} \tag{10}$$

for (γ_0, T_0) in $\Gamma_m \cap \Delta_0$, i.e., $q_0 \in (2m\pi, 2(m+1)\pi)$. Their persistence for the full system (exact solutions) can be analyzed on the finite-dimensional normal form using the approach of Lombardi [10], and will be considered in future works. We choose $(\gamma, T) \approx (\gamma_0, T_0)$ ($(\gamma_0, T_0) \in \Delta_0$), in such a way that the linearized operator L has four symmetric eigenvalues close to $\pm \mathrm{i}q_0$ and having nonzero real parts ($s_1(\gamma, T) > 0$). The existence of homoclinic orbits to 0 for the truncated normal form (with $C = D = 0$) is linked to the sign of s_2. Following the classical normal form computation scheme (see [7], p. 457), we find

$$\left(2 - \frac{q_0}{\tan(q_0/2)}\right)s_2 = T_0^2\left(6b + 8a^2 - \frac{4a^2T_0^2}{2\gamma_0 T_0^2 \cos(q_0) - T_0^2(1 + 2\gamma_0) + 4q_0^2}\right). \tag{11}$$

Note that for an even potential V ($a = 0$), s_2 has the sign of b on the left branch of Γ_m (i.e., at the left of the cusp, see Fig. 1) and the sign of $-b$ on the right branch.

Provided $s_2(\gamma_0, T_0) < 0$, the truncated normal form system admits homoclinic solutions to 0. By analogy with the results of Lombardi [10] on the $(\mathrm{i}q_0)^2(\mathrm{i}q_1)$ resonance case, we conjecture that the reversible homoclinic orbits to 0 do not persist generically for the full vector field (9). In addition the truncated normal form admits solutions which are homoclinic to small quasi-periodic orbits. We briefly describe the associated approximate solutions of the Klein–Gordon system. Homoclinic solutions bifurcating in the neighborhood of $\Gamma_{2m} \cap \Delta_0$ can be seen as superpositions of a travelling wave of permanent form $x_{TW}(\tau) = (A + D)(\frac{\tau}{T} - \frac{n-1}{2})$ and a travelling pulse $x_{TP}(\tau) = (-1)^n C(\frac{\tau}{T} - \frac{n-1}{2})$. If $|C| \ll |A(0)|$, the pulsating part x_{TP} is mainly visible at the wave tail. Note that pure travelling waves (with $C = 0$, $D \neq 0$) exist for the full system [7]. Moreover, homoclinic solutions bifurcating in the neighborhood of $\Gamma_{2m+1} \cap \Delta_0$ can be seen as superpositions of a travelling pulse $x_{TP}(\tau) = (-1)^n (A + C)(\frac{\tau}{T} - \frac{n-1}{2})$ and a travelling wave of permanent form $x_{TW}(\tau) = D(\frac{\tau}{T} - \frac{n-1}{2})$. For $|C|, |D| \ll |A(0)|$, the wave consists mainly in a $\mathrm{O}(|A|)$ pulsating part localized at the centre (travelling breather) and superposed to a small quasi-periodic tail. We sum up our findings in the following theorem.

Theorem 2.4. *Assume $s_2(\gamma_0, T_0) < 0$ for $(\gamma_0, T_0) \in \Delta_0$, and consider $(\gamma, T) \approx (\gamma_0, T_0)$ such that L has four symmetric eigenvalues close to $\pm \mathrm{i}q_0$ and having nonzero real parts. Then Eq.* (8) *written in normal form and truncated at order* 4 *admits homoclinic orbits to* 2*-dimensional tori. Such solutions should correspond to the principal part of travelling breather solutions of* (1)*, superposed at infinity to an oscillatory tail.*

Note that the existence of modulated plane waves in Klein–Gordon chains has been studied by Remoissenet [12] using formal multiscale expansions. Under this approximation, the wave enveloppe satisfies the nonlinear Schrödinger (NLS) equation. The condition obtained by the author for the existence of NLS solitons (with the specific wave number $k = q_0/2 - m\pi$) is exactly the condition $s_2 < 0$ previously derived.

Acknowledgements

Y. Sire is grateful to S. Aubry for his hospitality at the CEA Saclay and stimulating discussions. This work has been supported by the European Union under the RTN project LOCNET (HPRN-CT-1999-00163).

References

[1] S. Aubry, T. Crétégny, Physica D 119 (1998) 34–46.
[2] S. Flach, K. Kladko, Physica D 127 (1999) 61–72.
[3] G. Friesecke, R.L. Pego, Nonlinearity 12 (1999) 1601–1627.
[4] G. Friesecke, J.A. Wattis, Commun. Math. Phys. 161 (1994) 391–418.
[5] G. Iooss, Nonlinearity 13 (2000) 849–866.
[6] G. Iooss, M. Adelmeyer, Topics in Bifurcation Theory and Applications, in: Adv. Ser. Nonlinear Dynam., vol. 3, World Sci. Publishing, 1992.
[7] G. Iooss, K. Kirchgässner, Commun. Math. Phys. 211 (2000) 439–464.
[8] G. Iooss, M.-C. Pérouème, J. Differential Equations 102 (1993) 62–88.
[9] G. James, J. Nonlinear Sci. 13 (2003) 27–63.
[10] E. Lombardi, Phenomena beyond all orders and bifurcations of reversible homoclinic connections near higher resonances, in: Lecture Notes in Math., vol. 1741, Springer-Verlag, 2000.
[11] R.S. MacKay, S. Aubry, Nonlinearity 7 (1994) 1623–1643.
[12] M. Remoissenet, Phys. Rev. B 33 (1986) 2386.
[13] A.V. Savin, Y. Zolotaryuk, J.C. Eilbeck, Physica D 138 (2000) 267–281.
[14] A. Vanderbauwhede, G. Iooss, Center manifold theory in infinite dimensions, Dynam. Report (N.S.) (1992) 125–163.