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Dynamical shape gradient for the Navier–Stokes system

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Abstract

This Note deals with the sensitivity analysis of a newtonian incompressible fluid driven by the Navier–Stokes equations with respect to the dynamic of the fluid domain boundary. The structure of the gradient with respect to the velocity of the domain for a given cost function is established. This result is obtained using new shape derivation tools for Eulerian functionals and the Min–Max derivation principle. ***To cite this article: R. Dziri et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Gradient dynamique de forme pour le système de Navier–Stokes. Dans cette Note, nous nous intéressons à l'analyse de sensibilité de l'évolution d'un fluide newtonien incompressible régi par les équations de Navier–Stokes vis-à-vis de la dynamique de la frontière du domaine fluide. Nous établissons la structure du gradient d'une fonctionnelle coût spécifique par rapport à la vitesse du domaine mobile. Ce résultat est obtenu en utilisant, de façon combinée, des techniques nouvelles de dérivation de forme pour des fonctionnelles eulériennes et le principe de dérivation du Min–Max. ***Pour citer cet article : R. Dziri et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

1. Introduction

This Note deals with the dynamical shape sensitivity analysis for a fluid inside a moving domain. This problem is the basic tool in the design and the control of many industrial devices such as aircraft wings, cable-stayed bridges, automobile shapes, satellite reservoir tanks and more generally of systems involving fluid–solid interactions.

The control variable is the shape of the moving domain, and the objective is to minimize a given cost functional that may be chosen by the designer.

Recently, a new methodology to obtain eulerian derivative for non-cylindrical functionals has been introduced in [5,4]. This methodology was applied in [3] to perform a dynamical shape control analysis of the Navier–Stokes

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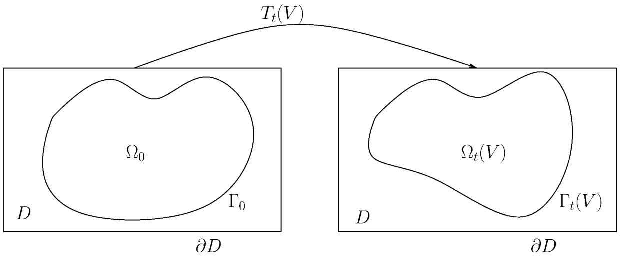

Fig. 1. Fluid domain deformation.

equations thanks to the state differentiation. The aim of this article is to show how the Min–Max principle allows, at least formally, to bypass the tedious obtention of the state differentiability with respect to the shape of the moving domain.

2. Mechanical problem

Let D be a convex, smooth and bounded open set in $\mathbb{R}^d$ with boundary ∂D and $\Omega_0 \subset D$ a smooth reference domain. We consider a smooth vector field V defined over $\overline{D} \times [0,T]$ with $T > 0$ and $\langle V, n\rangle = 0$ on ∂D where n stands for the unit normal vector. The Lagrangian flow

$$\begin{aligned} &T_t(V) : \overline{D} \to \overline{D},\\ &\qquad x_0 \mapsto x(t, x_0) \equiv T_t(V)(x_0) \end{aligned}$$

associated to V is a one-to-one mapping solution of the following dynamical system:

$$\begin{aligned} &\frac{\mathrm{d}x}{\mathrm{d}\tau} = V\big(\tau, x(\tau)\big), \quad \tau \in [0,T],\\ &x(\tau = 0) = x_0, \quad \text{in } D. \end{aligned} \tag{1}$$

We assume that a viscous incompressible Newtonian fluid fills the moving domain $\Omega_t(V) \stackrel{\text{def}}{=} T_t(V)(\Omega_0)$ whose boundary $\Gamma_t(V) \stackrel{\text{def}}{=} T_t(V)(\Gamma_0)$ evolves with speed $V|_{\Gamma_t}$. The fluid is described by its velocity u and its pressure p, satisfying the classical Navier–Stokes equations written in non-conservative form with no-slip boundary conditions on $\Gamma_t(V)$,

$$\begin{cases} \partial_t u + \mathrm{D}u \cdot u - \nu \Delta u + \nabla p = 0, & Q(V) \stackrel{\text{def}}{=} \displaystyle\bigcup_{0<t<T} \{t\} \times \Omega_t(V),\\ \operatorname{div}(u) = 0, & Q(V),\\ u = V, & \Sigma(V) \stackrel{\text{def}}{=} \displaystyle\bigcup_{0<t<T} \{t\} \times \Gamma_t(V),\\ u(t=0) = u_0, & \Omega_0, \end{cases} \tag{2}$$

where ν stands for the kinematic viscosity.

The quantity $\sigma(u,p) = -p\mathrm{I} + \nu(\mathrm{D}u + {}^*\mathrm{D}u)$ stands for the fluid stress tensor inside $\Omega_t(V)$, with $(\mathrm{D}u)_{i,j} = \partial_j u_i$.

3. Gradient computation

We are interested in performing the differentiation with respect to $V \in \mathcal{U}_{\text{ad}}$ of the following functional,

$$j(V) \stackrel{\text{def}}{=} J_V\big(u(V), p(V)\big), \tag{3}$$

where $(u(V), p(V))$ is a weak solution of problem (2) and $J_V(u, p)$ is a real functional of the following form:

$$J_V(u, p) = \frac{\alpha}{2}\int_0^T\int_{\Omega_t(V)} |\mathrm{curl}\, u|^2 + \frac{\gamma}{2}\int_0^T\int_{\Gamma_t(V)} |V|^2 + \frac{\theta}{2}\int_0^T\int_D |\mathcal{B}\cdot V|^2 \tag{4}$$

with

$$\mathcal{U}_{\mathrm{ad}} = \left\{V \in H^1\left(0, T; \left(H_0^m(D)\right)^d\right),\ m > 5/2,\ \mathrm{div}\, V = 0 \text{ in } D\right\} \tag{5}$$

and $\mathcal{B} \in \mathcal{L}(\mathcal{U}_{\mathrm{ad}}, L^2((0, T) \times D))$ with adjoint ${}^*\mathcal{B} \in \mathcal{L}(L^2((0, T) \times D), \mathcal{U}_{\mathrm{ad}}^*)$. We note $\gamma_{\Sigma(V)}$ the trace operator on $\Sigma(V)$ and we state the following result,

Proposition 3.1. *For $V \in \mathcal{U}_{\mathrm{ad}}$ and Ω_0 of class $\mathcal{C}^2$, the functional $j(V)$ admits a gradient $\nabla j(V)$ given by the following expression,*

$$\nabla j(V) = {}^*\gamma_{\Sigma(V)} \cdot \left[-\lambda n - \sigma(\varphi, \pi)\cdot n + \alpha\,(\mathrm{curl}\, u) \wedge n + \gamma V\right] + \theta[{}^*\mathcal{B}\mathcal{B}]\cdot V, \quad \text{in } \mathcal{U}_{\mathrm{ad}}^*, \tag{6}$$

where (φ, π) stands for the adjoint fluid state solution of the following backward system,

$$\begin{cases} -\partial_t\varphi - \mathrm{D}\varphi\cdot u + {}^*\mathrm{D}u\cdot\varphi - \nu\Delta\varphi + \nabla\pi = -\alpha\,\Delta u, & Q(V), \\ \mathrm{div}(\varphi) = 0, & Q(V), \\ \varphi = 0, & \Sigma(V), \\ \varphi(T) = 0, & \Omega_T \end{cases} \tag{7}$$

and λ is the adjoint transverse boundary field, solution of the backward tangential dynamical system,

$$\begin{cases} -\partial_t\lambda - \nabla_\Gamma\lambda\cdot V = f, & \Sigma(V), \\ \lambda(T) = 0, & \Gamma_T(V) \end{cases} \tag{8}$$

with

$$f = \left[-\nu\mathrm{D}\varphi\cdot n + \alpha(\mathrm{curl}\, u)\wedge n\right]\cdot(\mathrm{D}V\cdot n - \mathrm{D}u\cdot n) + \frac{1}{2}\left[\alpha|\mathrm{curl}\, u|^2 + \gamma\, H|V|^2\right], \tag{9}$$

where H stands for the additive curvature of $\Gamma_t(V)$ defined as the trace of the second fundamental form.

Remark 1. If $\theta > 0$ and ${}^*\mathcal{B}\mathcal{B}$ is coercive on $\mathcal{U}_{\mathrm{ad}}$, the problem of minimizing $j(V)$ over $\mathcal{U}_{\mathrm{ad}}$ admits at least one solution satisfying the optimality condition $\nabla j(V^*) = 0$.

Proof. The differentiability of the fluid state (u, p) with respect to V, as described in [3] can be tedious for dimension $d = 3$ and it can happen that even without state differentiability the gradient of the cost function exists. Here, we suggest to use a Min–Max formulation involving a Lagrangian functional coupled with a function space embedding, particulary suited for non-homogeneous Dirichlet boundary problems.

$$j(V) = \min_{(y,p)\in X\times P}\ \max_{(v,q)\in Y\times Q}\ \mathcal{L}_V(u, p; v, q) \tag{10}$$

with

$$\mathcal{L}_V(u, p; v, q) = J_V(u, p) - e_V(u, p; v, q) \tag{11}$$

and

$$e_V(y, p; v, q) = \int_{Q(V)} [\partial_t y + \mathrm{D}y\cdot y - \nu\Delta y + \nabla p]\cdot v - \int_{Q(V)} q\,\mathrm{div}\, y - \int_{\Sigma(V)} (y - V)\cdot\sigma(v, q)\cdot n$$

stands for the weak fluid state operator. The state and multiplier variables are defined on the hold-all domain D, i.e.,

$$X = Y \stackrel{\text{def}}{=} H^1\big(0,T;H^2(D)\big), \qquad P = Q \stackrel{\text{def}}{=} H^1\big(0,T;H^1(D)\big).$$

First-order optimality with respect to the multipliers (v,q) and the state variables (y,p) leads respectively to the primal system (2) and to the adjoint system (7).

The crucial point concerns the derivation with respect to the design variable $V \in \mathcal{U}_{\text{ad}}$. We consider a perturbation vector field $W \in \mathcal{U}_{\text{ad}}$ with an increment parameter $\rho \geqslant 0$. Since the state and multiplier variables are defined in the hold-all domain D, the pertubed Lagrangian only involves perturbed supports. Let us assume that we can apply the Min–Max derivation principle [1], then

$$\big\langle j'(V), W\big\rangle \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}\rho} j(V+\rho W)\bigg|_{\rho=0} = \frac{\mathrm{d}}{\mathrm{d}\rho}\mathcal{L}^{\rho}\bigg|_{\rho=0}(u,p;\varphi,\pi). \tag{12}$$

Using non-cylindrical shape derivative framework [2,5], we can state

$$\big\langle j'(V), W\big\rangle = \int_{\Sigma(V)} \big[f\langle Z_t, n\rangle + \big(-\sigma(\varphi,\pi)\cdot n + \alpha(\operatorname{curl} u)\wedge n + \gamma V\big)\cdot W\big] + \langle {}^*\mathcal{B}\mathcal{B}\cdot V, W\rangle_{\mathcal{U}_{\text{ad}}^*,\mathcal{U}_{\text{ad}}}$$

with f given by Eq. (9) and where Z_t stands for the transverse vector field [5,4], solution of a dynamical system involving the Lie brackets $[Z_t, V] \stackrel{\text{def}}{=} \mathrm{D}Z_t\cdot V - \mathrm{D}V\cdot Z_t$. We finally use the following fundamental adjoint identity,

Lemma 3.2 [4].

$$\int_{\Sigma(V)} f\,\langle Z_t, n\rangle = -\int_{\Sigma(V)} \lambda\,\langle W, n\rangle, \quad \forall W \in \mathcal{U}_{\text{ad}}, \tag{13}$$

where λ is solution of Eq. (8) *which corresponds to the adjoint system associated to the transverse dynamical system satisfied by Z_t. This adjoint variable is only supported by the moving boundary $\Gamma_t(V)$ over $(0,T)$.*

4. Conclusion

The main result of this Note concerns the computation of dynamical shape gradient for general cost functions involving the solution of the Navier–Stokes system. It can serve to build gradient based optimization algorithms dynamical shape optimal control and stability problems. It is also a first step towards the design of optimization strategies applied to mechanical systems involving moving fluid–solid interfaces.

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