## Algebraic Geometry

# On the irreducibility of multivariate subresultants 

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#### Abstract

Let $P_{1}, \ldots, P_{n}$ be generic homogeneous polynomials in $n$ variables of degrees $d_{1}, \ldots, d_{n}$ respectively. We prove that if $v$ is an integer satisfying $\sum_{i=1}^{n} d_{i}-n+1-\min \left\{d_{i}\right\}<v$, then all multivariate subresultants associated to the family $P_{1}, \ldots, P_{n}$ in degree $v$ are irreducible. We show that the lower bound is sharp. As a byproduct, we get a formula for computing the residual resultant of $\binom{\rho-v+n-1}{n-1}$ smooth isolated points in $\mathbb{P}^{n-1}$. To cite this article: L. Busé, C. D'Andrea, C. R. Acad. Sci. Paris, Ser. I 338 (2004). © 2004 Académie des sciences. Published by Elsevier SAS. All rights reserved.


## Résumé

Sur l'irréductibilité des sous-résultants multivariés. Soient $P_{1}, \ldots, P_{n}$ des polynômes homogènes génériques en $n$ variables de degré respectif $d_{1}, \ldots, d_{n}$. Nous montrons que si $v$ est un entier tel que $\sum_{i=1}^{n} d_{i}-n+1-\min \left\{d_{i}\right\}<v$, tous les sous-résultants multivariés de degré $v$ des polynômes $P_{1}, \ldots, P_{n}$ sont irréductibles. Nous montrons également que cette borne est atteinte dans des cas particuliers. Comme conséquence directe nous obtenons une nouvelle formule pour le calcul du résultant résiduel de $\binom{\rho-v+n-1}{n-1}$ points lisses isolés dans $\mathbb{P}^{n-1}$. Pour citer cet article: L. Busé, C. D'Andrea, C. R. Acad. Sci. Paris, Ser. I 338 (2004).
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Classical subresultants of two univariate polynomials have been studied by Sylvester in the foundational work [13]. Multivariate subresultants, introduced in [2], provide a criterion for over-constrained polynomial systems to have Hilbert function of prescribed value, generalizing the classical case. To be more precise, let $\mathbb{K}$ be a field. If $P_{1}, \ldots, P_{s}$ are homogeneous polynomials in $\mathbb{K}\left[X_{1}, \ldots, X_{n}\right]$ with $d_{i}=\operatorname{deg}\left(P_{i}\right)$ and $s \leqslant n, H_{d_{1}, \ldots, d_{s}}(\cdot)$ is the Hilbert function of a complete intersection given by $s$ homogeneous polynomials in $n$ variables of degrees $d_{1}, \ldots, d_{s}$, and $S$ is a set of $H_{d_{1}, \ldots, d_{s}}(\nu)$ monomials of degree $v$, the subresultant $\Delta_{S}^{v}$ is a polynomial in the coefficients of the $P_{i}$ 's of degree $H_{d_{1}, \ldots, d_{i-1}, d_{i+1}, \ldots, d_{n}}\left(\nu-d_{i}\right)$ in the coefficients of $P_{i}(i=1, \ldots, s)$ having the following universal property: $\Delta_{S}^{v} \neq 0$ if and only if $I_{v}+\mathbb{K}\langle S\rangle=\mathbb{K}\left[X_{1}, \ldots, X_{n}\right]_{v}$, where $I_{v}$ is the degree $v$ part of the ideal generated by the $P_{i}$ 's (see [2]).

[^0]Multivariate subresultants have been used in computational algebra for polynomial system solving [10,14] as well as for providing explicit formulas for the representation of rational functions [11,6,7,12]. The study of their properties is an active area of research [3,4,6-8]. In particular, it is important to know which $S$ verify $\Delta_{S}^{v} \neq 0$, and which of these $\Delta_{S}^{v}$ are irreducible (see the final remarks and open questions in [2] and the conjectures in [7]). Partial results have been obtained in this direction. In [5] it is shown that, if $s=n$ and $\sum_{i=1}^{n} d_{i}-n-\min \left\{d_{i}\right\}<v$, then for every set $S$ of monomials of degree $v$ and cardinal $H_{d_{1}, \ldots, d_{n}}(\nu)$, the polynomial $\Delta_{S}^{v}$ is not identically zero. Moreover, in [4], it is also proved that if $s=n, v=\sum_{i=1}^{n} d_{i}-n$, and $S=\left\{x_{j}^{\nu}\right\}$ for $j=1, \ldots, n$, then $\Delta_{S}^{v}$ is an irreducible polynomial in the coefficients of the $P_{i}$ 's. In [8, Lemma 4.2] the irreducibility of $\Delta_{S}^{v}$ is shown for $s=n=2, \max \left\{d_{1}, d_{2}\right\} \leqslant v$, and $S=\left\{X_{2}^{\nu}, X_{1} X_{2}^{\nu-1}, \ldots, X_{1}^{H_{d_{1}, d_{2}}(\nu)-1} X_{2}^{\nu-H_{d_{1}, d_{2}}(\nu)+1}\right\}$.

In this Note we study the irreducibility problem in the case $s=n$. Let us introduce some notations in order to state our result. Let $\rho:=\sum_{i=1}^{n}\left(d_{i}-1\right)$. For $i=1, \ldots, n$ and $\alpha \in \mathbb{Z}_{\geqslant 0}^{n}$ such that $|\alpha|=d_{i}$, introduce a new variable $c_{i, \alpha}$. Let $\mathbb{A}:=\mathbb{Z}\left[c_{i, \alpha}, i=1, \ldots, n,|\alpha|=d_{i}\right]$ and set

$$
\begin{equation*}
P_{i}\left(x_{1}, \ldots, x_{n}\right):=\sum_{|\alpha|=d_{i}} c_{i, \alpha} x^{\alpha} \tag{1}
\end{equation*}
$$

Theorem. For every $v$ such that $\rho-\min \left\{d_{i}\right\}+1<v$ and every set $S$ of monomials of degree $v$ and cardinality $H_{d_{1}, \ldots, d_{n}}(\nu)$, the subresultant $\Delta_{S}^{\nu}\left(P_{1}, \ldots, P_{n}\right)$ is irreducible in $\mathbb{A}$.

Observe that, if $n=2$, then $\rho-\min \left\{d_{i}\right\}+1=d_{1}+d_{2}-2-\min \left\{d_{i}\right\}+1=\max \left\{d_{i}\right\}-1$, and this is equivalent to $\max \left\{d_{i}\right\} \leqslant v$, so our result contains those in [8].

Proof. For simplicity we assume hereafter that $d_{1} \geqslant \cdots \geqslant d_{n} \geqslant 1$. First observe that if $v>\rho$ then $\Delta_{S}^{v}$ is simply a resultant, and is hence known to be irreducible. So, we can suppose w.l.o.g. that $d_{n}>1$. We thus only have to consider integers $v$ such that

$$
\begin{equation*}
\rho \geqslant v>\rho-d_{n}+1=\sum_{i=1}^{n-1}\left(d_{i}-1\right) \tag{2}
\end{equation*}
$$

where we recall that $\rho=\sum_{i=1}^{n}\left(d_{i}-1\right)$. We begin by computing the multi-degree of the subresultants $\Delta_{S}^{v}$; we know (see [2]) that

$$
\operatorname{deg}_{P_{i}}\left(\Delta_{S}^{v}\right)=H_{d_{1}, \ldots, d_{i-1}, d_{i+1}, \ldots, d_{n}}\left(v-d_{i}\right)
$$

But from the standard short exact sequence

$$
0 \rightarrow \frac{R}{\left(f_{1}, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{n}\right)}\left(-d_{i}\right) \xrightarrow{\times f_{i}} \frac{R}{\left(f_{1}, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{n-1}\right)} \rightarrow \frac{R}{\left(f_{1}, \ldots, f_{n}\right)} \rightarrow 0
$$

where $f_{1}, \ldots, f_{n}$ are homogeneous polynomials of respective degree $d_{i}$ in a graded polynomial ring $R$ and $f_{1}, \ldots, f_{n}$ is a complete intersection in $R$, we deduce

$$
H_{d_{1}, \ldots, d_{i-1}, d_{i+1}, \ldots, d_{n}}\left(t-d_{i}\right)=H_{d_{1}, \ldots, d_{i-1}, d_{i+1}, \ldots, d_{n}}(t)-H_{d_{1}, \ldots, d_{n}}(t)
$$

for all integer $t$. It follows that for all integer $v \geqslant \rho-d_{n}+1$,

$$
\begin{equation*}
\operatorname{deg}_{P_{i}}\left(\Delta_{S}^{v}\right)=\frac{d_{1} \ldots d_{n}}{d_{i}}-H_{d_{1}, \ldots, d_{n}}(v)=\frac{d_{1} \ldots d_{n}}{d_{i}}-\binom{\rho-v+n-1}{n-1} \tag{3}
\end{equation*}
$$

where that last equality comes from the facts that $H_{d_{1}, \ldots, d_{n}}(\rho-t)=H_{d_{1}, \ldots, d_{n}}(t)$ for all integer $t$, and $H_{d_{1}, \ldots, d_{n}}(t)=$ $\binom{t+n-1}{n-1}$ for all $0 \leqslant t<d_{n}$. We define $\mathbf{a}:=\binom{\rho-v+n-1}{n-1}$. As a does not depend on $i \in\{1, \ldots, n\}$ and residual (or reduced) resultants of a isolated points in $\mathbb{P}^{n-1}$ have the same degree in the coefficients of $P_{i}$ as the right-hand side of (3), this suggest that we compare $\Delta_{S}^{v}$ with residual resultants.

We will work with an ideal $G$ defining a points in $\mathbb{P}^{n-1}$ which is generated in degree at most $d_{n}$ and such that $G_{d_{n}-1} \neq 0$. Ideals defining a points in sufficiently generic position are generated in degree exactly $\rho-v+1$ (see [9, Proposition 4]). Since by (2) we have $d_{n}>\rho-v+1$, we thus choose such an ideal $G=\left(g_{1}, \ldots, g_{m}\right)$, where $\operatorname{deg}\left(g_{i}\right)=\rho-v+1$ for all $i=1, \ldots, m$, defining a points in generic position (see [9] for the definition of "generic position"), and hence locally a complete intersection.

Now consider the following specialization of polynomials $P_{i}$ 's

$$
\begin{equation*}
P_{i} \mapsto \bar{P}_{i}:=\sum_{j=1}^{m} p_{i j}(x) g_{j}(x) \tag{4}
\end{equation*}
$$

where $p_{i j}(x)=\sum_{|\alpha|=d_{i}-\underline{\rho}+v-1} c_{\underline{i j}}^{|\alpha|} x^{\alpha}$ is a generic polynomial of degree $d_{i}-\rho+v-1$. There exists a resultant associated to the system $\bar{P}_{1}, \ldots, \bar{P}_{n}$, called the residual resultant. We denote it by $\operatorname{Res}_{G}\left(\bar{P}_{1}, \ldots, \bar{P}_{n}\right)$. Let us recall its main properties (see [1], §3.1).

- $\operatorname{Res}_{G}\left(\bar{P}_{1}, \ldots, \bar{P}_{n}\right)$ is a homogeneous and irreducible polynomial in the ring of all the coefficients $\mathbb{Q}\left[c_{i j}^{|\alpha|}\right]$;
- For any given specialization of the coefficients $c_{i j}^{|\alpha|}$,s sending $\overline{P_{i}}$ to $Q_{i}$, we have

$$
\operatorname{Res}_{G}\left(Q_{1}, \ldots, Q_{n}\right)=0 \quad \text { if and only if } \quad\left(Q_{1}, \ldots, Q_{n}\right)^{\text {sat }} \varsubsetneqq G=G^{\text {sat }}
$$

- $\operatorname{Res}_{G}\left(\overline{P_{1}}, \ldots, \overline{P_{n}}\right)$ is multi-homogeneous: it is homogeneous in the coefficients of each polynomials $\overline{P_{i}}$, $i=1, \ldots, n$, and we have

$$
\operatorname{deg}_{\bar{P}_{i}}\left(\operatorname{Res}_{G}\left(\bar{P}_{1}, \ldots, \bar{P}_{n}\right)\right)=\frac{d_{1} \ldots d_{n}}{d_{i}}-\mathbf{a}
$$

We are now going to compare this residual resultant with the specialized subresultant $\Delta_{S}^{v}\left(\bar{P}_{1}, \ldots, \bar{P}_{n}\right)$, which is non-zero as proved in [4]. We claim that we have the following implications:

$$
\begin{equation*}
\Delta_{S}^{v}\left(Q_{1}, \ldots, Q_{n}\right) \neq 0 \Rightarrow H_{(\underline{Q})}(v)=\mathbf{a} \Rightarrow H_{(\underline{Q})}(t)=\mathbf{a} \text { for all } t \geqslant v \Rightarrow \operatorname{Res}_{G}\left(Q_{1}, \ldots, Q_{n}\right) \neq 0 \tag{5}
\end{equation*}
$$

where $H_{(\underline{Q})}(\cdot)$ denotes the Hilbert function associated to the ideal $\left(Q_{1}, \ldots, Q_{n}\right)$. Only the second implication needs to be proved, the others follow directly from the algebraic properties of resultants and subresultants. We know that $H_{G}(t)=\mathbf{a}$ for all $t \geqslant \rho-v+1$ (see [9]), and since we have supposed (2), it is a straightforward computation to show that $v \geqslant \rho-v+1$. It follows that, by hypothesis, the ideals $G$ and $(\underline{Q})$ coincide in degree $v$ and have Hilbert function $\mathbf{a}$ in this degree. As they are both generated in degree at most $v$ this implies that they coincide in all higher degrees, and therefore they both have Hilbert function equal to a in these degrees, because $G$ is the defining ideal of a set of points.

Due to (5) and the irreducibility of the residual resultant, we deduce that $\operatorname{Res}_{G}\left(\bar{P}_{1}, \ldots, \bar{P}_{n}\right)$ divides $\Delta_{S}^{v}\left(\bar{P}_{1}, \ldots, \bar{P}_{n}\right)$. But both polynomials have the same degree, so they must be equal up to a rational number (giving a new formula for computing this residual resultant using [3]). Since this residual resultant is irreducible, and since $\Delta_{S}^{v}$ and $\Delta_{S}^{v}\left(\bar{P}_{1}, \ldots, \bar{P}_{n}\right)$ have the same multi-degree, this shows that $\Delta_{S}^{v}$ is irreducible in $\mathbb{Q}\left[\operatorname{coeff}\left(P_{i}\right)\right]$.

It remains to prove that $\Delta_{S}^{\nu}$ is irreducible in $\mathbb{Z}\left[\operatorname{coeff}\left(P_{i}\right)\right]$. As it is irreducible in $\mathbb{Q}\left[\operatorname{coeff}\left(P_{i}\right)\right]$, we only have to show that $\Delta_{S}^{v}$ has content $\pm 1$. Suppose that this is not the case, and let $p \in \mathbb{Z}$ be a prime dividing the content of $\Delta_{S}^{\nu}$. Let $k$ be the algebraic closure of $\mathbb{Z}_{p}$. This implies that $\Delta_{S}^{\nu}=0$ in $K:=k\left(\operatorname{coeff}\left(P_{i}\right)\right)$, and hence $S$ is linearly dependent in $K\left[x_{1}, \ldots, x_{n}\right] /\left\langle P_{1}, \ldots, P_{n}\right\rangle$, contradicting the main result of [4].

Reducibility in lower degrees: We now exhibit some sets $S$ of degree $\nu=\rho-\min \left\{d_{i}\right\}+1$ such that $\Delta_{S}^{\nu}$ factorizes. This shows that the lower bound in our theorem is sharp.

- $n=2, d_{1}>d_{2}$ : In this case, $v=d_{1}-1 \geqslant d_{2}$, and $H_{d_{1}, d_{2}}(v)=d_{2}$. Thus $\Delta_{S}^{v}$ can be here computed with Sylvester type matrices [13]. However, setting $f_{2}=c_{0} x_{1}^{d_{2}}+c_{1} x_{1}^{d_{2}-1} x_{2}+\cdots+c_{d_{2}} x_{2}^{d_{2}}$, the universal property
of the subresultant $\Delta_{S}^{v}$ shows immediatly that it is a power of $c_{0}$, and we have already seen that its degree is $d_{1}-d_{2}+1$; it follows that $\Delta_{S}^{\nu}=c_{0}^{d_{1}-d_{2}+1}$, so it cannot be irreducible.
- $n>2, d_{1}-1>d_{2}=d_{3}=\cdots=d_{n}=1$ : Again in this case, $v=d_{1}-1$ and $H_{d_{1}, d_{2}}(v)=1$. Choose $S=\left\{x_{1}^{\nu}\right\}$ and, if $f_{i}=c_{1 i} x_{1}+\cdots+c_{n i} x_{n}, i=2, \ldots, n$, we set $\delta:=\operatorname{det}\left(c_{i j}\right)_{2 \leqslant i, j \leqslant n}$. Applying Lemma 4.4 in [6] to this situation, we get that $\Delta_{S}^{\nu}=\delta^{\nu}$. So, $\Delta_{S}^{v}$ is not irreducible.


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