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Harmonic Analysis

# Spectral multipliers on some rank one *NA*-groups with roots not all positive

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**Abstract**

We consider a particular class of (non-unimodular) rank one *NA*-groups with roots not all positive, and we show that, on each of these groups, there exists a distinguished left invariant sub-Laplacian which admits a $L^p$-differentiable functional calculus for every $p \in [1, +\infty]$. ***To cite this article: E. David-Guillou, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***


**Résumé**

**Multiplicateurs spectraux sur certains groupes de type *NA* de rang un avec des racines non-toutes positives.** Nous considérons une certaine classe de groupes de Lie (non-unimodulaires) de type *NA* de rang un avec des racines non toutes positives, et nous montrons que, sur chacun de ces groupes, il existe un sous-laplacien invariant à gauche particulier qui admet un calcul fonctionnel $L^p$-différentiable pour tout $p \in [1, +\infty]$. ***Pour citer cet article : E. David-Guillou, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***


## 1. Introduction

Let $G$ be a real connected Lie group and let $X_j$, $j = 1, \ldots, n$, be left invariant vector fields on $G$. Let us assume that these vector fields generate the Lie algebra of $G$ (i.e. they satisfy Hörmander's condition) and let us denote by $\Delta = -\sum_{j=1}^{n} X_j^2$ the corresponding sub-Laplacian. The operator $\Delta$ is formally self-adjoint and nonnegative on the space $L^2(G)$ of square integrable functions with respect to the right invariant Haar measure on $G$. Every Borel function $m$ bounded on $\mathbb{R}^+$ defines therefore a bounded operator $m(\Delta)$ on $L^2(G)$, via the spectral theorem. One question which arises naturally is that of finding sufficient conditions on the multiplier $m$ so that $m(\Delta)$ extends to a bounded operator on $L^p(G)$ for some $p \neq 2$ (cf. [4] for $\mathbb{R}^n$).

When $G$ is a solvable Lie group with exponential volume growth, two different situations are known: on some groups, $\Delta$-like sub-Laplacians admit a differentiable $L^p$-functional calculus; while on other groups, $\Delta$-like sub-Laplacians are of holomorphic $L^p$-type.

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The aim of this Note is to show that for a non-trivial class of groups $G$ which are semidirect products of a real nilpotent (non necessarily Euclidian) Lie group $N$ and the real line $\mathbb{R}$, and for which the adjoint action is semisimple and has non-zero eigenvalues not all positive, there exists a distinguished $\Delta$-like sub-Laplacian on $G$ which admits a differentiable $L^p$-functional calculus for every $p \in [1, +\infty]$ (known when $N$ is Euclidian [2,3]).

Note that when $N$ is not Euclidian, previous results (in the case of roots not all positive) concern exclusively sub-Laplacians of holomorphic $L^p$-type [1,5].

## 2. Multipliers theorem

Let $H$ be a stratified group, that is a connected simply connected nilpotent Lie group whose Lie algebra $\mathfrak{h}$ admits a vector space decomposition

$$\mathfrak{h} = \bigoplus_{j=1}^{n} V_j$$

such that the subspaces $V_j$ satisfy $[V_1, V_j] = V_{j+1}$ for every $j = 1, \dots, n-1$.

There is a natural family of dilations on $\mathfrak{h}$, namely the one parameter group of automorphisms of $\mathfrak{h}$

$$\sigma_t\left(\sum_{j=1}^{n} X_j\right) = \sum_{j=1}^{n} \mathrm{e}^{jt} X_j, \quad X_j \in V_j,\ t \in \mathbb{R}.$$

We endow $H$ with the corresponding family of dilations on $H$

$$\sigma_t = \exp_H \circ \sigma_t \circ \exp_H^{-1}, \quad t \in \mathbb{R},$$

where $\exp_H$ denotes the exponential map from $\mathfrak{h}$ to $H$. Then $H$ is a *homogeneous group* whose *homogeneous dimension* is

$$Q = \sum_{j=0}^{n} j \dim V_j.$$

We fix $\beta$ real negative. Let $G = H \times \mathbb{R} \times \mathbb{R}$ be the Lie group whose product is given by

$$g_1 \cdot g_2 = (h_1, a_1, t_1) \cdot (h_2, a_2, t_2) = (h_1 \cdot \sigma_{t_1} h_2, a_1 + \mathrm{e}^{\beta t_1} a_2, t_1 + t_2), \quad g_i = (h_i, a_i, t_i) \in G,\ i = 1, 2.$$

*Note that $G$ is a solvable Lie group with exponential volume growth, and, when $\beta$ equals not $-Q$, that $G$ is moreover non-unimodular.*

Let $\{e_1, \dots, e_d\}$ be a basis of the vector space $V_1$. We identify $\mathfrak{h}$ with an ideal of the Lie algebra $\mathfrak{g}$ of $G$ and define the following left invariant vector fields on $G$

$$\begin{aligned} &X_0 = \partial_t, \\ &X_j \phi(g) = \partial_s \phi\big(g \cdot \exp(s e_j)\big)\big|_{s=0}, \quad g \in G,\ \phi \in C^1(G),\ j = 1, \dots, d, \\ &X_{d+1} = \mathrm{e}^{\beta t} \partial_a, \end{aligned}$$

where exp denotes the exponential map from $\mathfrak{g}$ to $G$.

The system of vector fields $\chi = \{X_0, \dots, X_{d+1}\}$ satisfies Hörmander's condition on $G$. We shall denote

$$\Delta = -\sum_{j=0}^{d} X_j^2$$

the left invariant sub-Laplacian on $G$.

We endow $G$ with the right invariant Haar measure $\mathrm{d}g = \mathrm{d}h\,\mathrm{d}a\,\mathrm{d}t$ where $\mathrm{d}h$ denotes a bi-invariant Haar measure on $H$, and $\mathrm{d}a$ and $\mathrm{d}t$ denote the Lesbegue measures on $\mathbb{R}$ corresponding to the variables $a$ and $t$ respectively. We shall assume that $\Delta$ is the Friedrichs extension (i.e., the smallest self-adjoint extension) of $-\sum_{j=0}^{d} X_j^2$ considered as an operator on $L^2(G)$ defined on the set of smooth functions compactly supported on $G$.

The operator $\Delta$ is non-negative, and admits therefore a spectral resolution

$$\Delta = \int_0^{+\infty} \lambda\,\mathrm{d}E_\lambda,$$

where $E_\lambda$ are self-adjoint projections. Each Borel function $m$ bounded on $\mathbb{R}^+$ can then be associated with a bounded operator on $L^2(G)$ via the formula

$$m(\Delta) = \int_0^{+\infty} m(\lambda)\,\mathrm{d}E_\lambda. \tag{1}$$

We can now enunciate the multipliers theorem which motivates this note:

**Theorem 2.1.** *Let $G$ and $\Delta$ be as above. Suppose that $m$ is a real function compactly supported in $]0, +\infty[$ which belongs to the Sobolev space $H^{3Q/4+11/2+\varepsilon}(\mathbb{R}^+)$ for some $\varepsilon > 0$. Then the operator $m(\Delta)$ defined by (1) extends to an operator bounded on $L^p(G)$ for all $p \in [1, \infty]$.*

## 3. $L^1$ estimate on the heat kernel

Let $p_t$ be the heat kernel associated with the sub-Laplacian $\Delta$

$$\mathrm{e}^{-t\Delta}\phi = p_t *_l \phi, \quad \phi \in C_0^\infty(G),\ t > 0,$$

where $*_l$ denotes the product of convolution in the space $L^2(G, \mathrm{d}^l g)$ defined with respect to the left invariant Haar measure on $G$ $\mathrm{d}^l g = \mathrm{e}^{-(Q+\beta)t}\,\mathrm{d}g$.

We shall consider $p_z$ the analytical extension on $\Re z > 0$ of the heat kernel $p_t$.

We derive Theorem 2.1 (see, e.g., [6]) from the $L^1$ estimate on the heat kernel given by the following theorem:

**Theorem 3.1.** *Let $G$ and $p_z$ be as above. There exists $C > 0$ such that*

$$\|p_{1+\mathrm{i}s}\|_{L^1(G)} \leqslant C\big(1+|s|\big)^{3Q/4+5}, \quad s \in \mathbb{R}.$$

## 4. Weighted $L^2$ estimates

Let us fix a homogeneous norm $|\cdot|_H$ on the homogeneous group $H$. We define the function

$$\omega(g) = |h|_H^Q \cdot |a|, \quad g = (h, a, t) \in G.$$

We estimate the $L^1$ norm of $p_{1+\mathrm{i}s}$ by means of two terms, on the one hand a weighted $L^2$ norm of $p_{1+\mathrm{i}s}$ (the weight being $\omega$), and on the other hand a term which compensate the introduction of $\omega$:

**Proposition 4.1.** *There exists $C > 0$ such that*

$$\big\|\omega^{1/2} p_{1+\mathrm{i}s}\big\|_{L^2(G)} \leqslant C\big(1+|s|\big)^{3Q/4+2}, \quad s \in \mathbb{R}.$$

**Proposition 4.2.** *Let $B_R$ denote the ball centered in the neutral element of $G$ and of radius $R$ for the Carnot–Caratheodory distance on $G$ related to $\chi$, where $R$ is real positive. If $c > 0$ is large enough, there exists $C > 0$ such that*

$$\left\|(1+\omega)^{1/2}\right\|_{L^2(B_{cs^2})} \leqslant C\left(1+|s|\right)^3, \quad s \in \mathbb{R}.$$

Propositions 4.1 and 4.2 imply Theorem 3.1 (see [2]). Proposition 4.2 follows from easy properties of the homogeneous norm.

Proposition 4.1 is much more delicate to establish. We base its proof on various intermediate estimates, the most relevant one being given by the following technical lemma:

**Lemma 4.3.** *There exists $C > 0$ such that*

$$\begin{aligned}&\left|\partial_s \left\| |h|_H^{Q/2} |a|^{1/2} p_{1+\mathrm{i}s} \right\|^2\right| \\ &\quad \leqslant C\left(\sum_{k=0}^{Q} \sup_{|\xi-1|\leqslant 1/2} \left\|\mathrm{e}^{\frac{Q-k+\beta}{2}t} |h|^{k/2} p_{\xi+\mathrm{i}s}\right\|^2 + \sum_{k=0}^{Q-1} \sup_{|\xi-1|\leqslant 1/2} \left\|\mathrm{e}^{\frac{Q-k}{2}t} |h|^{k/2} |a|^{1/2} p_{\xi+\mathrm{i}s}\right\|^2\right), \quad s \in \mathbb{R}.\end{aligned}$$

Detailed proofs of Theorems 2.1 and 3.1 will appear later on.

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