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C. R. Acad. Sci. Paris, Ser. I 338 (2004) 121–126
COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

Biharmonic problem in exterior domains

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Received 12 September 2003; accepted 1 October 2003

Presented by Philippe G. Ciarlet

Abstract

We study here a biharmonic equation in an exterior domain of $\mathbb{R}^n$. We give in L^p theory, with $1 < p < \infty$ existence, uniqueness and regularity results. ***To cite this article: C. Amrouche, M. Fontes, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***
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Résumé

Problème biharmonique dans un ouvert extérieur. Nous étudions ici un problème biharmonique dans un ouvert extérieur de $\mathbb{R}^n$ avec $n \geqslant 2$. Nous donnons des résultats d'existence, d'unicité et de régularité en théorie L^p, avec $1 < p < \infty$. ***Pour citer cet article : C. Amrouche, M. Fontes, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***
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Version française abrégée

L'objet de cette Note est de résoudre le problème biharmonique (1) dans un ouvert extérieur. Ce problème étant posé dans un ouvert extérieur, une approche adaptée à la résolution est l'utilisation d'espaces de Sobolev avec poids (2). Dans de précédents travaux, des résultats semblables ont été établis pour le Laplacien (cf. [2]). Les principaux résultats de notre travail sont d'une part la caractérisation du noyau de l'opérateur bilaplacien associé au problème (1) (Théorèmes 2.5 et 2.6) et d'autre part le résultat d'existence et d'unicité donné par le Théorème 2.8.

1. Introduction

Let Ω' be an bounded open region of $\mathbb{R}^n$ ($n \geqslant 2$), with positive measure and a Lipschitz-continuous boundary Γ. We denote by Ω the complement of $\overline{\Omega'}$, we assume that Ω' has a finite number of connected components and that each connected component has a connected boundary, so that Ω is connected.

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doi:10.1016/j.crma.2003.10.026

The purpose of this work is to solve the biharmonic problem:

$$\Delta^2 u = f \quad \text{in } \Omega, \qquad u = g_0 \quad \text{on } \Gamma, \qquad \frac{\partial u}{\partial \boldsymbol{n}} = g_1 \quad \text{on } \Gamma, \tag{1}$$

where $\boldsymbol{n}$ is the unit vector normal to Γ and pointing outside Ω. Since this problem is posed in an exterior domain, an approach adapted to the solution is the use of weighted Sobolev spaces.

Let us begin by introducing these spaces. For any integer q, we denote by $\mathcal{P}_q$ the space of polynomials in n variables, of degree smaller than or equal to q, with the convention that $\mathcal{P}_q$ is reduced to $\{0\}$ when q is negative. For any real number $p > 1$, we denote by p' the dual exponent of p:

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

A point of $\mathbb{R}^n$ will be denoted by $\boldsymbol{x} = (x_1, \ldots, x_n)$ and its distance to the origin by $r = |\boldsymbol{x}| = (x_1^2 + \cdots + x_n^2)^{1/2}$. We will use two basics weights:

$$\rho = \rho(r) = \left(1 + r^2\right)^{1/2} \quad \text{et} \quad \lg r = \ln\left(2 + r^2\right).$$

For any nonnegative integer m and real numbers $p > 1$, α and β we define the following space:

$$\begin{aligned} W^{m,p}_{\alpha,\beta}(\Omega) = \big\{ & u \in \mathcal{D}'(\Omega);\ \forall \lambda \in \mathbb{N}^n\colon\ 0 \leqslant |\lambda| \leqslant k,\ \rho^{\alpha-m+|\lambda|}(\lg r)^{\beta-1} D^\lambda u \in L^p(\Omega); \\ & \forall \lambda \in \mathbb{N}^n\colon\ k+1 \leqslant |\lambda| \leqslant m,\ \rho^{\alpha-m+|\lambda|}(\lg r)^{\beta} D^\lambda u \in L^p(\Omega) \big\}, \end{aligned} \tag{2}$$

where $k = -1$ if $n/p + \alpha \notin \{1, \ldots, m\}$ and $k = m - n/p - \alpha$ if $n/p + \alpha \in \{1, \ldots, m\}$.

It is a reflexive Banach space equipped with its natural norm:

$$\|u\|_{W^{m,p}_{\alpha,\beta}(\Omega)} = \left(\sum_{0 \leqslant |\lambda| \leqslant k} \left\| \rho^{\alpha-m+|\lambda|}(\lg r)^{\beta-1} D^\lambda u \right\|^p_{L^p(\Omega)} + \sum_{k+1 \leqslant |\lambda| \leqslant m} \left\| \rho^{\alpha-m+|\lambda|}(\lg r)^{\beta} D^\lambda u \right\|^p_{L^p(\Omega)} \right)^{1/p},$$

and we denote by $|\cdot|_{W^{m,p}_{\alpha,\beta}(\Omega)}$ the associated semi-norm.

When $\beta = 0$, we denote simply the space by $W^{m,p}_{\alpha}(\Omega)$.

2. Biharmonic problem

We propose to solve the following biharmonic problem:

For f given in $W_0^{-2,p}(\Omega)$, g_0 given in $W^{2-1/p,p}(\Gamma)$ and g_1 given in $W^{1-1/p,p}(\Gamma)$, find u in $W_0^{2,p}(\Omega)$ solution of:

$$\Delta^2 u = f \quad \text{dans } \Omega, \qquad u = g_0 \quad \text{sur } \Gamma, \qquad \frac{\partial u}{\partial \boldsymbol{n}} = g_1 \quad \text{sur } \Gamma.$$

First of all, in the case $p = 2$, we have:

$$\forall v \in \mathring{W}_0^{2,2}(\Omega), \quad |v|_{W_0^{2,2}(\Omega)} = \|\Delta v\|_{L^2(\Omega)}.$$

Consequently, thanks to Lax–Milgram's lemma, we have the

Proposition 2.1. *For f given in $W_0^{-2,2}(\Omega)$, g_0 in $W^{3/2,2}(\Gamma)$ and g_1 in $W^{1/2,2}(\Gamma)$, problem* (1) *has a unique solution u in $W_0^{2,2}(\Omega)$.*

Hence problem (1) is solved when $p = 2$. Since it is also solved in $\mathbb{R}^n$ (cf. [1]), we shall reduce problem (1) for $p > 2$ to two simpler problems, one in Ω (with $f = 0$) and another in $\mathbb{R}^n$. Then a duality argument will solve problem (1) for $p < 2$.

We first solve a biharmonic equation in $\mathbb{R}^n$ with a right-hand side that has a compact support. By the means of a partition of unity, we use regularity results of the bilaplacian in an bounded domain (cf. [3,4]).

Lemma 2.2. *Let $p > 2$, $f \in W_0^{-2,p}(\mathbb{R}^n)$ with compact support such that*

$$\langle f, 1\rangle = 0 \quad \text{if } n \leqslant 4 \quad \text{and satisfying} \quad \langle f, x_i\rangle = 0 \quad \text{for } i = 1, 2 \text{ if } n = 2.$$

Then there exists $u \in W_0^{2,2}(\mathbb{R}^n) \cap W_0^{2,p}(\mathbb{R}^n)$ such that

$$\Delta^2 u = f \quad \text{in } \mathbb{R}^n.$$

Moreover u is unique if $n > 4$, up to an additive constant if $n \in \{3, 4\}$ and up to a polynomial of degree smaller or equal to one if $n = 2$.

The following lemma solves problem (1) with homogeneous boundary conditions and a right-hand side f with compact support.

Lemma 2.3. *Assume $p > 2$, Γ of class $\mathcal{C}^{3,1}$ and $f \in W_0^{-2,p}(\Omega)$ with compact support. The problem*

$$\Delta^2 u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma, \qquad \frac{\partial u}{\partial \boldsymbol{n}} = 0 \quad \text{on } \Gamma,$$

has a unique solution u in $W_0^{2,2}(\Omega) \cap W_0^{2,p}(\Omega)$.

Lemma 2.3 has the following consequence:

Corollary 2.4. *Let $p > 2$ and Γ of class $\mathcal{C}^{3,1}$. For any $g_0 \in W^{2-1/p,p}(\Gamma)$ and any $g_1 \in W^{1-1/p,p}(\Gamma)$, problem*

$$\Delta^2 u = 0 \quad \text{in } \Omega, \qquad u = g_0 \quad \text{on } \Gamma, \qquad \frac{\partial u}{\partial \boldsymbol{n}} = g_1 \quad \text{on } \Gamma,$$

has a unique solution u in $W_0^{2,2}(\Omega) \cap W_0^{2,p}(\Omega)$.

We propose now to characterize the kernel $\mathcal{D}_0^p(\Omega)$ of biharmonic operator with Dirichlet boundary conditions:

$$\mathcal{D}_0^p(\Omega) = \left\{ z \in W_0^{2,p}(\Omega),\ \Delta^2 z = 0 \text{ in } \Omega,\ z = \frac{\partial z}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma \right\}.$$

For this purpose, let us introduce first of all some notations:

- Let λ be the unique solution in $W_0^{2,2}(\Omega) \cap W_0^{2,p}(\Omega)$ of:

$$\Delta^2 \lambda = 0 \quad \text{in } \Omega, \qquad \lambda = 1 \quad \text{on } \Gamma, \qquad \frac{\partial \lambda}{\partial \boldsymbol{n}} = 0 \quad \text{on } \Gamma.$$

- Let α_i $(1 \leqslant i \leqslant n)$ be the unique solution in $W_0^{2,2}(\Omega) \cap W_0^{2,p}(\Omega)$ of:

$$\Delta^2 \alpha_i = 0 \quad \text{in } \Omega, \qquad \alpha_i = x_i \quad \text{on } \Gamma, \qquad \frac{\partial \alpha_i}{\partial \boldsymbol{n}} = \frac{\partial x_i}{\partial \boldsymbol{n}} \quad \text{on } \Gamma.$$

We first deal with the case $n > 4$:

Theorem 2.5. *Let $p > 2$ and assume that Γ is of class $\mathcal{C}^{3,1}$. If $n > 4$, we have*:

- $\mathcal{D}_0^p(\Omega) = 0$ *if* $p < n/2$,
- $\mathcal{D}_0^p(\Omega) = \{c(\lambda - 1),\ c \in \mathbb{R}\}$ *if* $n/2 \leqslant p < n$,
- $\mathcal{D}_0^p(\Omega) = \{\sum_{i=1}^n c_i(\alpha_i - x_i) + d(\lambda - 1);\ c_i,\ d \in \mathbb{R}\}$ *if* $p \geqslant n$.

We are now interested in the case where $n \in \{2, 3, 4\}$. Let U_n be the fundamental solution of the biharmonic operator in $\mathbb{R}^n$ and

$$u_n = U_n * \left(\frac{1}{|\Gamma|}\delta_\Gamma\right),$$

where δ_Γ denotes the distribution defined by

$$\forall \varphi \in \mathcal{D}\big(\mathbb{R}^n\big), \quad \langle \delta_\Gamma, \varphi\rangle = \int_\Gamma \varphi \,\mathrm{d}\sigma.$$

We denote by μ the unique solution in $W_0^{2,2}(\Omega) \cap W_0^{2,p}(\Omega)$ of:

$$\Delta^2 \mu = 0 \quad \text{in } \Omega, \qquad \mu = u_n \quad \text{on } \Gamma, \qquad \frac{\partial \mu}{\partial \boldsymbol{n}} = \frac{\partial u_n}{\partial \boldsymbol{n}} \quad \text{on } \Gamma.$$

Finally, when $n = 2$, we introduce the distributions T_i $(1 \leqslant i \leqslant 2)$ defined by:

$$\forall \varphi \in \mathcal{D}\big(\mathbb{R}^2\big), \quad \langle T_i, \varphi\rangle = \int_\Gamma \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}\sigma$$

and

$$d_i = U_2 * T_i, \quad 1 \leqslant i \leqslant 2.$$

For $i = 1$ or 2, ν_i will indicate the unique solution in $W_0^{2,2}(\Omega) \cap W_0^{2,p}(\Omega)$ of:

$$\Delta^2 \nu_i = 0 \quad \text{in } \Omega, \qquad \nu_i = d_i \quad \text{on } \Gamma, \qquad \frac{\partial \nu_i}{\partial \boldsymbol{n}} = \frac{\partial d_i}{\partial \boldsymbol{n}} \quad \text{on } \Gamma.$$

Now we have

Theorem 2.6. *Let $p > 2$ and assume that Γ is of class $\mathcal{C}^{3,1}$.*

- *If $n = 4$, then* $\mathcal{D}_0^p(\Omega) = \{c(\mu - u_4);\ c \in \mathbb{R}\}$.
- *If $n = 3$, then* $\mathcal{D}_0^p(\Omega) = \begin{cases} 0 & \text{if } p \leqslant 3, \\ \{c(\mu - u_3);\ c \in \mathbb{R}\} & \text{if } p > 3. \end{cases}$
- *If $n = 2$, then* $\mathcal{D}_0^p(\Omega) = \{\sum_{i=1}^2 c_i(\nu_i - d_i);\ c_i \in \mathbb{R}\}$.

Proof. We prove the result for $n = 3$. Let us note first that:

$$U_3 = -\frac{r}{8\pi},$$

hence

$$u_3(\boldsymbol{x}) = -\frac{1}{8\pi|\Gamma|}\int_\Gamma |\boldsymbol{x} - \boldsymbol{y}| \,\mathrm{d}\sigma_{\boldsymbol{y}}. \tag{3}$$

Now let $z \in \mathcal{D}_0^p(\Omega)$ and h the element of $\mathcal{D}'(\mathbb{R}^3)$ defined by $\Delta^2 z$:

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^3), \quad \langle h, \varphi \rangle = \langle \Delta^2 z, \varphi \rangle = -\left\langle \frac{\partial \Delta z}{\partial \boldsymbol{n}}, \varphi \right\rangle_\Gamma + \left\langle \Delta z, \frac{\partial \varphi}{\partial \boldsymbol{n}} \right\rangle_\Gamma .$$

The function u_3 belongs to $W_0^{2,s}(\mathbb{R}^3)$ iff $s > 3$, but an easy calculation shows that $\Delta^2 u_3$ belongs to $W_0^{-2,p}(\mathbb{R}^3)$. Moreover, u_3 satisfies:

$$\Delta^2 u_3 = 0 \quad \text{in } \Omega, \qquad \Delta^2 u_3 = 0 \quad \text{in } \Omega', \qquad \langle \Delta^2 u_3, 1 \rangle = 1,$$

where the duality pairing makes sense because $\Delta^2 u_3$ has a compact support. Therefore $h - \langle h, 1 \rangle \Delta^2 u_3$ has a compact support, belongs to $W_0^{-2,p}(\mathbb{R}^3)$ and is orthogonal to constants. Thanks to Lemma 2.2, there exists $w \in W_0^{2,2}(\mathbb{R}^3) \cap W_0^{2,p}(\mathbb{R}^3)$, unique up to an additive constant such that:

$$\Delta^2 w = h - \langle h, 1 \rangle \Delta^2 u_3 \quad \text{in } \mathbb{R}^3;$$

so $\Delta^2(w + \langle h, 1 \rangle u_3 - z) = 0$ in $\mathbb{R}^3$. We have to discuss according the value of p.

(a) If $2 < p < 3$, then the distribution defined by $\theta = \Delta(w + \langle h, 1 \rangle u_3 - z)$ belongs to $W_{-1/2}^{0,p}(\mathbb{R}^3)$ and satisfies $\Delta\theta = 0$ in $\mathbb{R}^3$ so θ is a polynomial of degree $[\frac{1}{2} - \frac{3}{p}]$, i.e., $\theta = 0$ since $2 < p < 3$. Therefore $\Delta(w - z) = \langle h, 1 \rangle \Delta u_3$ in $\mathbb{R}^3$. If $\langle h, 1 \rangle \neq 0$, then Δu_3 belongs to $L^p(\mathbb{R}^3)$ which is not true. So $\langle h, 1 \rangle = 0$ and, as $w - z$ belongs to $W_0^{2,p}(\mathbb{R}^3)$, there exists $c \in \mathbb{R}$ such that $w - z = c$. Then z belongs to $W_0^{2,2}(\Omega) \cap W_0^{2,p}(\Omega)$ and, according to Lemma 2.3, $z = 0$.

(b) If $p = 3$, the function θ belongs to $W_{-\varepsilon}^{0,3}(\mathbb{R}^3)$ for all $\varepsilon > 0$. As θ is harmonic, $\theta = 0$ and the proof used in (a) shows that $z = 0$.

(c) If $p > 3$, then θ belongs to $L^p(\mathbb{R}^3)$ and satisfies $\Delta\theta = 0$ in $\mathbb{R}^3$ so $\theta = 0$ in $\mathbb{R}^3$. Furthermore, for i, j in $\{1, 2, 3\}$,

$$\Delta\left[\partial_{ij}^2\left(w + \langle h, 1 \rangle u_3 - z\right)\right] = 0 \quad \text{in } \mathbb{R}^3,$$

and $\partial_{ij}^2(w + \langle h, 1 \rangle u_3 - z)$ belongs to $L^p(\mathbb{R}^3)$, so $\partial_{ij}^2(w + \langle h, 1 \rangle u_3 - z) = 0$ in $\mathbb{R}^3$.

Consequently, there exists $q \in \mathcal{P}_1$ such that $w + \langle h, 1 \rangle u_3 - z = q$ in $\mathbb{R}^3$. We have in particular $w + \langle h, 1 \rangle u_3 = q$ in $\overline{\Omega'}$. But, owing to (3), u_3 is negative so $\langle h, 1 \rangle u_3$ has a constant sign; for instance $\langle h, 1 \rangle u_3 \geqslant 0$. Furthermore, w belongs to $W_0^{2,2}(\mathbb{R}^3) \cap W_0^{2,p}(\mathbb{R}^3)$, so $w \in L^\infty(\overline{\Omega'})$. As w is defined up to an additive constant, we can assume that $w \geqslant 0$ in $\overline{\Omega'}$ and, therefore, $q \geqslant 0$ in $\overline{\Omega'}$. Even if it means using a translation, we can assume that $0 \in \Omega'$. As $q \geqslant 0$ in $\overline{\Omega'}$, this implies that q is constant in Ω' so q belongs to $\mathcal{P}_0$. As a consequence, q belongs to $W_0^{2,2}(\Omega) \cap W_0^{2,p}(\Omega)$. Moreover, the restriction of w to Ω belongs to $W_0^{2,2}(\Omega) \cap W_0^{2,p}(\Omega)$ and we write $w = w_1 + q$ where w_1 is the unique solution in $W_0^{2,2}(\Omega) \cap W_0^{2,p}(\Omega)$ of:

$$\begin{cases} \Delta^2 w_1 = 0 \quad \text{in } \Omega, \\ w_1 = -\langle h, 1 \rangle u_3 \quad \text{on } \Gamma, \\ \dfrac{\partial w_1}{\partial \boldsymbol{n}} = -\langle h, 1 \rangle \dfrac{\partial u_3}{\partial \boldsymbol{n}} \quad \text{on } \Gamma, \end{cases}$$

so $w_1 = -\langle h, 1 \rangle u_3$ and $z = -\langle h, 1 \rangle (\mu - u_3)$.

Conversely, $(\mu - u_3)$ belongs to $\mathcal{D}_0^p(\Omega)$. □

We are now in position to solve problem (1) for $p \geqslant 2$.

Theorem 2.7. *Let $p \geqslant 2$; assume that Γ is of class $\mathcal{C}^{3,1}$ if $p > 2$; $f \in W_0^{-2,p}(\Omega)$, $g_0 \in W^{2-1/p,p}(\Gamma)$ et $g_1 \in W^{1-1/p,p}(\Gamma)$. Then problem* (1) *has a unique solution $u \in W_0^{2,p}(\Omega)/\mathcal{D}_0^p(\Omega)$.*

Now a duality argument allows to solve (1) for $p < 2$.

First, if $n > 4$ and $\frac{n}{n-2} < p < 2$, we show existence and uniqueness of the solution of problem (1).

On the other hand, when $n \leqslant 4$ or $p \leqslant \frac{n}{n-2}$, there is necessarily a compatibility condition between the data f, g_0 and g_1 in order that a solution exists; if (1) has a solution then f, g_0 and g_1 must satisfy

$$\forall \varphi \in \mathcal{D}_0^{p'}(\Omega), \quad \langle f, \varphi \rangle = \langle g_1, \Delta\varphi \rangle_\Gamma - \left\langle g_0, \frac{\partial \Delta \varphi}{\partial \boldsymbol{n}} \right\rangle_\Gamma. \tag{4}$$

The next theorem summarizes the previous results.

Theorem 2.8. *Let Γ be of class $\mathcal{C}^{3,1}$ if $p \neq 2$ or Lipschitz-continuous if $p = 2$, $f \in W_0^{-2,p}(\Omega)$, $g_0 \in W^{2-1/p,p}(\Gamma)$ and $g_1 \in W^{1-1/p,p}(\Gamma)$.*

If $p \geqslant 2$, problem (1) *has a unique solution u in $W_0^{2,p}(\Omega)/\mathcal{D}_0^p(\Omega)$ and there exists a constant C, independent of u, f, g_0 and g_1 such that*

$$\|u\|_{W_0^{2,p}(\Omega)/\mathcal{D}_0^p(\Omega)} \leqslant C\big\{\|f\|_{W_0^{-2,p}(\Omega)} + \|g_0\|_{W^{2-1/p,p}(\Gamma)} + \|g_1\|_{W^{1/p',p}(\Gamma)}\big\}.$$

In particular, u is unique in $W_0^{2,p}(\Omega)$ when $p = 2$ or $2 < p < n/2$ or ($n = 3$ and $p \leqslant 3$).

If $p < 2$ then

(1) *If $n > 4$ and $\frac{n}{n-2} < p$, problem* (1) *has a unique solution u in $W_0^{2,p}(\Omega)$.*

(2) *If $n \leqslant 4$ or $p \leqslant \frac{n}{n-2}$ and if f, g_0 and g_1 satisfy* (4) *then problem* (1) *has a unique solution u in $W_0^{2,p}(\Omega)$.*

In both cases, there exists a constant C, independent of u, f, g_0 and g_1 such that

$$\|u\|_{W_0^{2,p}(\Omega)} \leqslant C\big\{\|f\|_{W_0^{-2,p}(\Omega)} + \|g_0\|_{W^{2-1/p,p}(\Gamma)} + \|g_1\|_{W^{1-1/p,p}(\Gamma)}\big\}.$$

Finally, we give a regularity result.

Proposition 2.9. *The assumptions are that of Theorem* 2.8 *with, in addition of that, Γ of class $\mathcal{C}^{3,1}$ and $\frac{n}{p}, \frac{n}{p'} \notin \{1, 2\}$.*

Moreover we assume that the data have the following regularity: $f \in W_1^{-1,p}(\Omega)$, $g_0 \in W^{3-1/p,p}(\Gamma)$ *and* $g_1 \in W^{2-1/p,p}(\Gamma)$.

Then, the solution u discussed in Theorem 2.8 *belongs to $W_1^{3,p}(\Omega)$.*

Also, if $f \in W_2^{0,p}(\Omega)$, $g_0 \in W^{4-1/p,p}(\Gamma)$ and $g_1 \in W^{3-1/p,p}(\Gamma)$, then u belongs to $W_2^{4,p}(\Omega)$.

Remark 1. We can obtain similar results with mixed boundary conditions on Δu and $\frac{\partial \Delta u}{\partial \boldsymbol{n}}$.

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