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C. R. Acad. Sci. Paris, Ser. I 338 (2004) 521–526
COMPTES RENDUS MATHEMATIQUE

Potential Theory/Probability Theory

Beurling–Deny formula of semi-Dirichlet forms ☆

Ze-Chun Hu [a], Zhi-Ming Ma [b]

[a] *College of Mathematics, Sichuan University, Chengdu 610064, China*
[b] *Inst. Appl. Math., Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, China*

Received 4 January 2004; accepted 7 January 2004

Presented by Paul Malliavin

Abstract

In this Note we announce a structure result for non-symmetric Dirichlet forms and semi-Dirichlet forms. Our result is regarded as an extension of the celebrated Beurling–Deny formula which is up to now available only for symmetric Dirichlet forms. The result can also be regarded as an extension of Lévy–Khinchine formula or more generally, an extension of Courrège's Theorem in the semi-Dirichlet forms setting. ***To cite this article: Z.-C. Hu, Z.-M. Ma, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Formule de Beurling–Deny pour demi-forme de Dirichlet. Dans cette Note nous annonçons un résultat de structure pour formes de Dirichlet non-symétrique et demi-forme de Dirichlet. Notre résultat peut être considéré comme une extension de la célèbre formule de Beurling–Deny qui est, jusqu'à maintenant, valable pour formes de Dirichlet symétrique seulement. Le résultat peut être considéré aussi comme une extension de la formule de Lévy–Khinchine, ou plus généralement, une extension du Théorème de Courrège. ***Pour citer cet article : Z.-C. Hu, Z.-M. Ma, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version francaise abrégée

Soient E un espace de Lusin métrisable (ou plus généralement, un espace de co-Souslin métrisable), m une mesure σ-finie sur la tribu Borelienne $\mathcal{B}=\mathcal{B}(E)$. Soit $(\mathcal{E}, D(\mathcal{E}))$ une demi-forme de Dirichlet quasi-régulière sur $L^2(E;m)$, au sens de [11] et [7]. Nous montrons qu'il existe une mesure (positive) Borélienne σ-finie unique J sur $E\times E\setminus d$ et une mesure (positive) Borélienne unique K sur E, ne chargeant aucun sous-ensemble dont la projection est $\mathcal{E}$-exceptionelle, telles que $\mathcal{E}(u,v)=\int_{E\times E\setminus d}2(u(y)-u(x))v(y)J(\mathrm{d}x,\mathrm{d}y)+\int_E u(x)v(x)K(\mathrm{d}x)$, pourvu que u soit une constante $\mathcal{E}$-quasi-partout sur un ensemble quasi-ouvert contenant un quasi-support de v. En outre, il existe une métrique ρ quasi-compatible et un noyau $\widetilde{D}_b$ tels que pour tous $u,v\in\widetilde{D}_b$ et $\varepsilon>0$, $\mathcal{E}(u,v)=\mathcal{E}^{\rho,\varepsilon}(u,v)+\int_{\{\rho>\varepsilon\}}2(u(y)-u(x))v(y)J(\mathrm{d}x,\mathrm{d}y)+\int_E u(x)v(x)K(\mathrm{d}x)$. Si $(u(y)-u(x))v(y)$ est $S.P.V.$ intégrable par rapport à J, alors $\lim_{\varepsilon\downarrow 0}\mathcal{E}^{\rho,\varepsilon}(u,v)=\mathcal{E}^c(u,v)$, où $\mathcal{E}^c(u,v)$ est indépendant de ρ et a la property locale forte de gauche.

☆ Research supported by 973 project and NSFC.
E-mail addresses: huzc@mail.amss.ac.cn (Z.-C. Hu), mazm@mail.amt.ac.cn (Z.-M. Ma).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2004.01.020

De façon semblable au cas des processus de Lévy, nous avons quelques conditions suffisantes imposées sur la mesure de saut J qui assurent que $(u(y)-u(x))v(y)$ est $S.P.V.$ intégrable pour tous u, v dans un noyau spécial inclus dans $\widetilde{D}_b$.

1. Semi-Dirichlet forms

Let E be a metrizable Lusin space (or more generally a metrizable co-Souslin space), m a σ-finite measure on the Borel σ-field $\mathcal{B}=\mathcal{B}(E)$. Let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-regular semi-Dirichlet form on $L^2(E;m)$ in the sense of [11] and [7]. (Note that by the above quoted references $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular if and only if it is associated with a right process.) Recall [12,11] that an increasing sequence $\{F_k\}_{k\in\mathbb{N}}$ of closed subsets of E is called an *$\mathcal{E}$-nest* if $\bigcup_{k\geqslant 1} D(\mathcal{E})_{F_k}$ is $\widetilde{\mathcal{E}}_1$-dense in $D(\mathcal{E})$, where $D(\mathcal{E})_{F_k} := \{u\in D(\mathcal{E})\colon\ u=0\ m\text{-}a.e.\ on\ F_k^c := E\setminus F_k\}$ and $\widetilde{\mathcal{E}}_1(u,v) := \frac{1}{2}[\mathcal{E}(u,v)+\mathcal{E}(v,u)]+(u,v)_{L^2(E;m)}$. A subset $N\subset E$ is called *$\mathcal{E}$-exceptional* if $N\subset\bigcap_{k\geqslant 1} F_k^c$ for some $\mathcal{E}$-nest $\{F_k\}_{k\in\mathbb{N}}$. A property is said to hold *$\mathcal{E}$-q.e.*(*$\mathcal{E}$-quasi-everywhere*) on E, if it holds on $E\setminus N$ for some $\mathcal{E}$-exceptional set N. A function f $\mathcal{E}$-$q.e.$ defined on E is said to be *$\mathcal{E}$-quasi-continuous* if there is an $\mathcal{E}$-nest $\{F_k\}_{k\in\mathbb{N}}$ such that $f\in C(\{F_k\})$, i.e. f is well defined and continuous on F_k for each $k\in\mathbb{N}$. By the virtue of quasi-regularity [12,11], every element $u\in D(\mathcal{E})$ admits an $\mathcal{E}$-quasi-continuous m-version. We shall denote by $\widetilde{D}(\mathcal{E})$ all the $\mathcal{E}$-quasi-continuous versions of elements in $D(\mathcal{E})$.

Following [8], we say that a subset $A\subset E$ is *quasi-open* (resp. *quasi-closed*) if there exists an $\mathcal{E}$-nest $\{F_k\}_{k\in\mathbb{N}}$ such that $F_k\cap A$ is relatively open (resp. relatively closed) in F_k for each $k\in\mathbb{N}$. Let u be an m-a.e. defined function on E, then there exists a smallest (up to an $\mathcal{E}$-exceptional set) quasi-closed set F, which is called the quasi-support of u and is denoted by $\mathrm{supp}_q[u]$, such that $\int_{E\setminus F}|u(x)|m(\mathrm{d}x)=0$.

In this Note we introduce the following concepts.

A metric ρ on E will be called a *quasi-compatible metric* if the Borel σ-algebra induced by ρ coincides with $\mathcal{B}(E)$ and there exists an $\mathcal{E}$-nest $\{F_k\}_{k\in\mathbb{N}}$ such that ρ is compatible with the original trace topology on F_k for each $k\in\mathbb{N}$. Note that all the above quasi-notions are invariant under the change of topology with a quasi-compatible metric.

A subset D of $\widetilde{D}(\mathcal{E})$ is called a *core* of $(\mathcal{E}, D(\mathcal{E}))$ if the following conditions are fulfilled:

(C.1) D is a linear lattice and $u\in D$ implies $u^+\wedge 1\in D$;
(C.2) D is dense in $D(\mathcal{E})$ w.r.t. $\widetilde{\mathcal{E}}_1$-norm;
(C.3) There exist a countable family $\{u_n\}_{n\in\mathbb{N}}\subset D$ and an $\mathcal{E}$-exceptional set N such that $\{u_n\}$ separates the points of $E\setminus N$ and $\sup_n u_n(x)>0$ for all $x\in E\setminus N$. D is said to be a *special core* if in addition to (C.1)–(C.3), it satisfies also;
(C.4) For all $v\in D$ there exists $u\in D\cap I(v)$ such that $u=1$ $\mathcal{E}$-$q.e.$ on $\mathrm{supp}_q[v]$, where

$$I(v) := \bigl\{u\in\widetilde{D}(\mathcal{E}) : u \text{ is constant } \mathcal{E}\text{-}q.e. \text{ on a quasi-open set containing } \mathrm{supp}_q[v]\bigr\}. \tag{1}$$

Let J be a σ-finite Borel measure on $E\times E\setminus d$ (d: diagonal). A measurable function f is said to be integrable with respect to J in the sense of *symmetric principle value* (abbreviated by *S.P.V. integrable*), if for any increasing sequence $\{A_n\}_{n\geqslant 1}$ of subsets of $E\times E\setminus d$ with $J((E\times E\setminus d)\setminus(\bigcup_n A_n))=0$, $J(A_n)<\infty$, and $I_{A_n}(x,y)=I_{A_n}(y,x)$ for all $x,y\in E$, $n\in\mathbb{N}$, it holds that f is integrable on each A_n and the limit

$$S.P.V.\int_{E\times E\setminus d} f(x,y)J(\mathrm{d}x,\mathrm{d}y) := \lim_{n\to\infty}\int_{A_n} f(x,y)J(\mathrm{d}x,\mathrm{d}y)$$

exists and is independent of the sequence $\{A_n\}_{n\in\mathbb{N}}$.

We can now state our results.

Theorem 1.1. *There exist unique (positive) σ-finite Borel measure J on $E \times E \setminus d$ and unique (positive) σ-finite Borel measure K on E satisfying the following two properties.*

(i) $J(N \times E \setminus d) = J(E \times N \setminus d) = 0$ *and* $K(N) = 0$ *for any $\mathcal{E}$-exceptional set N.*
(ii) *For any $v \in \widetilde{D}(\mathcal{E})$ and $u \in I(v)$, we have*

$$\mathcal{E}(u,v) = \int_{E\times E\setminus d} 2\big(u(y)-u(x)\big)v(y)J(\mathrm{d}x,\mathrm{d}y) + \int_E u(x)v(x)K(\mathrm{d}x), \tag{2}$$

where $I(v)$ is defined by (1).

Remark 1. As in the situation of Beurling–Deny formula, J describes the jumps and K describes the killing inside E of the sample paths of the associated Markov process. For details see [10].

Proposition 1.2. *Denote by D^* all the elements $u \in \widetilde{D}(\mathcal{E})$ such that*

$$\int_{E\times\{u\neq 0\}\setminus d} \big(u(y)-u(x)\big)^2 J(\mathrm{d}x,\mathrm{d}y) + \int_E u^2(x)K(\mathrm{d}x) < \infty. \tag{3}$$

Then D^* is dense in $D(\mathcal{E})$. Moreover, D^* contains a special core $\widetilde{D}$ (i.e., $\widetilde{D}$ satisfies (C.1)–(C.4)).

Proposition 1.3. *There exists a quasi-compatible metric ρ and a special core $\widetilde{D} \subset D^*$ satisfying the following two properties*:

(ρ.1) $\int_{E\times\{u\neq 0\}\setminus d} \rho^2(x,y)J(\mathrm{d}x,\mathrm{d}y) < \infty$, *for all $u \in \widetilde{D}$.*
(ρ.2) *Let D_ρ be all the bounded elements $u \in \widetilde{D}$ which are $\mathcal{E}$-q.e. ρ-Lipschitz in the sense that*

$$\big|u(y)-u(x)\big| \leqslant C\rho(x,y), \quad \forall x,y \in E\setminus N$$

for some constant C and some $\mathcal{E}$-exceptional set N, then D_ρ is again a special core.

In what follows we fix a metric ρ and a special core $\widetilde{D}$ satisfying Proposition 1.3. Denote by $\widetilde{D}_b$ (resp. D_b^*) all the bounded elements in $\widetilde{D}$ (resp. D^*).

Theorem 1.4. (i) *Let $u, v \in D_b^*$. If $(u(y)-u(x))v(y)$ is S.P.V. integrable w.r.t. J, then we have the following unique decomposition*

$$\mathcal{E}(u,v) = \mathcal{E}^c(u,v) + S.P.V.\int_{E\times E\setminus d} 2\big(u(y)-u(x)\big)v(y)J(\mathrm{d}x,\mathrm{d}y) + \int_E u(x)v(x)K(\mathrm{d}x), \tag{4}$$

where $\mathcal{E}^c$ satisfies left strong local property in the sense that if $u \in I(v)$, then decomposition (4) *holds and $\mathcal{E}^c(u,v) = 0$.*

(ii) *For any $u, v \in \widetilde{D}_b$ and $\varepsilon > 0$, we have the decomposition*:

$$\mathcal{E}(u,v) = \mathcal{E}^{\rho,\varepsilon}(u,v) + \int_{\{(x,y)\in E\times E\setminus d:\ \rho(x,y)>\varepsilon\}} 2\big(u(y)-u(x)\big)v(y)J(\mathrm{d}x,\mathrm{d}y) + \int_E u(x)v(x)K(\mathrm{d}x). \tag{5}$$

If $(u(y)-u(x))v(y)$ is S.P.V. integrable w.r.t. J, then

$$\lim_{\varepsilon\downarrow 0} \mathcal{E}^{\rho,\varepsilon}(u,v) = \mathcal{E}^c(u,v), \tag{6}$$

where $\mathcal{E}^c(u,v)$ is specified in (4).

The above theorem is an extension of Beurling–Deny formula [3,8,1,6]. It can also be regarded as an extension of Lévy–Khinchine formula [2,13], or more generally Courrège's Theorem ([5]) in semi-Dirichlet forms setting. As in the case of Lévy processes, we can find some sufficient conditions to ensure that the decomposition (4) holds for all u, v in a special core. To this end let us write $\hat{J}(\mathrm{d}x, \mathrm{d}y) := J(\mathrm{d}y, \mathrm{d}x)$. We say that J is *symmetric* if $J = \hat{J}$. In general J is not symmetric and $J - \hat{J}$ is a generalized signed measure which is well defined and finite on each A_n for some countable partition $\{A_n\}_{n\in\mathbb{N}}$ of $E \times E \setminus d$. Denote by $J_1 := (J - \hat{J})^+$ the positive part of the Jordan decomposition of $(J - \hat{J})$. Let $J_0 := J - J_1$. One can check that J_0 is the largest symmetric σ-finite positive measure dominated by J. In particular if J itself is symmetric then $J = J_0$.

Theorem 1.5. (i) *If $J_1(E \times E \setminus d) < \infty$, then $(u(y) - u(x))v(y)$ is S.P.V. integrable w.r.t. J and hence* (4) *holds for all $u, v \in D_b^*$.*

(ii) *If we can find a quasi-compatible metric ρ satisfying $(\rho.1)$ and $(\rho.2)$ of Proposition 1.3 and satisfying further*

$(\rho.3)$ $\int_{E\times\{v\neq 0\}\setminus d} \rho(x, y) \wedge 1 \, J_1(\mathrm{d}x, \mathrm{d}y) < \infty$ *for all $v \in D_\rho$,*

then $(u(y) - u(x))v(y)$ is S.P.V. integrable w.r.t. J and hence (4) *holds for all $u, v \in D_\rho$, where D_ρ is specified by $(\rho.2)$.*

Remark 2. Theorem 1.5(i) can be slightly weakened as follows:

Let $D_0 \subset D_b^*$ be a special core. If $J_1(E \times \{v \neq 0\} \setminus d) < \infty$ for all $v \in D_0$, then $(u(y) - u(x))v(y)$ is S.P.V. integrable w.r.t. J and hence (4) holds for all $u \in D_b^*$ and $v \in D_0$.

Corollary 1.6. *Suppose that J specified in Theorem* 1.1 *is symmetric, then the decomposition* (4) *holds for all $u, v \in D_b^*$.*

2. (Non-symmetric) Dirichlet forms

Let $L^2(E; m)$ and $(\mathcal{E}, D(\mathcal{E}))$ be as above. In this section we assume further that the dual form $(\hat{\mathcal{E}}, D(\mathcal{E}))(\hat{\mathcal{E}}(u, v) := \mathcal{E}(v, u))$ satisfies also the semi-Dirichlet property, i.e., $(\mathcal{E}, D(\mathcal{E}))$ is a (non-symmetric) Dirichlet form.

Proposition 2.1. (i) (3) *holds for all $u \in \widetilde{D}(\mathcal{E})$, i.e., $D^* = \widetilde{D}(\mathcal{E})$. Moreover, for any $u \in D^*$ we have*

$$\int_{E\times E\setminus d} \big(u(y) - u(x)\big)^2 J(\mathrm{d}x, \mathrm{d}y) + \int_E u^2(x) K(\mathrm{d}x) < \infty.$$

(ii) *The metric ρ in Proposition* 1.3 *can be constructed to satisfy* $(\rho.1')$ *below*:

$(\rho.1')$ $\int_{E\times E\setminus d} \rho^2(x, y) J(\mathrm{d}x, \mathrm{d}y) < \infty.$

Combining the decompositions obtained for $\mathcal{E}$ and $\hat{\mathcal{E}}$ respectively, we have the following results.

Theorem 2.2. (i) *Let ρ be a quasi-compatible metric satisfying $(\rho.1')$. Then for any $u, v \in D_b^*$ and any $\varepsilon > 0$, we have*

$$\begin{aligned}\mathcal{E}(u, v) = \widetilde{\mathcal{E}}^c(u, v) &+ \int_{E\times E\setminus d} \big(u(y) - u(x)\big)\big(v(y) - v(x)\big) J(\mathrm{d}x, \mathrm{d}y) + \int_E u(x)v(x)\widetilde{K}(\mathrm{d}x) \\ &+ \check{\mathcal{E}}^{\rho,\varepsilon}(u, v) + \int_{\{(x,y)\in E\times E\setminus d:\ \rho(x,y)>\varepsilon\}} \big(u(y)v(x) - u(x)v(y)\big) J(\mathrm{d}x, \mathrm{d}y),\end{aligned} \tag{7}$$

where $\widetilde{\mathcal{E}}^c(u,v)$ *is a nonnegative definite symmetric form having strong local property (i.e.,* $u \in I(v) \Rightarrow \widetilde{\mathcal{E}}^c(u,v) = 0$*),* $\widetilde{K} = \frac{1}{2}(K + \hat{K})$*, where* $\hat{K}$ *is specified by Theorem* 1.1 *with respect to the dual form* $(\hat{\mathcal{E}}, D(\mathcal{E}))$, $\check{\mathcal{E}}^{\rho,\varepsilon}$ *is an anti-symmetric form.*

(ii) *Let* $u, v \in D_b^*$ *be such that*

$$\big(u(y)v(x) - u(x)v(y)\big) \text{ is S.P.V. integrable w.r.t. } J, \tag{8}$$

then

$$\mathcal{E}(u,v) = \widetilde{\mathcal{E}}^c(u,v) + \int_{E\times E\backslash d} \big(u(y)-u(x)\big)\big(v(y)-v(x)\big)J(\mathrm{d}x,\mathrm{d}y) + \int_E u(x)v(x)\widetilde{K}(\mathrm{d}x)$$
$$+ \check{\mathcal{E}}^c(u,v) + S.P.V. \int_{E\times E\backslash d} \big(u(y)v(x) - u(x)v(y)\big)J(\mathrm{d}x,\mathrm{d}y), \tag{9}$$

where $\widetilde{\mathcal{E}}^c$, $\widetilde{K}$ *are the same as in* (7), $\check{\mathcal{E}}^c$ *is an anti-symmetric form having local property, i.e.,* $\mathrm{supp}_q[u] \cap \mathrm{supp}_q[v] = \emptyset$ *implies* $\check{\mathcal{E}}^c(u,v) = 0$.

Remark 3. If both $(u(y)-u(x))v(y)$ and $(v(y)-v(x))u(y)$ are *S.P.V.* integrable w.r.t. J, then (8) is fulfilled.

Theorem 2.3. *Let* $J = J_0 + J_1$ *be as in Theorem* 1.5.

(i) *If* $J_1(E \times E \setminus d) < \infty$ *then* (8) *is fulfilled and hence decomposition* (9) *holds for all* $u, v \in D_b^*$*. In particular, if* J *is symmetric, then* (9) *holds for all* $u, v \in D_b^*$.

(ii) *If we can find a quasi-compatible metric* ρ *satisfying* $(\rho.1')$, $(\rho.2)$ *and* $(\rho.3)$*, then the decomposition* (9) *holds for all* $u, v \in D_\rho$*, where* D_ρ *is specified by* $(\rho.2)$.

Remark 4. If $(\mathcal{E}, D(\mathcal{E}))$ is regular (i.e., the symmetric part of $\mathcal{E}$ is regular in the sense of [8]), then we can take $\widetilde{D} = C_0(E) \cap D(\mathcal{E})$ ($C_0(E)$: continuous functions with compact support on E). Moreover, we can improve Proposition 1.3 to find a compatible (rather than quasi-compatible) metric ρ satisfying $(\rho.1')$ and $(\rho.2')$ below.

$(\rho.2')$ Let D_ρ be all the elements $u \in C_0(E) \cap D(\mathcal{E})$ which are ρ-Lipschitz (rather than $\mathcal{E}$-q.e. ρ-Lipschitz), then D_ρ is again a special core.

Remark 5. Theorems 2.2 and 2.3 extend the classical Beurling–Deny formula to non-symmtric case. In [4, (9.2)] the author gave a representation which is essentially the same as in (9) but without the notion of S.P.V. integral and without the restriction of condition (8). We remark that condition (8) can not be dropped. See [9] for a counter example.

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