Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
C. R. Acad. Sci. Paris, Ser. I 338 (2004) 447–452
Comptes Rendus Mathematique

Ordinary Differential Equations

On the recovery of a curve isometrically immersed in a Euclidean space

Marcela Szopos

Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, 4, place Jussieu, 75005 Paris, France

Received 18 December 2003; accepted 8 January 2004

Presented by Philippe G. Ciarlet

Abstract

It is known from differential geometry that one can reconstruct a curve with $n-1$ prescribed curvature functions, if these functions can be differentiated a certain number of times in the usual sense and if the first $n-2$ functions are strictly positive. We establish here that this result still holds under the assumption that the curvature functions belong to some Sobolev spaces, by using the notion of derivative in the distributional sense. We also show that the mapping that associates with such prescribed curvature functions the reconstructed curve is of class $\mathcal{C}^{\infty}$. ***To cite this article: M. Szopos, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Sur la reconstruction d'une courbe isométriquement immergée dans un espace Euclidien. Il est connu en géométrie différentielle que l'on peut reconstruire une courbe à partir de ses $n-1$ fonctions de courbure, si l'on peut dériver ces fonctions suffisamment de fois dans le sens classique et si les premières $n-2$ fonctions sont strictement positives. On montre ici que ce résultat reste vrai sous l'hypothèse que les fonctions de courbure appartiennent à des espaces de Sobolev, en utilisant la notion de dérivée au sens des distributions. On montre aussi que l'application qui associe à ces fonctions de courbure la courbe ainsi construite est de classe $\mathcal{C}^{\infty}$. ***Pour citer cet article : M. Szopos, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Le théorème fondamental de la théorie des courbes affirme que l'on peut reconstruire une courbe immergée dans $\mathbb{R}^n$ si on préscrit ses $n-1$ fonctions de courbure et que cette courbe est unique aux isométries de $\mathbb{R}^n$ près. L'existence d'une telle courbe est classiquement établie sous l'hypothèse que les fonctions de courbure sont continûment dérivables un certain nombre de fois et que les premières $n-2$ sont strictement positives.

Le premier objectif de cette Note est d'établir que le théorème fondamental de la théorie des courbes reste vrai sous des hypothèses de régularité plus faibles que celles utilisées classiquement. Plus précisément, l'on montre (Théorème 2.1) qu'il existe une courbe unique $c \in H(I;\mathbb{R}^n)$, isométriquement immergée dans $\mathbb{R}^n$, dont les fonctions de courbure sont $n-1$ fonctions données de $H^{n-2}(I;\mathbb{R}) \times H^{n-3}(I;\mathbb{R}) \times \cdots \times H^1(I;\mathbb{R}) \times L^2(I;\mathbb{R})$.

E-mail address: szopos@ann.jussieu.fr (M. Szopos).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2004.01.018

Cette généralisation est motivée par des questions rencontrées en élasticité non linéaire. Dans cet esprit, d'autres généralisations de résultats de géométrie différentielle dans le cadre fonctionnel des espaces de Sobolev ont été obtenues récemment dans [5] et [6] pour un espace de Riemann de dimension n immergé dans $\mathbb{R}^n$ et pour une surface immergée dans $\mathbb{R}^3$.

Le Théorème 2.1 établie l'existence d'une application qui, à chaque $(n-1)$-uple de fonctions de courbure, associe une courbe immergée dans $\mathbb{R}^n$. Le deuxième objectif de cette Note est d'établir que l'application ainsi construite est de classe $\mathcal{C}^\infty$ (Théorème 3.1). D'autres résultats dans cette directions ont été obtenu dans le cas d'un ouvert de $\mathbb{R}^n$ (voir [1]) et dans le cas d'une surface (voir [2]), pour des données régulières.

Les résultats d'existence classique sont ensuite retrouvés comme conséquences des deux théorèmes cités ci-dessus.

Les démonstrations complètes des Théorèmes 2.1 et 3.1 se trouvent dans [7].

1. Preliminaries

For any integer $n \geqslant 1$, the n-dimensional Euclidean space will be identified with $\mathbb{R}^n$ and will be endowed with the Euclidean norm defined by $|a| = \sqrt{\langle a, a\rangle}$, where $\langle a, b\rangle$ denotes the Euclidean inner product of $a, b \in \mathbb{R}^n$. The notations $\mathbb{M}^{n\times n}$ and $\mathbb{O}^n_+$, respectively designate the set of all real square matrices and of all orthogonal matrices of order n with $\det Q = 1$. A mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\varphi(x) = x_0 + Qx$, where $x_0 \in \mathbb{R}^n$ and $Q \in \mathbb{O}^n_+$ is called a proper isometry.

If X is a Hilbert space, we denote by $|\cdot|_X$ its norm induced by the inner product and by $\mathcal{D}'((0,T); X)$ the space of X-valued distributions. For all integer $m \geqslant 1$, the mth derivative of $f \in \mathcal{D}'((0,T); X)$ is denoted $f^{(m)}$. The first derivative is also denoted f'. The function spaces used in this paper are denoted as follows: $L^p((0,T); X)$ for $1 \leqslant p < +\infty$ is the space of all mesurable functions $f : (0,T) \to X$ such that

$$|f|_{L^p((0,T);X)} := \left(\int_0^T |f(t)|_X^p \mathrm{d}t\right)^{1/p} < +\infty,$$

$H^m((0,T); X)$ denotes the usual Sobolev space and $H^0((0,T); X) := L^2((0,T); X)$. In what follows, the derivatives are to be understood in the distributional sense and classes of functions in $H^1((0,T); \mathbb{M}^{n\times n})$ are identified with their continuous representative, as allowed by the Sobolev imbedding theorem.

The following lemma is a key ingredient in the proof of our first result (Theorem 2.1).

Lemma 1.1. *Consider the system of differential equations*:

$$Y'(t) = A(t)Y(t) + B(t) \quad \textit{for almost all } t \textit{ in } (0,T), \tag{1}$$

$$Y(0) = Y_0, \tag{2}$$

where the matrix fields A and B belong to the space $L^2((0,T); \mathbb{M}^{n\times n})$ and Y_0 is a matrix of $\mathbb{M}^{n\times n}$. Then there exists a unique solution $Y \in H^1((0,T); \mathbb{M}^{n\times n})$ to this system.

This result can be obtained by combinying Theorem 4.1 and Remark 4.3 of [4].

2. Existence and uniqueness of the curve

An integer $n \geqslant 3$ is fixed throughout this section. Let $I = (0,T)$ be a bounded interval of $\mathbb{R}$. Let $c : I \to \mathbb{R}^n$ be a *regular curve* of class $\mathcal{C}^{n-1}$ over I, in the sense that the vectors $c^{(1)}(t), c^{(2)}(t), \ldots, c^{(n-1)}(t)$ are linearly independent for all $t \in I$. In this setting, one can show (see, e.g., [3]) that there exists a unique Frenet frame associated with this curve, denoted $\{e_1(t), \ldots, e_n(t)\}$.

Then Frenet formulas can be written:

$$e_i'(t) = \sum_{j=1}^{n} a_{ij}(t) e_j(t), \quad \forall i \in \{1, \ldots, n\},$$

where the functions $a_{ij} : I \to \mathbb{R}$ satisfy

$$a_{ij}(t) + a_{ji}(t) = 0, \quad \forall i, j \in \{1, \ldots, n\},\ \forall t \in I,$$
$$a_{ij}(t) = 0, \quad \forall j \geqslant i + 2,\ \forall t \in I$$

The curvature functions of the regular curve c at the point $t \in I$ are then defined by:

$$k_i(t) := \frac{a_{i,i+1}(t)}{|c'(t)|}, \quad \forall i \in \{1, \ldots, n-1\} \tag{3}$$

or equivalently, by means of the Frenet formulas, as

$$k_i(t) = \frac{\langle e_{i+1}(t), e_i'(t)\rangle}{|c'(t)|}. \tag{4}$$

The objective of this section is to prove the existence and uniqueness up to proper isometries of a curve isometrically immersed in $\mathbb{R}^n$, with prescribed curvature functions. While this result is classical if the curvature functions are regular enough (see the assumptions of Corollary 2.2), it will be assumed here that they only belong to some specific Sobolev spaces. More precisely, we have the following theorem:

Theorem 2.1 (existence and uniqueness). *Let* $(F_1, \ldots, F_{n-1}) \in H^{n-2}(I; \mathbb{R}) \times \cdots \times H^1(I; \mathbb{R}) \times L^2(I; \mathbb{R})$ *be such that* $F_j(t) > 0$ *for all* $t \in I$ *and for all* $j \leqslant n-2$. *Then*:

(a) *There exists a regular curve* $c \in H^n(I; \mathbb{R}^n)$ *such that* $|c'(t)| = 1$ *for all* $t \in I$ *and its curvature functions are* $F_1, \ldots, F_{n-1}$, *i.e.,* $k_i(t) = F_i(t)$ *for all* $i \in \{1, \ldots, n-1\}$ *and* $t \in I$.

(b) *If* c *and* $\tilde{c}$ *are two curves satisfying the conditions of part* (a), *then there exists a rigid motion* $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $\tilde{c} = \varphi \circ c$.

(c) *If* $x_0 \in \mathbb{R}^n$ *is fixed, then there exists a unique curve* c *satisfying the properties of part* (a), *such that* $c(0) = x_0$ *and its Frenet frame at the origin is given by* $e_1(0) = (1, 0, \ldots, 0), \ldots, e_n(0) = (0, 0, \ldots, 1)$.

Sketch of proof. The proof is broken into six steps. We first prove (step (i)) the existence of a family of vector fields $(e_1, \ldots, e_n)$, which will turn out to be the Frenet frame of the sought curve c, then we establish that this family is orthonormal at all $t \in I$ (step (ii)) and that $e_n \in H^1(I; \mathbb{R}^n)$ and $e_i \in H^{n-i}(I; \mathbb{R}^n)$ for all $i = 1, 2, \ldots, n-1$ (step (iii)). The existence of a curve satisfying part (a) of the theorem is obtained in step (iv). Finally, we establish the uniqueness (step (vi)), or the uniqueness up to rigid motions if the assumptions are weakened (step (v)) of a curve with the prescribed curvatures.

(i) *There exists a unique solution* $(e_1, \ldots, e_n) \in H^1(I; \mathbb{R}^n) \times \cdots \times H^1(I; \mathbb{R}^n)$ *to the Cauchy problem*:

$$e_i'(t) = -F_{i-1}(t) e_{i-1}(t) + F_i(t) e_{i+1}(t) \quad \text{a.e. in } I,\ i \in \{1, \ldots, n\}, \tag{5}$$
$$e_1(0) = (1, 0, \ldots, 0), \ldots, e_n(0) = (0, 0, \ldots, 1), \tag{6}$$

where $F_0 = F_n := 0, e_0 = e_{n+1} := 0$.

To this end, we rewrite the system of ordinary differential equations as a matrix equation of the form:

$$Y' = AY \quad \text{a.e. in } I, \tag{7}$$
$$Y(0) = I_n. \tag{8}$$

It suffices then to apply Lemma 1.1 to the above system, the assumptions of this lemma being satisfied since the functions $F_1, \dots, F_{n-1}$ are at least in $L^2(I;\mathbb{R})$.

(ii) *The family* $\{e_1(t), \dots, e_n(t)\}$ *is orthonormal at all* $t \in I$. We first prove that the functions $\alpha_{ij}(t) := \langle e_i(t), e_j(t)\rangle$ and $\beta_{ij}(t) = \delta_{ij}$ where $t \in I$ and $i, j \in \{1, \dots, n\}$ satisfy the same system of ordinary differential equations. Then, the uniqueness of the solution to this system implies that $\langle e_i(t), e_j(t)\rangle = \delta_{ij}$ for all $t \in I$ and for all $i, j \in \{1, \dots, n\}$.

(iii) *Regularity result for the solution* $\{e_1, \dots, e_n\}$ *to the system* (5). The fact that $H^m(I;\mathbb{R})$ is an algebra for all $m > \frac{1}{2}$ allows to prove that $e_n \in H^1(I;\mathbb{R}^n)$ and $e_i \in H^{n-i}(I;\mathbb{R}^n)$ for all $i = 1, 2, \dots, n-1$.

(iv) *We establish the existence of a curve satisfying part* (a) *of the theorem*. Define the function $c : [0, T] \to \mathbb{R}^n$ by $c(t) := x_0 + \int_0^t e_1(s)\,\mathrm{d}s,\ 0 \leqslant t \leqslant T$. We show that the orthonormal family of vectors $\{e_1(t), \dots, e_n(t)\}$ constitutes the Frenet frame of the curve c at $t \in I$. Then, relations (4) and (5) imply that $k_i(t) = F_i(t)$ for all $t \in I$ and for all $i \in \{1, \dots, n-1\}$, which in turn shows that the curvature functions of the curve c are exactly the prescribed functions $F_1, \dots, F_{n-1}$.

(v) The uniqueness of the curve c under the assumptions of part (c) of the Theorem 2.1 can be established as a consequence of the uniqueness result of Lemma 1.1.

(vi) The uniqueness up to proper isometries of $\mathbb{R}^n$ of the curve c satisfying the conditions of part (a) of the theorem is established as in the classical framework, by using the uniqueness result of Lemma 1.1 instead of the classical result of uniqueness for ordinary differential equations. □

As a consequence of Theorem 2.1, one can obtain a similar result for curves of class $\mathcal{C}^n$. More specifically, one has the following:

Corollary 2.2. *Let* $(F_1, \dots, F_{n-1}) \in \mathcal{C}^{n-2}(I;\mathbb{R}) \times \dots \times \mathcal{C}^1(I;\mathbb{R}) \times \mathcal{C}^0(I;\mathbb{R})$ *be such that* $F_j(t) > 0$ *for all* $t \in I$ *and for all* $j \leqslant n-2$. *Then*:

(a) *There exists a regular curve* $c \in \mathcal{C}^n(I;\mathbb{R}^n)$ *such that* $|c'(t)| = 1$ *for all* $t \in I$ *and its curvature functions are* $F_1, \dots, F_{n-1}$, *i.e.,* $k_i(t) = F_i(t)$ *for all* $i \in \{1, \dots, n-1\}$ *and* $t \in I$.

(b) *If* c *and* $\tilde{c}$ *are two curves satisfying the conditions of part* (a), *then there exists a rigid motion* $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $\tilde{c} = \varphi \circ c$.

(c) *If* $x_0 \in \mathbb{R}^n$ *is fixed, then there exists a unique curve* c *satisfying the conditions of part* (a), *such that* $c(0) = x_0$ *and its Frenet frame at the origin is given by* $e_1(0) = (1, 0, \dots, 0), \dots, e_n(0) = (0, 0, \dots, 1)$.

3. Regularity of a curve as a function of its curvatures

In order to simplify the presentation, we introduce the following notations:

$$\mathcal{H}(I;\mathbb{R}) := \prod_{k=0}^{n-2} H^{n-k-2}(I;\mathbb{R}),$$

$$\mathcal{H}(I;\mathbb{R})_> := \big\{(F_1, \dots, F_{n-1}) \in \mathcal{H}(I;\mathbb{R});\ F_i(t) > 0,\ \forall t \in I,\ \forall i \in \{1, \dots, n-2\}\big\},$$

and

$$\mathbf{H}\big(I;\mathbb{R}^n\big) := \left(\prod_{k=1}^{n-1} H^{n-k}\big(I;\mathbb{R}^n\big)\right) \times H^1\big(I;\mathbb{R}^n\big).$$

Note that the set $\mathcal{H}(I;\mathbb{R})_>$ is open in the Hilbert space $\mathcal{H}(I;\mathbb{R})$.

Under some appropriate assumptions, we have proved in the previous section the existence and uniqueness of a curve c with prescribed curvature functions. In this way, we have constructed a mapping

$$\mathcal{F} : (F_1, \ldots, F_{n-1}) \in \mathcal{H}(I; \mathbb{R})_> \to c \in H^n\big(I; \mathbb{R}^n\big).$$

The aim of this section is to study the regularity properties of this mapping. Our result is established in the theorem below:

Theorem 3.1. *Define the mapping*

$$\mathcal{F} : (F_1, \ldots, F_{n-1}) \in \mathcal{H}(I; \mathbb{R})_> \to c \in H^n\big(I; \mathbb{R}^n\big),$$

where the curve c is defined in part (c) *of Theorem* 2.1. *Then the mapping $\mathcal{F}$ is of class $\mathcal{C}^\infty$.*

Sketch of proof. For clarity, we break the proof in three steps: in step (i) we construct a function f, related to $\mathcal{F}$, and prove that it is of class $\mathcal{C}^\infty$, in step (ii) we apply the implicit function theorem to this function, and in step (iii) we conclude the proof.

(i) *Let* $e_1^0 = (1, 0, \ldots, 0)$, $e_2^0 = (0, 1, \ldots, 0), \ldots,$ *and* $e_n^0 = (0, 0, \ldots, 1)$. *Define the function*

$$f : \mathcal{H}(I; \mathbb{R})_> \times \mathbf{H}\big(I; \mathbb{R}^n\big) \to \mathbf{H}\big(I; \mathbb{R}^n\big),$$
$$f\big((F_1, \ldots, F_{n-1}), (e_1, \ldots, e_n)\big) = \big(w^1, \ldots, w^n\big),$$

where, for all $t \in I$ and for all $k \in \{1, \ldots, n\}$,

$$w^k(t) = e_k^0 - e_k(t) + \int_0^t \big(F_k(s)e_{k+1}(s) - F_{k-1}(s)e_{k-1}(s)\big)\,\mathrm{d}s \tag{9}$$

(*with the convention that* $F_0 = F_n = 0$ *and* $e_0 = e_{n+1} = 0$). *Then the function f is well-defined and of class $\mathcal{C}^\infty$.* To see this, it suffices to prove that each component of f is well-defined and of class $\mathcal{C}^\infty$. This is a direct consequence of the fact that the mappings

$$(g, h) \in H^k\big((0, T); \mathbb{R}\big) \times H^m\big((0, T); \mathbb{R}^n\big) \to (gh) \in H^k\big((0, T); \mathbb{R}^n\big)$$

and

$$g \in H^k\big((0, T); \mathbb{R}^n\big) \to G \in H^{k+1}\big((0, T); \mathbb{R}^n\big),$$

where $m \geqslant 1$, $0 \leqslant k \leqslant m$, and $G(t) = \int_0^t g(s)\,\mathrm{d}s$, are well-defined and of class $\mathcal{C}^\infty$.

(ii) *The implicit function theorem can be applied to the function f defined in step* (i). Let

$$\tilde{p} := \big((\widetilde{F}_1, \ldots, \widetilde{F}_{n-1}), (\tilde{e}_1, \ldots, \tilde{e}_n)\big) \in \mathcal{H}(I; \mathbb{R})_> \times \mathbf{H}\big(I; \mathbb{R}^n\big)$$

be such that $f(\tilde{p}) = 0$. Note that such an element $\tilde{p}$ always exists, as showed in steps (i) and (iii) of the proof of Theorem 2.1. By applying the same method as that used in the proof of steps (i) and (iii) of Theorem 2.1, we then show that the derivative $D_{(e_1,\ldots,e_n)} f(\tilde{p})$ is an isomorphism of the space $\mathbf{H}(I; \mathbb{R}^n)$. The notation $D_{(e_1,\ldots,e_n)} f$ is used for the partial derivative of the function f with respect to $(e_1, \ldots, e_n)$.

(iii) *Conclusion*. According to the implicit function theorem applied to the function f, there exist an open subset U of $\mathcal{H}(I; \mathbb{R})_>$ containing $(\widetilde{F}_1, \ldots, \widetilde{F}_{n-1})$, an open subset V of $\mathbf{H}(I; \mathbb{R}^n)$ containing $(\tilde{e}_1, \ldots, \tilde{e}_n)$, and an implicit function $g : U \to V$ such that the conditions

$$f\big((F_1, \ldots, F_{n-1}), (e_1, \ldots, e_n)\big) = 0 \quad \text{and} \quad \big((F_1, \ldots, F_{n-1}), (e_1, \ldots, e_n)\big) \in U \times V$$

are equivalent to

$$(e_1, \ldots, e_n) = g(F_1, \ldots, F_{n-1}) \quad \text{for all } (F_1, \ldots, F_{n-1}) \in U.$$

Moreover, the same theorem shows that the mapping $g: U \to V$ is of class $\mathcal{C}^\infty$.

However, we have seen in the proof of Theorem 2.1 that, for any $(F_1, \ldots, F_{n-1}) \in \mathcal{H}(I; \mathbb{R})_>$, the equation

$$f\big((F_1, \ldots, F_{n-1}), (e_1, \ldots, e_n)\big) = 0$$

has a unique solution $(e_1, \ldots, e_n) \in \mathbf{H}(I; \mathbb{R}^n)$. This shows that the mapping $\bar{g}: \mathcal{H}(I; \mathbb{R})_> \to \mathbf{H}(I; \mathbb{R}^n)$ defined in this fashion, i.e., by $\bar{g}(F_1, \ldots, F_{n-1}) = (e_1, \ldots, e_n)$, is well-defined. Therefore, the uniqueness part of the implicit function theorem shows that $\bar{g} = g$ on U, hence that $\bar{g}$ is of class $\mathcal{C}^\infty$ over U. Since the $(n-1)$-tuple $(\tilde{F}_1, \ldots, \widetilde{F}_{n-1})$ was arbitrarily chosen in $\mathcal{H}(I; \mathbb{R})_>$, we deduce that the mapping $\bar{g}$ is of class $\mathcal{C}^\infty$ over the open set $\mathcal{H}(I; \mathbb{R})_>$. Then, by first applying a projection on the first component, then integrating over the interval $[0, t]$ (these operations preserve the $\mathcal{C}^\infty$ regularity), we finally obtain that the mapping $\mathcal{F}$ is of class $\mathcal{C}^\infty$, as the composition of the three above mappings. □

In a similar manner, we can prove the following result for curves of class $\mathcal{C}^n$:

Corollary 3.2. *Let the mapping*

$$\mathcal{F}: \mathcal{C}^{n-2}\big(I; \mathbb{R}_+^*\big) \times \mathcal{C}^{n-3}\big(I; \mathbb{R}_+^*\big) \times \cdots \times \mathcal{C}^1\big(I; \mathbb{R}_+^*\big) \times \mathcal{C}^0(I; \mathbb{R}) \to \mathcal{C}^n\big(I; \mathbb{R}^n\big)$$

be defined by

$$(F_1, \ldots, F_{n-1}) \to c,$$

where the curve c is defined in part (c) *of Corollary* 2.2. *Then $\mathcal{F}$ is of class $\mathcal{C}^\infty$.*

Acknowledgements

The author is grateful to Ph.G. Ciarlet for bringing this problem to her attention and for his interest in her work. The author would also like to thank C. Mardare for interesting and helpful discussions.

References

[1] P.G. Ciarlet, On the continuity of a surface as a function of its two fundamental forms, J. Math. Pures Appl. 82 (2003) 253–274.
[2] P.G. Ciarlet, F. Laurent, Continuity of a deformation as a function of its Cauchy–Green tensor, Arch. Rational Mech. Anal. 167 (2003) 255–269.
[3] W. Klingenberg, A Course in Differential Geometry, Springer-Verlag, 1978.
[4] J.-L. Lions, E. Magenes, Problèmes aux limites non homogènes et applications, vol. I, Dunod, Paris, 1968.
[5] S. Mardare, On isometric immersions of a Riemannian space under weak regularity assumptions, C. R. Acad. Sci. Paris, Ser. I 337 (2003) 785–790.
[6] S. Mardare, On the fundamental theorem of surface theory under weak regularity assumptions, C. R. Acad. Sci. Paris, Ser. I, in press.
[7] M. Szopos, On the recovery of a curve isometrically immersed in a Euclidean space, in press.