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The wavelet Mortar method in the adaptative framework

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Abstract

This paper is concerned with the extension to the case of a nonuniform discretization of the definition of the Mortar wavelet method. Given a (biorthogonal) non-uniform wavelet space, satisfying a suitable cone (or tree) condition, we construct a multiplier space satisfying the requirements for stability and approximation. ***To cite this article: S. Bertoluzza, A.-S. Piquemal, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

La méthode de Mortar en ondelettes dans le cas adaptatif. Nous définissons l'extension de la méthode de Mortar en ondelettes dans le cadre d'une discrétisation non-uniforme, et construisons un espace de multiplicateurs, satisfaisant des hypothèses d'approximation et de stabilité, associé à des espaces d'ondelettes reliés par une condition de cône. ***Pour citer cet article : S. Bertoluzza, A.-S. Piquemal, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Wavelet bases have proven to be particularly well suited to the design of adaptive schemes for the solution of partial differential equations. The good localization properties of such bases (both in space and in frequency) and the consequent norm equivalences in terms of wavelet coefficients allow to use the coefficients themselves as a criterion for deciding whether refining or de-refining, and the refinement/de-refinement procedure is particularly simple. On the other hand, such bases suffer from serious drawbacks, which limit their actual applicability to real life scientific computing problems. One of such drawback is the inherent tensorial nature of such bases when considering dimensions $\geqslant 2$. To overcome such problem it is then necessary to resort to some form of domain decomposition. The use of wavelet bases in the framework of the mortar domain decomposition method [1,2] has been studied in [3,4] in the case of uniform wavelet discretization. The aim of this Note is to extend the study to the case of non-uniform wavelet discretizations under the assumption that the spaces considered satisfy a *cone* type condition.

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1. The wavelet approximation and multiplier spaces

Here, we work with a couple $(V_j)_{j\geqslant j_0}$, and $(\widetilde{V}_j)_{j\geqslant j_0}$ of biorthogonal multiresolution analyses (MRA) of $L^2([0,1])$, [6]. The corresponding compactly supported scaling function bases $\{\phi_{j,k},\ k=0,\ldots,2^j+1\}$ and $\{\tilde{\phi}_{j,k},\ k=0,\ldots,2^j+1\}$, are assumed to be biorthogonal. We assume that for some $r>1,\ r'>0$, one has $\phi_{j,k}\in H^r([0,1])$, and $\tilde{\phi}_{j,l}\in H^{r'}([0,1])$, and the polynomials of degree $d-1$ and $\tilde{d}-1$ are included in V_{j_0} and $\widetilde{V}_{j_0}$ respectively. Finally, we can also suppose that all the scaling functions of V_j vanish at the edges 0 and 1, except one function at each edge: $\phi_{j,0}(0)\neq 0$ and, $\phi_{j,2^j+1}(1)\neq 0$. The two complement (or *wavelet*) spaces W_j and $\widetilde{W}_j$ are defined by $W_j=(P_{j+1}-P_j)V_{j+1}$, and $\widetilde{W}_j=(\widetilde{P}_{j+1}-\widetilde{P}_j)\widetilde{V}_{j+1}$, where P_j and $\widetilde{P}_j$ are the projectors on V_j and $\widetilde{V}_j$ orthogonal to $\widetilde{V}_j$ and V_j respectively. Following [6], it exists two biorthogonal Riesz bases $\{\psi_{j,k},\ k=0,\ldots,2^j-1\}$, $\{\tilde{\psi}_{j,k},\ k=0,\ldots,2^j-1\}$ for these spaces, constructed in such a way that at each scale j, only one wavelet does not vanish at the each edge, i.e., $\psi_{j,0}(0)\neq 0$, and $\psi_{j,2^j-1}(1)\neq 0$.

Let now $V_j^0=V_j\cap H_0^1(0,1)=\operatorname{span}\{\phi_{j,k},\ k=1,\ldots,2^j\}$. Following [3], we can construct a biorthogonal space $\widetilde{V}_j^*=\operatorname{span}\{\tilde{\phi}_{j,k}^*,\ k=1,\ldots,2^j\}$ which contains all the polynomials of degree $\tilde{d}-1$. As previously, assuming that π_j and $\tilde{\pi}_j$ are respectively the projectors on V_j^0 and $\widetilde{V}_j^*$ orthogonal to $\widetilde{V}_j^*$ and V_j^0, the corresponding wavelet spaces are defined by $W_j^0=(\pi_{j+1}-\pi_j)V_{j+1}^0$, and $\widetilde{W}_j^*=(\tilde{\pi}_{j+1}-\tilde{\pi}_j)\widetilde{V}_{j+1}^*$.

We next consider a 2-D biorthogonal MRA $(\mathcal{V}_j,\widetilde{\mathcal{V}}_j)_{j\geqslant j_0}$ of $L^2(]0,1[^2)$ constructed using tensor products of the one dimensional multiresolution analyses, that is $\mathcal{V}_j=V_j\otimes V_j$ and $\widetilde{\mathcal{V}}_j=\widetilde{V}_j\otimes\widetilde{V}_j$. The corresponding wavelet spaces $\mathcal{W}_j,\ \widetilde{\mathcal{W}}_j,\ j\geqslant j_0$, are respectively spanned by two families of functions $\{\Psi_{j,\kappa}^{\varepsilon}\}_\kappa$ and $\{\tilde{\Psi}_{j,\kappa}^{\varepsilon}\}_\kappa$, with $\kappa=(k,l), 0\leqslant k,l\leqslant 2^j-1,\ \varepsilon=1,2,3$, defined by

$$\Psi_{j,\kappa}^1(x,y)=\psi_{j,k}(x)\psi_{j,l}(y),\qquad \Psi_{j,\kappa}^2(x,y)=\psi_{j,k}(x)\phi_{j,l}(y),\qquad \Psi_{j,\kappa}^3(x,y)=\phi_{j,k}(x)\psi_{j,l}(y).$$

Let now an approximation space $\mathcal{U}_J\subset\mathcal{V}_J$ be given, of the form

$$\mathcal{U}_J=\mathcal{V}_{j_0}\oplus\left(\bigoplus_{j=j_0}^{J-1}\mathcal{X}_j\right),$$

with

$$\begin{cases}\mathcal{X}_j=\operatorname{span}\{\Psi_{j,\kappa}^{\varepsilon},(\varepsilon,\kappa)=(\varepsilon,k,l)\in D_j\}\subset\mathcal{W}_j,\\ D_j\subset\{(\varepsilon,k,l),\varepsilon\in\{1,2,3\},\ k,l\in\{0,\ldots,2^j-1\}\}.\end{cases}$$

Definition 1.1. We say that a subspace $\mathcal{U}_J\subseteq\mathcal{V}_J$, satisfies a strong cone condition if $\Psi_{j,\kappa}^{\varepsilon}\in\mathcal{X}_j$ implies $\Psi_{m,\eta}^{\varepsilon}\in\mathcal{X}_m$ for all wavelets $\Psi_{m,\eta}^{\varepsilon}\in\mathcal{W}_m$ with $m<j$ and $\operatorname{supp}\Psi_{j,\kappa}^{\varepsilon}\cap\operatorname{supp}\tilde{\Psi}_{m,\eta}^{\varepsilon}\neq\emptyset$.

To fix the ideas let us construct the multiplier space associated to the edge $\gamma_y=\{(x,0),x\in(0,1)\}$. The trace on γ_y of a funcion $u\in\mathcal{U}_J$ will have contributions from the following basis functions (the ones not identically vanishing on the edge): for $j_0\leqslant j\leqslant J-1$

$$\psi_{j,k}(x)\psi_{j,l}(0),\quad (1,k,l)\in D_j,\qquad \psi_{j,k}(x)\phi_{j,l}(0),\quad (2,k,l)\in D_j,\qquad \phi_{j,k}(x)\psi_{j,l}(0),\quad (3,k,l)\in D_j.$$

Then, the trace space is spanned by functions of the form $\psi_{j,k}(x)$ and $\phi_{j,k}(x)$ and can be defined by

$$T_y(\mathcal{U}_J)=V_{j_0}+\operatorname{span}\{\psi_{j,k},\ j_0\leqslant j\leqslant J-1,\ k\in d_j\}+\operatorname{span}\{\phi_{j,k},\ j_0\leqslant j\leqslant J-1,\ k\in c_j\},$$

where $d_j = \{k, (1,k,0) \in D_j,\} \cup \{k, (2,k,0) \in D_j,\}$ and $c_j = \{k, (3,k,0) \in D_j\}$. Remark that the functions $\psi_{j,k},\ j_0 \leqslant j \leqslant J-1,\ k \in d_j$ and $\phi_{j,k},\ j_0 \leqslant j \leqslant J-1,\ k \in c_j$ are in general not linearly independent. We can however prove the following result [5].

Proposition 1.1. *If $\mathcal{U}_J$ satisfies a strong cone condition then the trace space $T_y(\mathcal{U}_J)$ verifies*

$$T_y(\mathcal{U}_J) = V_{j_0} \oplus \operatorname{span}\{\psi_{j,k},\ j_0 \leqslant j \leqslant J-1,\ k \in d_j\}.$$

Let us now consider the subspace $T_y^0(\mathcal{U}_J) = \{u \in T_y(\mathcal{U}_J);\ u(0) = u(1) = 0\} \subset T_y(\mathcal{U}_J)$. In particular we are interested in a basis $\{h_{j,k}, j_0 \leqslant j \leqslant J-1, k \in d_j\}$, for $T_y^0(\mathcal{U}_J)$, in such a way that

$$T_y^0(\mathcal{U}_J) = V_{j_0}^0 \oplus \left(\bigoplus_{j=j_0}^{J-1} X_j^0\right), \quad \text{with } X_j^0 = \operatorname{span}\{h_{j,k},\ k \in d_j\}, \quad \text{and} \quad \forall k \in d_j,\ h_{j,k} \perp \widetilde{V}_j^*.$$

Under the assumption that $\mathcal{U}_J$ satisfies the strong cone condition, this holds [5] if, for $j_0 \leqslant j \leqslant J-1$, we define the functions $h_{j,k}$ by

$$\begin{cases} h_{j,k} = \psi_{j,k}, & k \neq 0,\ k \neq 2^j - 1, \\ h_{j,0} = (\mathrm{Id} - \pi_j)\big(\psi_{j,0} - \frac{\psi_{j,0}(0)}{\phi_{j,0}(0)}\phi_{j,0}\big), & \\ h_{j,2^j-1} = (\mathrm{Id} - \pi_j)\big(\psi_{j,2^j-1} - \frac{\psi_{j,2^j-1}(1)}{\phi_{j,2^j+1}(-1)}\phi_{j,2^j+1}\big). & \end{cases}$$

In addition, it is not difficult to prove the following proposition [5].

Proposition 1.2. *The set $\{h_{j,k},\ k = 0, \ldots, 2^j - 1\}$ constitutes a basis for W_j^0.*

It is then possible to construct a basis $\{\tilde{h}_{j,k},\ k = 0, \ldots, 2^j - 1\}$ for W_j^* such that the following biorthogonality property holds $\langle h_{jk}, \tilde{h}_{mn}\rangle = \delta_{jm}\delta_{kn}$. Moreover, the functions $\tilde{h}_{j,k}$ can be constructed in such a way that they have the same space localization properties as the functions $\tilde{\psi}_{j,k}$. We even have $\tilde{h}_{j,k} = \tilde{\psi}_{j,k}$ for $j_0 \leqslant j \leqslant J-1$, $k \neq 0$ and $k \neq 2^j - 1$.

We now define the multiplier space $M_y(\mathcal{U}_J)$ as the dual of the trace space $T_y^0(\mathcal{U}_J)$: $M_y(\mathcal{U}_J) = \widetilde{V}_{j_0}^* \oplus \big(\bigoplus_{j=j_0}^{J-1} \widetilde{X}_j^*\big)$, with $\widetilde{X}_j^* = \operatorname{span}\{\tilde{h}_{j,k}, k \in d_j\}$. For such a space it is possible to prove the existence and boundedness of a projection and of a lifting operator, which play a key role in the error estimate for the Mortar method described in the next section. More precisely we have the following theorem.

Theorem 1.2. *Let $\check{\pi}_J : L^2(0,1) \to T_y^0(\mathcal{U}_J)$ be defined for all $u \in L^2(0,1)$ by*

$$\check{\pi}_J u = \pi_{j_0} u + \sum_{j=j_0}^{J-1} \sum_{k \in d_j} \langle u, \tilde{h}_{j,k}\rangle h_{j,k}.$$

Then, for all $\eta \in H_{00}^{1/2}(0,1)$ the bound $\|\check{\pi}_J \eta\|_{H_{00}^{1/2}(0,1)} \lesssim \|\eta\|_{H_{00}^{1/2}(0,1)}$ holds, and for all $\eta \in L^2(0,1)$ and $\lambda \in M_y(\mathcal{U}_J)$ we have

$$\int_0^1 (\eta - \check{\pi}_J \eta)\lambda = 0.$$

Moreover, there exist a lifting $\mathcal{L}_J : T_y^0(\mathcal{U}_J) \to \mathcal{U}_J$ such that for all $\eta_h \in T_y^0(\mathcal{U}_J)$ we have $\mathcal{L}_J \eta_h = \eta_h$ on γ_y and $\|\mathcal{L}_J \eta_h\|_{1,(0,1)^2} \lesssim \|\eta_h\|_{H_{00}^{1/2}(0,1)}$.

2. The non-uniform Mortar wavelet method

Let now $\Omega \in \mathbb{R}^2$ be a polygonal domain. We consider a geometrically conforming decomposition of Ω as the union of L subdomains Ω_l ($l = 1, \ldots, L$), of side $\Gamma_{ln} = \partial\Omega_l \cup \partial\Omega_n$ for simplicity we will assume to be rectangular. Let $\Sigma = \cup\Gamma_{ln}$. Correspondingly we consider the broken norm $\|u\|_{1,*}^2 = \sum_l \|u_l\|_{1,\Omega_l}^2$. We consider the following simple problem. Given $f \in L^2(\Omega)$, find $u : \Omega \to \mathbb{R}$ such that

$$-\Delta u = f, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega. \tag{1}$$

We now consider the Mortar wavelet method with nonuniform discretization. After identifying (using a suitable affine transformation) each sub-domain with the reference square $]0,1[^2$, for each subdomain we consider a wavelet discretization space $\mathcal{U}_J^l$ (with $J = J(l)$) satisfying the strong cone condition. We can assemble the global multiplier space $\mathcal{M}_\delta$ from the local multiplier spaces defined on the slave sides according to the construction introduced in the previous section. Letting $\mathcal{V}_\delta = \prod_l \mathcal{U}_J^l$ we then look for an approximation $u_\delta \in \mathcal{V}_\delta$ to the solution u of problem (1) satisfying the weak continuity condition $\int_\Sigma [u_h]\lambda = 0$ for all $\lambda \in \mathcal{M}_\delta$ ($[\cdot]$ denoting the jump taken with the proper sign) and such that for all $v_h \in \mathcal{V}_\delta$ satisfying the same weak continuity condition one has $\sum_l \int_{\Omega_l} \nabla u_h \cdot \nabla v_h = \int_\Omega f v_h$. It is possible to prove the following error estimate [3,4]:

Theorem 2.1. *We have that*

$$\|u - u_h\|_{1,*} \lesssim \left(\max_l J(l)\right)\left(\inf_{v_h \in \mathcal{V}_\delta} \|u - v_h\|_{1,*} + \inf_{\lambda_h \in \mathcal{M}_\delta} \|\partial u \partial \nu - \lambda_h\|_{-1/2,\Sigma}\right),$$

where $\partial u/\partial\nu$ denotes the trace of the normal derivative on the interface Σ taken with the proper sign and $\|\cdot\|_{-1/2,\Sigma}^2 = \sum_l \|\cdot\|_{-1/2,\partial\Omega_l}^2$.

We want to underline that, though we did not explicitely prove an error estimate – it is always difficult to formally express the approximation properties of nonuniform approximation spaces, since the multiplier space contains the polynomials of degree less or equal to $\tilde{d} - 1$ and since it has the same localization properties of the trace space $T_y(\mathcal{U}_J)$, we believe that, assuming that the space $\mathcal{U}_J$ has been taylored to approximate well the function u through a suitable adaptive procedure, we can expect that the multiplier space is itself well suited to the approximation of the outer normal derivative $\partial u/\partial\nu$.

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