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Mathematical Analysis

Estimating the first zero of a characteristic function

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Abstract

For a characteristic function (Fourier transform of a probability distribution), the first zero encodes important information. We present a general lower bound estimation of the first zero in terms of a moment of any order. The result proves the complementary nature between the first zero and moments, and has interesting implications for quantum mechanical uncertainty relations. *To cite this article: S. Luo, Z. Zhang, C. R. Acad. Sci. Paris, Ser. I 338 (2004).* © 2003 Académie des sciences. Published by Elsevier SAS. All rights reserved.

Résumé

Estimation du premier zero d'une fonction caractéristique. Pour une fonction caractéristique (la transformation de Fourier d'une mesure de probabilité), le premier zéro contient des informations importantes. Nous allons présenter une formule générale pour l'estimée inférieure du premier zéro en terme de moments de tout ordre. Le résultat obtenu illustre l'aspect de complémentarité entre le premier zéro et les moments et sera utilisé pour étudier le principe d'incertitude en mécanique quantique. *Pour citer cet article : S. Luo, Z. Zhang, C. R. Acad. Sci. Paris, Ser. I 338 (2004).* © 2003 Académie des sciences. Published by Elsevier SAS. All rights reserved.

1. Introduction

For any probability distribution function F(x) on $\mathbb{R}^+ = [0, \infty)$, its moment of order p > 0 is:

$$M_p = \int x^p \,\mathrm{d}F(x),$$

and its characteristic function is:

$$\phi(t) = \int \mathrm{e}^{-\mathrm{i}tx} \,\mathrm{d}F(x).$$

Let

 $\tau = \inf\{t > 0: \phi(t) = 0\}$

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be the first positive zero of the characteristic function $\phi(t)$. In some statistical problems (e.g., parameter estimation) and some physical problems (e.g., estimation of the evolution speed of quantum systems and time-energy uncertainty relations), it is required to estimate τ , at least to give a positive lower bound. We will present such an estimation. The main result is as follows:

Theorem 1.1. If F is supported on the positive axis, that is, $[0, \infty)$, then for any positive p, it holds that

$$\tau \geqslant \frac{\pi}{(2M_p)^{1/p}}.$$

From this theorem we see readily that τ and M_p are complementary to each other and satisfy an uncertainty relation: if M_p is small, then τ must be large; if τ is small, then M_p must be large.

This theorem depends crucially on the following subtle inequality.

Proposition 1.2. For any $x \ge 0$ and p > 0, it holds that

$$\cos x + \frac{2p}{\pi} \sin x \ge 1 - 2\left(\frac{x}{\pi}\right)^p.$$

This inequality, though elementary, is rather intriguing and its proof requires delicate and lengthy analysis. It is clear that when x = 0 or π , the equality holds.

2. Proof of Theorem 1.1

By Proposition 1.2, we have:

$$\Re\phi(t) - \frac{2p}{\pi} \Im\phi(t) = \int \left(\cos(tx) + \frac{2p}{\pi}\sin(tx)\right) dF(x)$$
$$\geq \int \left(1 - 2\left(\frac{tx}{\pi}\right)^p\right) dF(x)$$
$$= 1 - \frac{2}{\pi^p} t^p M_p.$$

Here \Re and \Im denote the real part and imaginary part of a complex number, respectively.

Now take $t = \tau$. Since $\phi(\tau) = 0$ implies $\Re \phi(\tau) = 0$ and $\Im \phi(\tau) = 0$, we come to

$$0 \ge 1 - \frac{2}{\pi^p} \tau^p M_p,$$

and the conclusion of Theorem 1.1 follows.

3. Applications to time-energy uncertainty relations

We now present a physical application of Theorem 1.1: it can be used to establish a whole family of quantum mechanical time-energy uncertainty relations. We will use Dirac's notation of bras and kets [1], and proceed in a physically heuristic way.

Time-energy uncertainty relations are of fundamental importance in quantum mechanics, especially in quantum dynamics, and are widely studied in both mathematics and physics [2–4,6,7]. The mathematics behind such relations are intimately related to Fourier transform analysis.

Let *H* be a Hamiltonian (energy operator), that is, a self-adjoint, bounded from below operator on some Hilbert space \mathcal{H} describing a quantum system. Without loss of generality, we can shift *H* such that its ground state energy is zero. Thus we may assume that *H* is non-negative. Let $|\psi\rangle \in \mathcal{H}$ be a normalized wave function, then the *p*-th order standardized moment of *H* in the state $|\psi\rangle$ is defined as

$$\|H\|_p = \langle \psi | H^p | \psi \rangle^{1/p}$$

Let τ be the first time that the state $|\psi\rangle$ evolves into an orthogonal state according to Schrödinger dynamics. Mathematically, τ is defined as (we put the Planck constant $\hbar = 1$)

$$\tau = \inf\{t > 0: \langle \psi | \mathrm{e}^{-\mathrm{i}t H} | \psi \rangle = 0\}$$

Then we have the time-energy uncertainty relations:

$$\tau \|H\|_p \geqslant \frac{\pi}{2^{1/p}}.$$

For p = 1, we recover the Margolus–Levitin theorem [7], and this simple case is of particular importance because $||H||_1 = \langle \psi | H | \psi \rangle$ is the average energy. Since τ is just the first time for a quantum state to evolve into an orthogonal state (that is, maximum change), the above inequality places an upper bound on the evolution speed of a quantum state: in order to have fast evolution rate (small τ), it is necessary to have large *average energy*. This is in sharp contrast to the conventional uncertainty relations which are usually expressed in terms of variance. The estimation for p = 1 is used by Lloyd in calculating the ultimate physical limits of computation [5].

To see how the above uncertainty relations follow from Theorem 1.1, let $\{|E\rangle\}$ be the complete set of the energy eigenstates:

$$H|E\rangle = E|E\rangle, \qquad \langle E'|E\rangle = \delta(E'-E).$$

Let $|\psi\rangle$ be expanded in the energy eigenstates as (when the energy spectrum is discrete, the following integrals should be interpreted as discrete sums):

$$|\psi\rangle = \int \lambda(E) |E\rangle \,\mathrm{d}E,$$

then

$$e^{-itH}|\psi\rangle = e^{-itH}\int \lambda(E)|E\rangle dE = \int e^{-itE}\lambda(E)|E\rangle dE.$$

Consequently, by the Parseval theorem,

$$\langle \psi | \mathrm{e}^{-\mathrm{i}tH} | \psi \rangle = \int \mathrm{e}^{-\mathrm{i}tE} |\lambda(E)|^2 \mathrm{d}E, \qquad \langle \psi | H^p | \psi \rangle = \int E^p |\lambda(E)|^2 \mathrm{d}E.$$

Now just taking $dF(x) = |\lambda(x)|^2 dx$ in Theorem 1.1, we obtain the desired result.

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