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Topology

The extended mapping class group is generated by 3 symmetries

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Abstract

We prove that for $g \geqslant 1$ the extended mapping class group is generated by three orientation reversing involutions. ***To cite this article: M. Stukow, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Le groupe modulaire étendu est engendré par 3 symétries. Nous prouvons que pour chaque $g \geqslant 1$ le groupe modulaire étendu est éngendré par trois involutions qui inversent l'orientation. ***Pour citer cet article : M. Stukow, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

1. Introduction

Let S_g be a closed orientable surface of genus g. Denote by $\mathcal{M}_g^{\pm}$ the *extended mapping class group*, i.e., the group of isotopy classes of homeomorphisms of S_g. By $\mathcal{M}_g$ we denote the *mapping class group*, i.e., the subgroup of $\mathcal{M}_g^{\pm}$ consisting of orientation preserving maps. We will make no distinction between a map and its isotopy class, so in particular by the order of a homeomorphism $h: S_g \to S_g$ we mean the order of its class in $\mathcal{M}_g^{\pm}$.

By $C_i,\ U_i,\ Z_i$ we denote the right Dehn twists along the curves c_i, u_i, z_i indicated in Fig. 1. It is known that this set of generators of $\mathcal{M}_g$ is not minimal, and a great deal of attention has been paid to the problem of finding a minimal (or at least small) set of generators or a set of generators with some additional property. For different approaches to this problem see [3,5,7,8,10,11] and references there. The main purpose of this Note is to prove that for $g \geqslant 1$ the extended mapping class group $\mathcal{M}_g^{\pm}$ is generated by three symmetries, i.e. orientation reversing involutions. This generalises a well known fact for $\mathcal{M}_1^{\pm} \cong \mathrm{GL}(2,\mathbb{Z})$.

As was observed in [4], the fact that $\mathcal{M}_g^{\pm}$ is generated by symmetries is rather simple. Namely, suppose that S_g is embedded in $\mathbb{R}^3$ as shown in Fig. 1. Define the *sandwich symmetry* $\tau: S_g \to S_g$ as a reflection across the yz-plane. Now if u is any of the curves indicated in Fig. 1, then the twist U along this curve satisfies the relation:

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$\tau U \tau = U^{-1}$, i.e. the element τU is a symmetry. This proves that each of generating twists is a product of two symmetries. Note that for the composition of mappings we use the following convention: fg means that g is applied first.

2. Preliminaries

Suppose that S_g, for $g \geqslant 2$, is embedded in $\mathbb{R}^3$ as shown in Fig. 1. Let $\rho : S_g \to S_g$ be a *hyperelliptic involution*, i.e., the half turn about y-axis.

The *hyperelliptic mapping class group* $\mathcal{M}_g^h$ is defined to be the centraliser of ρ in $\mathcal{M}_g$. By [2] the quotient $\mathcal{M}_g^h/\langle\rho\rangle$ is isomorphic to the mapping class group $\mathcal{M}_{0,2g+2}$ of a sphere $S_{0,2g+2}$ with $2g+2$ marked points $P_1, \ldots, P_{2g+2}$. This set of marked points corresponds (under the canonical projection) to fixed points of ρ (Fig. 1). In a similar way, we define the *extended hyperelliptic mapping class group* $\mathcal{M}_g^{h\pm}$ which projects onto the extended mapping class group $\mathcal{M}_{0,2g+2}^{\pm}$ of $S_{0,2g+2}$. Denote this projection by $\pi : \mathcal{M}_g^{h\pm} \to \mathcal{M}_{0,2g+2}^{\pm}$. In case $g = 2$ it is known that $\mathcal{M}_2 = \mathcal{M}_2^h$ and $\mathcal{M}_2^{\pm} = \mathcal{M}_2^{h\pm}$.

Denote by $\sigma_1, \sigma_2, \ldots, \sigma_{2g+1}$ the images under π of twist generators $C_1, U_1, Z_1, U_2, Z_2, \ldots, U_g, Z_g$ respectively. These generators of $\mathcal{M}_{0,2g+2}$ are closely related to Artin braids, cf. [2].

Let $\widetilde{M} : S_{0,2g+2} \to S_{0,2g+2}$ be a rotation of order $2g+1$ with a fixed point P_1 such that: $\widetilde{M}(P_i) = P_{i+1}$, for $i = 2, \ldots, 2g+1$ and $\widetilde{M}(P_{2g+2}) = P_2$ (Fig. 2). In terms of the generators $\sigma_1, \ldots, \sigma_{2g+1}$ we have:

$$\widetilde{M} = \sigma_2\sigma_3 \cdots \sigma_{2g+1}. \tag{1}$$

If $M' \in \mathcal{M}_g$ is the lifting of $\widetilde{M}$ of order $2g+1$, then $M = \rho M'$ is the lifting of $\widetilde{M}$ for which $M^{2g+1} = \rho$. In particular M has order $4g+2$. Using the technique described in [10] it is easy to write M as a product of twists: $M = U_1Z_1U_2Z_2 \cdots U_gZ_g$.

Since every finite subgroup of $\mathcal{M}_g$ can be realised as the group of automorphisms of a Riemann surface [6], M has maximal order among torsion elements of $\mathcal{M}_g$ [12]. Geometric properties of M played a crucial role in the problem of finding particular sets of generators for $\mathcal{M}_g$ and $\mathcal{M}_g^{\pm}$, cf. [3,7,8,11].

Following [1], let $t_1, s_1, \ldots, t_g, s_g$ be generators of the fundamental group $\pi_1(S_g)$ as in Fig. 3. In terms of these generators, $\pi_1(S_g)$ has the single defining relation: $R = s_g^{t_g} s_{g-1}^{t_{g-1}} \cdots s_1^{t_1} s_1^{-1} s_2^{-1} \cdots s_g^{-1}$, where by a^b we denote the conjugation bab^{-1}.

It is well known [9] that the mapping class group $\mathcal{M}_g^{\pm}$ is isomorphic to the group $\mathrm{Out}(\pi_1(S_g))$ of outer automorphisms of $\pi_1(S_g)$. In terms of this isomorphism, elements of $\mathcal{M}_g$ correspond to the elements of $\mathrm{Out}(\pi_1(S_g))$ which map the relation R to its conjugate, and elements of $\mathcal{M}_g^{\pm} \setminus \mathcal{M}_g$ to those elements of $\mathrm{Out}(\pi_1(S_g))$ which map R to a conjugate of R^{-1}.

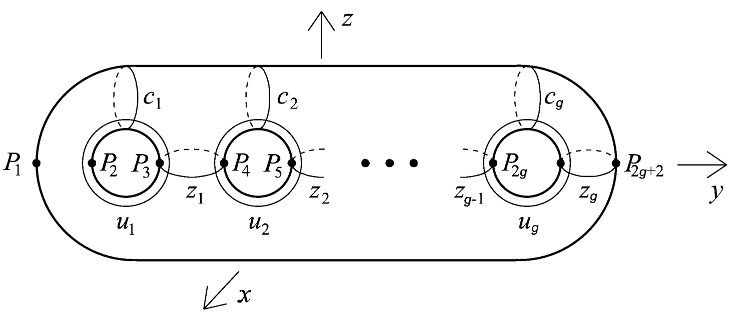

Fig. 1. Surface S_g embedded in $\mathbb{R}^3$.

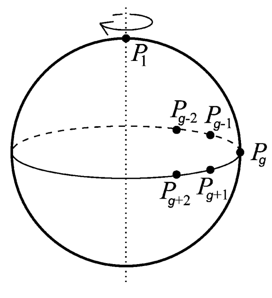

Fig. 2. Rotation $\widetilde{M}$.

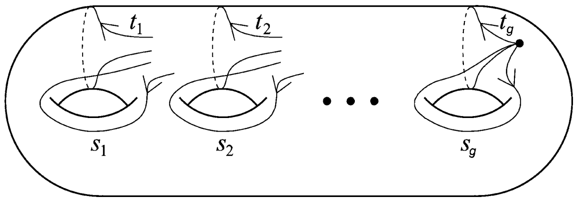

Fig. 3. Generators of $\pi_1(S_g)$.

Using representations of twist generators as automorphisms of $\pi_1(S_g)$ [1] we could derive the following representation for the rotation M:

$$\begin{aligned} M: t_i &\mapsto s_i^{t_i}\cdots s_1^{t_1}t_1 && \text{for } i=1,\dots,g,\\ s_i &\mapsto t_1^{-1}s_1^{-t_1}\cdots s_i^{-t_i}t_{i+1}t_i^{-1}s_i^{t_i}\cdots s_1^{t_1}t_1 && \text{for } i=1,\dots,g-1,\\ s_g &\mapsto t_1^{-1}s_1^{-t_1}\cdots s_g^{-t_g}t_g^{-1}s_g^{t_g}\cdots s_1^{t_1}t_1. \end{aligned}$$

As in the case of maps and their isotopy classes, we abuse terminology by identifying an element of $\mathrm{Out}(\pi_1(S_g))$ with its representative in $\mathrm{Aut}(\pi_1(S_g))$.

3. $\mathcal{M}_g^{\pm}$ is generated by 3 symmetries

If we represent the action of the rotation $\widetilde{M}$ as the orthogonal action on the unit sphere, it becomes obvious that $\widetilde{M}$ can be written as a product of two symmetries. To be more precise, if $\tilde{\varepsilon}_1$ is the symmetry across the plane passing through P_1, P_g and the center of the sphere (Fig. 2), then $\widetilde{M}=\tilde{\varepsilon}_1\tilde{\varepsilon}_2$, where $\tilde{\varepsilon}_2$ is another symmetry.

Tedious but straightforward computations show that one of the liftings $\varepsilon_1\in\mathcal{M}_g^{\pm}$ of $\tilde{\varepsilon}_1$ has the following representation as an automorphism of $\pi_1(S_g)$:

$$\begin{aligned} \varepsilon_1: t_i &\mapsto t_{g-1}^{-1}s_1^{-1}\cdots s_{g-1-i}^{-1},\quad s_i\mapsto t_{g-1-i}^{-1}t_{g-i}\quad \text{for } i=1,\dots,g-2,\\ t_{g-1} &\mapsto t_{g-1}^{-1},\quad s_{g-1}\mapsto s_g\cdots s_1t_1,\quad t_g\mapsto t_{g-1}^{-1}t_g,\quad s_g\mapsto s_g^{-1}. \end{aligned}$$

To obtain the above representation we proceed as follows: take a generator u of $\pi_1(S_g)$, find the image $\tilde{u}$ of u under projection $S_g\to S_{0,2g+2}$, find $\tilde{\varepsilon}_1(\tilde{u})$, lift back $\tilde{\varepsilon}_1(\tilde{u})$ to S_g and finally express the obtained loop as a product of generators $t_1,s_1,\dots,t_g,s_g$ of $\pi_1(S_g)$.

We would like to point out that although the above procedure is a bit subtle, it is quite simple to verify that the obtained formulas are correct. In fact, it is enough to check that $\varepsilon_1^2=1$ and $\varepsilon_1(R)$ is conjugate to R^{-1}. Moreover, the representation of $\varepsilon_2=\varepsilon_1M$ is given by the following formulas:

$$\begin{aligned} \varepsilon_2: t_i &\mapsto \big(t_{g-1}^{-1}s_1^{-1}\cdots s_{g-1-i}^{-1}t_{g-1-i}^{-1}\big)\big(s_{g-i}^{-t_{g-i}}\cdots s_{g-1}^{-t_{g-1}}\big)t_{g-1}s_{g-1}\quad \text{for } i=1,\dots,g-2,\\ t_{g-1} &\mapsto t_{g-1}^{-1}s_g^{t_g}t_{g-1}s_{g-1},\quad t_g\mapsto s_{g-1},\\ s_i &\mapsto s_{g-1}^{-1}t_{g-1}^{-1}\big(s_{g-1}^{t_{g-1}}\cdots s_{g-i}^{t_{g-i}}\big)\big(s_{g-1-i}^{t_{g-1-i}}\big)\big(s_{g-i}^{-t_{g-i}}\cdots s_{g-1}^{-t_{g-1}}\big)t_{g-1}s_{g-1}\quad \text{for } i=1,\dots,g-2,\\ s_{g-1} &\mapsto \big(s_{g-1}^{-1}t_{g-1}^{-1}s_g^{-t_g}\big)t_g\big(s_g^{t_g}t_{g-1}s_{g-1}\big),\quad s_g\mapsto s_{g-1}^{-1}t_g^{-1}t_{g-1}s_{g-1}. \end{aligned}$$

It is straightforward to verify that ε_2^2 is an identity in $\mathrm{Out}(\pi_1(S_g))$.

Theorem 3.1. *For each $g\geqslant 1$, the extended mapping class group $\mathcal{M}_g^{\pm}$ is generated by three symmetries.*

Proof. As observed in the introduction, the result is well known for $g = 1$, but for the sake of completeness let us prove this in more geometric way. Since $\mathcal{M}_1 = \langle U_1, C_1 \rangle$ (Fig. 1) and $\tau U_1 \tau = U_1^{-1}$, $\tau C_1 \tau = C_1^{-1}$, the group $\mathcal{M}_1^{\pm}$ is generated by the symmetries $\tau,\ \tau U_1,\ \tau C_1$.

Now suppose that $g \geqslant 2$. Let ε_1 and $\varepsilon_2 = \varepsilon_1 M$ be the symmetries defined above. Since $\varepsilon_1(t_{g-1}) = t_{g-1}^{-1}$ we have $\varepsilon_1 C_{g-1} \varepsilon_1 = C_{g-1}^{-1}$, i.e., $\varepsilon_3 = \varepsilon_1 C_{g-1}$ is a symmetry. In particular $\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle \supset \langle \varepsilon_1\varepsilon_2, \varepsilon_1\varepsilon_3 \rangle = \langle M, C_{g-1} \rangle$. But by [7] the latter group is equal to $\mathcal{M}_g$. Since $\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$ contains orientation reversing element, this proves that $\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle = \mathcal{M}_g^{\pm}$. $\square$

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