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Group Theory

# Profinite groups of finite cohomological dimension

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**Abstract**

Let $1 \to N \to G \to G/N \to 1$ be a short exact sequence of profinite groups, and let $p$ be a prime number. We prove that if $G$ is of finite cohomological $p$-dimension $n := cd_p(G) < \infty$ and if the order of $H^k(N, \mathbb{F}_p)$ is finite for $k := cd_p(N)$, the virtual cohomological $p$-dimension of $G/N$ equals $n - k$. ***To cite this article: T. Weigel, P. Zalesskii, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***
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**Résumé**

**Des groupes profinis de dimension cohomologique finie.** Soit $1 \to N \to G \to G/N \to 1$ une suite exacte courte de groupes profinis, et soit $p$ un nombre premier. Nous montrons que si $G$ a $p$-dimension cohomologique finie $n := cd_p(G)$ et si l'ordre du groupe $H^k(N, \mathbb{F}_p)$ est fini pour $k := cd_p(N)$, la $p$-dimension cohomologique virtuelle de $G/N$ est égale à $n - k$. ***Pour citer cet article : T. Weigel, P. Zalesskii, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***
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## Version française abrégée

Soit $1 \to N \to G \to G/N \to 1$ une suite exacte courte de groupes profinis. Pour un nombre premier $p$ la $p$-dimension cohomologique vérifie l'inégalité

$$cd_p(G) \leqslant cd_p(N) + cd_p(G/N). \tag{1}$$

Chaque groupe profini a cependant une résolution par des groupes projectifs profinis de longueur 1. Alors l'inégalité (3) est très loin d'être une égalité. Nous montrons malgré tout le théorème suivant :

**Théorème 0.1.** *Soient $G$ un groupe profini de $p$-dimension cohomologique finie $n := cd_p(G)$, $N$ un sous-groupe de $G$ normal fermé de $p$-dimension cohomologique $k$ tel que l'ordre de $H^k(N, \mathbb{F}_p)$ soit fini. Alors la $p$-dimension cohomologique virtuelle de $G/N$ est égale à $n - k$, autrement dit, $vcd_p(G/N) = n - k$.*

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Le Théorème 0.1 est une généralisation d'un travail de Engler et al. [4]. La démonstration est basée sur l'analyse des suites spectrales de Hochschild–Lyndon–Serre pour un $U$-module trivial $\mathbb{F}_p$ associé à une extension de groupes profinis $1 \to N \to U \to U/N \to 1$.

## 1. Introduction

For a prime number $p$ the cohomological $p$-dimension has some properties similar to a geometric dimension theory, i.e., if $1 \to N \to G \to G/N \to 1$ is a short exact sequence of profinite groups one has the inequality

$$cd_p(G) \leqslant cd_p(N) + cd_p(G/N) \tag{2}$$

(cf. [7], §I.3.3, Proposition 15). Every profinite group has a resolution by projective profinite groups of length 1. Thus the inequality (2) is failing being an equality very strongly. Nevertheless, we will prove the following theorem.

**Theorem 1.1.** *Let $G$ be a profinite group of finite cohomological $p$-dimension $n := cd_p(G)$ and let $N$ be a closed normal subgroup of $G$ of cohomological $p$-dimension $k$ such that the order of $H^k(N, \mathbb{F}_p)$ is finite. Then $G/N$ is of virtual cohomological $p$-dimension $n - k$, i.e., $vcd_p(G/N) = n - k$.*

At first sight one might think that Theorem 1.1 is a variant of a result of Serre (cf. [7], §I.4.1, Proposition 22(i)). However, in our theorem the finiteness of the virtual cohomological $p$-dimension of $G/N$ is a conclusion and not a hypothesis.

Theorem 1.1 can be considered as a generalization of the work of Engler et al. [4]. The proof of Theorem 1.1 is based on the analysis of the Hochschild–Lyndon–Serre spectral sequence for the trivial $U$-module $\mathbb{F}_p$ associated to an extension of profinite groups $1 \to N \to U \to U/N \to 1$.

## 2. The proof of Theorem 1.1

Let $p$ be a prime number and let $G$ be a profinite group. By ${}_G\mathbf{com}_p$ we denote the Abelian category the objects of which are (left) pro-$p$ $G$-modules. This is an Abelian category with enough projectives, and the right-derived functors of $\mathbf{Hom}_G(\mathbb{Z}_p, \_)$ evaluated on finite discrete $G$-modules of $p$-power order coincide with the Galois cohomology groups $H^\bullet(G, \_)$ on these modules. For further details see [3,8].

For the proof of the Theorem 1.1 we need two elementary facts about profinite groups. The first one is an elementary fact in Galois cohomology:

**Proposition 2.1.** *Let $G$ be a profinite group and let $A$ be a finite discrete (left) $G$-module. Let $k \geqslant 0$ and assume that the order of $H^k(G, A)$ is finite. Then there exists an open normal subgroup $U$ of $G$ acting trivially on $A$ such that*

$$\inf^k_{G/U,G} : H^k(G/U, A) \longrightarrow H^k(G, A) \tag{3}$$

*is surjective.*

**Proof.** This follows from of the finiteness of $|A|$, the finiteness of $|H^k(G, A)|$, and [7], §I.2.2, Corollary 1. □

The second fact concerns almost direct complementability of finite normal subgroups.

**Proposition 2.2.** *Let $G$ be a profinite group and let $N$ be a finite normal subgroup of $G$. Then there exists an open subgroup $U$ of $G$, such that $U \subseteq C_G(N)$ and $U \cap N = 1$. In particular, $UN$ is open in $G$ and $UN \simeq U \times N$.*

**Proof.** Let $\phi : G \to \operatorname{Aut}(N)$ be the canonical morphism induced by conjugation. Then $K := \ker(\phi)$ is open and normal in $G$, and $K \cap N = Z(N)$. For every element $z \in Z(N) \setminus \{1\}$ let $U_z$ be an open normal subgroup of $K$ such that $z \notin U_z$. Then $U := \bigcap_{z \in Z(N)\setminus\{1\}} U_z$ satisfies the claim. □

Using basically the same argument as presented in [4], §2, we can now prove Theorem 1.1.

**Theorem 2.3.** *Let $p$ be a prime number and let $G$ be a profinite group of finite cohomological $p$-dimension $n := cd_p(G)$. Let $N$ be a closed normal subgroup of $G$ of cohomological $p$-dimension $k$ such that the order of $H^k(N, \mathbb{F}_p)$ is finite. Then $G/N$ is of virtual cohomological $p$-dimension $n - k$.*

**Proof.** By substituting $G$ by the preimage of a Sylow pro-$p$ subgroup of $G/N$ under the canonical projection, we may assume that $G/N$ is a pro-$p$ group (cf. [7], §I.3.3, Proposition 14(i)).

By Proposition 2.1, there exists an open normal subgroup $M$ of $N$ such that

$$\inf^k_{N/M,N} : H^k(N/M, \mathbb{F}_p) \longrightarrow H^k(N, \mathbb{F}_p) \tag{4}$$

is surjective. Since $M$ is open in $N$, there exists an open normal subgroup $V$ of $G$ such that $V \cap N \subseteq M$. Thus we may assume that $M$ is normal in $G$.

Applying Proposition 2.2 to the finite normal subgroup $N/M$ of $G/M$ yields the existence of an open subgroup $U_0$ of $G$ such that $U_0 \cap N = M$ and $U_0N/M \simeq N/M \times U_0/M$. Since $H^k(N/M, \mathbb{F}_p)$ is a finite discrete $G$-module, we may also assume that $U_0$ is acting trivially on $H^k(N/M, \mathbb{F}_p)$, and thus by construction also on $H^k(N, \mathbb{F}_p)$. Hence for $U := U_0N$, $H^k(N, \mathbb{F}_p)$ is a finite trivial $U$-module, and thus isomorphic to $\mathbb{F}_p^m$ for some $m \geqslant 1$.

We consider the Hochschild–Lyndon–Serre spectral sequences $(E_\bullet^{\bullet,\bullet}, d_\bullet)$ for the trivial $U$-module $\mathbb{F}_p$ associated to the extension $1 \to N \to U \to U/N \to 1$, and $(\overline{E}_\bullet^{\bullet,\bullet}, \bar{d}_\bullet)$ for the trivial $U/M$-module $\mathbb{F}_p$ associated to the extension $1 \to N/M \to U/M \to U/N \to 1$. Note that by construction, the second extension is a direct product, and thus the spectral sequence $(\overline{E}_\bullet^{\bullet,\bullet}, \bar{d}_\bullet)$ collapses at the $\overline{E}_2$-term, i.e., $\bar{d}_t = 0$ for all $t \geqslant 2$ (cf. [6], p. 96, Example 7).

Let $P_\bullet \to \mathbb{Z}_p$ be a projective resolution of $\mathbb{Z}_p$ in ${}_{U/N}\mathbf{com}_p$, let $\overline{Q}_\bullet \to \mathbb{Z}_p$ be a projective resolution of $\mathbb{Z}_p$ in ${}_{U/M}\mathbf{com}_p$, and let $Q_\bullet \to \mathbb{Z}_p$ be a projective resolution of $\mathbb{Z}_p$ in ${}_U\mathbf{com}_p$. By the comparison theorem, there exists a mapping of chain complexes

$$\pi_\bullet : Q_\bullet \longrightarrow \overline{Q}_\bullet \tag{5}$$

in ${}_U\mathbf{com}_p$. This mapping induces a mapping of double complexes

$$\pi^{\bullet,\bullet} : \mathbf{Hom}_{U/N}\big(P_\bullet, \mathbf{Hom}_{N/M}(\overline{Q}_\bullet, \mathbb{F}_p)\big) \longrightarrow \mathbf{Hom}_{U/N}\big(P_\bullet, \mathbf{Hom}_N(Q_\bullet, \mathbb{F}_p)\big), \tag{6}$$

and thus induces a mapping of spectral sequences

$$\pi_\bullet^{\bullet,\bullet} : \big(\overline{E}_\bullet^{\bullet,\bullet}, \bar{d}_\bullet\big) \longrightarrow \big(E_\bullet^{\bullet,\bullet}, d_\bullet\big) \tag{7}$$

(cf. [1], Lemma 3.5.1). By hypothesis, $E_2^{\bullet,\bullet}$ has only $k+1$ non-trivial rows, hence $d_{k+2} = 0$. As in [4] we will show:

**Claim 2.4.** *For $t \geqslant 2$, the map $\pi_t^{n+1-k,k} : \overline{E}_t^{n+1-k,k} \to E_t^{n+1-k,k}$ is surjective, and $d_t^{n+1-k,k} : E_t^{n+1-k,k} \to E_t^{n+1+t-k,k-t+1}$ is the zero map.*

**Proof.** From the commutative diagrams

$$\begin{array}{ccc} \overline{E}_t^{n+1-k,k} & \xrightarrow{\;0\;} & \overline{E}_t^{n+1+t-k,k-t+1} \\ \Big\downarrow{\scriptstyle \pi_t^{n+1-k,k}} & & \Big\downarrow{\scriptstyle \pi_t^{n+1+t-k,k-t+1}} \\ E_t^{n+1-k,k} & \xrightarrow{\;d_t^{n+1-k,k}\;} & E_t^{n+1+t-k,k-t-1} \end{array} \tag{8}$$

one concludes that it suffices to prove that $\pi_t^{n+1-k,k}$ is surjective for all $t \geqslant 2$. For $t = 2$ this is obvious: by construction of $M$,

$$\inf^k_{N/M,N}\colon H^k(N/M,\mathbb{F}_p) \longrightarrow H^k(N,\mathbb{F}_p) \tag{9}$$

is a split epimorphism of discrete $U/N$-modules, and thus

$$\pi_2^{n+1-k,k}\colon H^{n+1-k}\big(U/N, H^k(N/M,\mathbb{F}_p)\big) \longrightarrow H^{n+1-k}\big(U/N, H^k(N,\mathbb{F}_p)\big) \tag{10}$$

is surjective.

For $t > 2$ we proceed by induction and assume we have proved the assertion for $2 \leqslant i < t-1$. In particular, $\pi_{t-1}^{n+1-k,k}$ is surjective, and as $\pi_t^{\bullet,\bullet}$ is the mapping induced by $\pi_{t-1}^{\bullet,\bullet}$, one concludes that $\pi_t^{n+1-k,k}$ is surjective. This yields the claim. □

From the claim we deduce that $E_\infty^{n+1-k,k} = H^{n+1-k}(U/N, H^k(N,\mathbb{F}_p))$, and as $cd_p(U) = n$, this group must be trivial. By construction, $H^k(N,\mathbb{F}_p)$ is a finite trivial $U$-module isomorphic to $\mathbb{F}_p^m$ for some $m \geqslant 1$. Thus $H^{n+1-k}(U/N,\mathbb{F}_p) = 0$, and this yields $cd_p(U/N) \leqslant n-k$ (cf. [7], §I.3.1, Proposition 11). The inequality $cd_p(U/N) \geqslant n-k$ is a direct consequence of (2). □

## 3. Implications for pro-$p$ groups

In this section we collect some consequences of Theorem 2.3 for pro-$p$ groups.

**Corollary 3.1.** *Let $G$ be a pro-$p$ group, $cd_p(G) = n < \infty$, and let $N$ be a closed normal subgroup of $G$ such that $cd_p(N) = cd_p(G)$ and that $|H^n(N,\mathbb{F}_p)|$ is finite. Then $N$ is open in $G$. In particular,*

(a) *If $N$ is of type $p$-$FP$, $G$ is also of type $p$-$FP$.*

(b) *If $N$ is of type $p$-$FP$ of (additive) Euler–Poincaré characteristic $\chi_N \neq 0$, $G$ has also non-trivial Euler–Poincaré characteristic $\chi_G \neq 0$. In this case one has*

$$|G/N| \leqslant |\chi_N|. \tag{11}$$

**Proof.** By Theorem 2.3, $G/N$ is a pro-$p$ group of virtual cohomological $p$-dimension 0 and thus is finite. For (a) see [8], Proposition 4.2.1.

(b) As $\chi_N = |G/N| \cdot \chi_G$ (cf. [7], §I.4.1, Example (b)) the first part is obvious. The inequality (11) follows from the fact that $\chi_G$ is a non-trivial integer. □

The group $G := \mathbb{Z}_p$ has finitely generated normal subgroups of arbitrary large index. Hence for normal subgroups $N$ of type $p$-*FP* with $\chi_N = 0$, there is no bound for the index of $N$ in $G$.

Every Abelian profinite group $A$ with $cd_p(A) < \infty$ is $p$-torsion free with $p$-rank equal to $cd_p(A)$. Therefore any Abelian normal subgroup of a profinite group $G$ of finite cohomological $p$-dimension satisfies the hypothesis of Theorem 2.3. Thus if $A$ is an Abelian normal subgroup $G$ we have the following equality

$$cd_p(A) + vcd_p(G/A) = cd_p(G). \tag{12}$$

Moreover, for central subgroups in $G$ one has even stronger implications.

**Corollary 3.2.** *Let $G$ be a profinite group with $cd_p(G) = 2$ and let $A_p$ denote the pro-$p$ Sylow subgroup of the center of $G$. Then one of the following holds*:

(i) *$A_p$ is trivial*;

(ii) *$A_p \simeq \mathbb{Z}_p$ and $cd_p(G') \leqslant 1$, where $G' = cl([G,G])$ denotes the closure of the derived group of $G$*;

(iii) *$A_p \simeq \mathbb{Z}_p \times \mathbb{Z}_p$ and the pro-$p$ Sylow subgroups of $G/A_p$ are finite.*

**Proof.** As $cd_p(A_p) \leqslant 2$, $A_p$ is isomorphic to either of the groups $1$, $\mathbb{Z}_p$, $\mathbb{Z}_p \times \mathbb{Z}_p$. If $A_p \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, Theorem 2.3 implies that $vcd_p(G/A_p) = 0$. Moreover, a profinite group has virtual cohomological $p$-dimension 0, if and only if its pro-$p$ Sylow subgroups are finite.

If $A_p \simeq \mathbb{Z}_p$, Theorem 2.3 implies that $vcd_p(G/A_p) = 1$. Hence the Hochschild–Lyndon–Serre spectral sequence implies that $H^2(G/A_p, \mathbb{Q}_p/\mathbb{Z}_p)$ is a finite Abelian $p$-group. Moreover, $H^1(A_p, \mathbb{Q}_p/\mathbb{Z}_p)$ is a trivial $G$-module isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$ and in particular $p$-divisible. Thus from the 5-term exact sequence one concludes that the restriction mapping $res : H^1(G, \mathbb{Q}_p/\mathbb{Z}_p) \to H^1(A_p, \mathbb{Q}_p/\mathbb{Z}_p)$ is surjective. The Pontryagin dual $res^* : A_p \to G/G'$, which coincides with the canonical map, is therefore injective. Hence $A_p \cap G' = 1$. This yields that $cd_p(G') \leqslant 1$. □

Recall that the deficiency of a finitely presented pro-$p$ group $G$ is given by

$$df(G) := d_G - r_G = 1 - \chi_G, \tag{13}$$

where $d_G$ denotes the minimal number of generators of $G$ and $r_G$ the minimal number of pro-$p$ relators of $G$. The following proposition is a pro-$p$ version of a result of Bieri (cf. [2], Corollary 4).

**Corollary 3.3.** *Let $G$ be a finitely presented pro-$p$ group with $cd(G) = 2$ and let $N$ be a non-trivial finitely presented normal subgroup such that $G/N$ is infinite. Suppose that the deficiency $df(G)$ of $G$ is positive. Then $df(G) = 1$ and either $N \simeq \mathbb{Z}_p$ or $N$ is a non-Abelian free pro-$p$ group and $G/N$ is virtually procyclic.*

**Proof.** Note that the hypothesis on the deficiency reads as $\chi_G \leqslant 0$. Therefore the hypothesis of the proposition is valid for any open subgroup of $G$ containing $N$. If $N \simeq \mathbb{Z}_p$, the claim follows. Suppose $N \not\simeq \mathbb{Z}_p$. Then by Theorem 2.3, $N$ is a non-Abelian free pro-$p$ group and there exists an open normal subgroup $N_1$ of $G$ containing $N$ such that $N_1/N$ is a free pro-$p$ group. The multiplicity of the Euler–Poincaré characteristic $\chi$ yields

$$\chi_{N_1} = \bigl(1 - rk(N)\bigr)\bigl(1 - rk(N_1/N)\bigr), \tag{14}$$

which by hypothesis is a non-positive number. However, the right-hand side of (14) shows that it is also non-negative. Thus $\chi_{N_1} = 0$, and as $rk(N) > 1$, this implies $N_1/N \simeq \mathbb{Z}_p$. This completes the proof. □

Combining the main result in [5] with Theorem 2.3 one obtains:

**Corollary 3.4.** *Let $G$ be a finitely generated pro-$p$ group with $cd_p(G) = n < \infty$ and let $N$ be a closed normal subgroup of $G$ with $cd_p(N) = cd_p(G) - 1$. Assume further that $|H^{n-1}(N, \mathbb{F}_p)| < \infty$. Then $G$ splits non-trivially as either a free amalgamated pro-$p$ product or pro-$p$ HNN-extension.*

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## References

[1] D.J. Benson, Representations and Cohomology, II. Cohomology of Groups and Modules, in: Cambridge Stud. Adv. Math., vol. 31, Cambridge University Press, 1991.

[2] R. Bieri, On groups of cohomological dimension 2, topol. and algebra, in: Proc. Colloq. in Honor of B. Eckmann, Zürich, 1977, 1978, pp. 55–62.

[3] A. Brumer, Pseudocompact algebras, profinite groups and class formations, J. Algebra 4 (1966) 442–470.

[4] A. Engler, D. Haran, D. Kochloukova, P.A. Zalesskii, Normal subgroups of profinite groups of finite cohomological dimension, J. London Math. Soc., in press.

[5] W.N. Herford, P.A. Zalesskii, Virtually free pro-$p$ groups, ANUM preprint No. 132/01, University of Technology, Vienna, 2001.

[6] J. Neukirch, A. Schmidt, K. Wingberg, Cohomology of Number Fields, in: Grundlehren Math. Wiss., vol. 323, Springer, Berlin, 1999.

[7] J.-P. Serre, Cohomologie Galoisienne, 5 cinquième édition, révisée et complétée, in: Lecture Notes in Math., Springer-Verlag, Berlin, 1994.

[8] P. Symonds, T. Weigel, Cohomology of $p$-adic analytic groups, in: M. du Sautoy, D. Segal, A. Shalev (Eds.), New Horizons in pro-$p$-Groups, in: Progr. Math., vol. 184, Birkhäuser, 2000, pp. 349–410.