A class of stochastic differential equations with non-Lipschitzian coefficients: pathwise uniqueness and no explosion

Shizan Fang, Tusheng Zhang

Abstract

A new result for the pathwise uniqueness of solutions of stochastic differential equations with non-Lipschitzian coefficients is established. Furthermore, we prove that the solution has no explosion under the growth $\xi \log \xi$. To cite this article: S. Fang, T. Zhang, C. R. Acad. Sci. Paris, Ser. I 337 (2003).

1. Introduction

Let $\sigma: \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^m$ and $b: \mathbb{R}^d \rightarrow \mathbb{R}^d$ be continuous functions. Consider the following Itô s.d.e.:

$$dx_t(w) = \sigma(x_t(w)) \, dw_t + b(x_t(w)) \, dt, \quad x_0(w) = x_0,$$

where $t \rightarrow w(t)$ is a $\mathbb{R}^m$-valued standard Brownian motion. It is well known (see [5], [2, p. 159]) that the s.d.e. (1) has a weak solution up to a lifetime $\zeta$, and $\zeta \equiv +\infty$ if $\sigma$ and $b$ are of linear growth. Moreover if the s.d.e. (1) has the pathwise uniqueness, then it admits a strong solution (see [2, p. 149], [4, p. 341]). So the study of pathwise uniqueness is of great interest. It is well known that the pathwise uniqueness holds for (1) in the case that the coefficients are locally Lipschitzian. However, there are few results on the pathwise uniqueness beyond the Lipschitz (or locally) conditions except in the one dimensional case, some results have been obtained for certain Hölder coefficients (see [4, Chapter IX-3], [2, p. 168]). The purpose of this work is to establish the pathwise uniqueness of the solutions under the following assumptions,
2. Pathwise uniqueness

Theorem 2.1. Let \((x_t(w))_{t \geq 0}\) and \((y_t(w))_{t \geq 0}\) be two solutions (of continuous samples, without explosion) of the s.d.e. (1) such that \(x_0(w) = y_0(w)\). Then under (H1), we have almost surely \(x_t(w) = y_t(w), \ t \geq 0\).

Proof. Let \(\eta_t(w) = x_t(w) - y_t(w)\) and \(\xi_t(w) = |\eta_t(w)|^2\). We have \(d\eta_t(w) = (\sigma(x_t) - \sigma(y_t)) dw_t + (b(x_t) - b(y_t)) dt\), \(\eta_0(w) = 0\). Then

\[
\frac{d\xi_t}{\xi_t} = 2\left(\sigma(x_t) - \sigma(y_t)\right) dw_t + 2\left(\eta_t, b(x_t) - b(y_t)\right) dt + \left\|\sigma(x_t) - \sigma(y_t)\right\|^2 dt,
\]
and the stochastic contraction \(d\xi_t \cdot d\xi_t\) is given by

\[
\frac{d\xi_t}{\xi_t} dt = 4\left(\sigma^*(x_t) - \sigma^*(y_t)\right)\eta_t dt,
\]
where \(\sigma^*\) denotes the transpose matrix of \(\sigma\). Let \(\rho > 0\). Define the function \(\psi_\rho : [0, 1] \rightarrow \mathbb{R}\) by

\[
\psi_\rho(\xi) = \int_0^\xi \frac{ds}{s \log (1/s) + \rho}.
\]

It is clear that for any \(0 < \xi < 1\), \(\psi_\rho(\xi) \uparrow \psi_0(\xi) = \int_0^\xi \frac{ds}{s \log (1/s)} = +\infty\), as \(\rho \downarrow 0\). Define

\[
\Phi_\rho(\xi) = e^{\psi_\rho(\xi)}.
\]

Then we have

\[
\Phi_\rho'(\xi)\left(\xi \log \frac{1}{\xi} + \rho\right) = \Phi_\rho(\xi),
\]
and

\[
\Phi_\rho''(\xi) = \frac{\Phi_\rho(\xi)(2 - \log (1/\xi))}{(\xi \log (1/\xi) + \rho)^2} \leq 0 \quad \text{if} \ \xi < e^{-2}.
\]

Let \(\tau = \inf\{t > 0, \ \xi_t \geq e^{-2}\}\).

By Itô formula, (2) and (3), we get

\[
\Phi_\rho(\xi_{t \wedge \tau}) = 1 + 2 \int_0^{t \wedge \tau} \Phi_\rho'(\xi_s)\left[\eta_s, (\sigma(x_s) - \sigma(y_s)) dt_s + 2 \int_0^{t \wedge \tau} \Phi_\rho'(\xi_s)\left[\eta_s, b(x_s) - b(y_s)\right] ds_s
\]

\[+ \int_0^{t \wedge \tau} \Phi_\rho''(\xi_s)\left[\sigma(x_s) - \sigma(y_s)\right]^2 ds_s + 2 \int_0^{t \wedge \tau} \Phi_\rho'(\xi_s)\left[(\sigma^*(x_s) - \sigma^*(y_s))\eta_s\right]^2 ds_s.
\]

Using (7) and hypothesis (H1), it follows that \(\mathbb{E}(\Phi_\rho(\xi_{t \wedge \tau})) \leq 1 + C\mathbb{E}(\int_0^{t \wedge \tau} \Phi_\rho'(\xi_s) \log \frac{1}{\xi_s} ds_s)\) which is smaller by (6) than \(1 + C \int_0^\infty \mathbb{E}(\Phi_\rho(\xi_{t \wedge \tau})) ds_s\).
Now by Gronwall lemma, we get \( E(\Phi(t, \xi \mid \tau)) \leq C t \) or
\[
E(e^{\Phi(t, \xi \mid \tau)}) \leq C t. \tag{8}
\]
Letting \( \rho \downarrow 0 \) in (8), \( E(e^{\Phi(t, \xi \mid \tau)}) \leq C t \) which implies that for any \( t \) given,
\[
\xi_{t, \tau} = 0 \quad \text{almost surely.} \tag{9}
\]
If \( P(\tau < +\infty) > 0 \), then for some \( T > 0 \) big enough \( P(\tau \leq T) > 0 \). It follows from (9) that on \( \{\tau \leq T\} \), \( \xi_{t} = 0 \) which is absurd by the definition of \( \tau \). Therefore \( \tau = +\infty \) almost surely and for any \( t \) given, \( \xi_{t} = 0 \) almost surely. Now by continuity of the sample paths, the two solutions are indistinguishable. \( \square \)

3. Criterion for non-explosion

**Theorem 3.1.** Let \( \sigma \) and \( b \) be continuous functions satisfying
\[
\begin{align*}
\left\|\sigma(x)\right\|^2 & \leq C \left( |x|^2 \log |x| + 1 \right), \\
\left|\sigma(x)\right| & \leq C \left( |x| \log |x| + 1 \right).
\end{align*}
\tag{H2}
\]
Then the s.d.e. (1) has no explosion: \( P(\xi = +\infty) = 1 \).

**Proof.** Consider \( \psi(\xi) = \int_{0}^{\xi} \frac{dx}{|x| \log |x| + 1} \) and \( \Phi(\xi) = e^{\psi(\xi)}, \xi \geq 0 \). We have
\[
\begin{align*}
\Phi'(\xi) & = e^{\psi(\xi)}, \\
\Phi''(\xi) & = -\frac{\Phi(\xi) \log \xi}{(\xi \log \xi + 1)^2}, \quad \text{for } \xi > 1.
\end{align*}
\tag{10}
\tag{11}
\]
Note that the function \( \Phi \) is not in \( C^{2}(\mathbb{R}^{+}) \). We need to modify \( \Phi \) slightly in the neighbourhood of 1. Fix a small \( \delta > 0 \), take \( \Phi \in C^{2}(\mathbb{R}^{+}) \) such that
\[
\begin{align*}
\widetilde{\Phi} & \geq \Phi, \\
\widetilde{\Phi}(\xi) & = \Phi(\xi), \quad \text{for } \xi \notin [1 - \delta, 1 + \delta]. \tag{12}
\end{align*}
\]
Denote \( K_{1} = \sup_{\xi \in [1 - \delta, 1 + \delta]} (|\widetilde{\Phi}'(\xi)| + |\widetilde{\Phi}''(\xi)|) \), \( K_{2} = \sup_{\xi \in [1 - \delta, 1 + \delta]} (|\xi| \log |\xi|) \). Then
\[
\begin{align*}
|\widetilde{\Phi}'(\xi)| & \leq K_{1} (K_{2} + 1) \frac{\Phi(\xi)}{\Phi(1 - \delta)}, \quad \xi \in [1 - \delta, 1 + \delta], \\
|\widetilde{\Phi}''(\xi)| & \leq K_{1} (K_{2} + 1)^2 \frac{\Phi(\xi)}{\Phi(1 - \delta)} \frac{1}{(\xi \log |\xi| + 1)^2}, \quad \xi \in [1 - \delta, 1 + \delta]. \tag{13}
\end{align*}
\]
Let \( \eta(w) = v(w) - x_{0} \) and \( \xi_{t}(w) = |\eta(w)|^{2} \). Define \( \tau_{R} = \inf\{t > 0, \xi_{t} \geq R\}, R > 0 \). Then \( \tau_{R} \uparrow \infty \) as \( R \uparrow +\infty \).
Let \( I_{w} = \{t > 0, \xi_{t}(w) \in [e^{-2}, 1 + \delta]\} \).

By (12) and (11),
\[
|\widetilde{\Phi}''(\xi_{t})| = \Phi''(\xi_{t}) \leq 0 \quad \text{for } t \notin I_{w}. \tag{15}
\]
Combining (12) and (14), there exists a constant \( C_{1} \) such that
\[
|\widetilde{\Phi}'(\xi_{t})| \leq \frac{C_{1} \Phi(\xi_{t})}{|\xi_{t} \log |\xi_{t}| + 1|^{2}}, \quad t \in I_{w}. \tag{16}
\]
By (10) and (13), for some constant \( C_{2} \), we have
\[
|\widetilde{\Phi}'(\xi_{t})| \leq \frac{C_{2} \Phi(\xi_{t})}{|\xi_{t} \log |\xi_{t}| + 1|}, \quad t > 0. \tag{17}
\]
Now by Itô formula, we have

\[
\tilde{\Phi}(\xi_{t \wedge T_R}) = 1 + 2 \int_0^{t \wedge T_R} \tilde{\Phi}'(\xi_s) \langle \eta_s, \sigma(x_s) \rangle \, dw_s + 2 \int_0^{t \wedge T_R} \tilde{\Phi}'(\xi_s) \langle \eta_s, b(x_s) \rangle \, ds \\
+ \int_0^{t \wedge T_R} \tilde{\Phi}'(\xi_s) \| \sigma(x_s) \|^2 \, ds + 2 \int_0^{t \wedge T_R} \tilde{\Phi}''(\xi_s) |\sigma^*(x_s)\eta_s|^2 \, ds.
\]

(18)

By (15) and (16),

\[
t \wedge T_R \int_0^t \tilde{\Phi}''(\xi_s) |\sigma^*(x_s)\eta_s|^2 \, ds \leq t \wedge T_R \int_0^t 1_{I_w}(s) \frac{C_1 \Phi(\xi_s)}{(\xi_s \log \xi_s + 1)^2} |\sigma^*(x_s)\eta_s|^2 \, ds.
\]

(19)

By (H2), there exists \(C_1 > 0\) such that

\[
\| \sigma(x) \|^2 \leq C_1 (|x - x_0|^2 \log |x - x_0| + 1),
\]

\[
|b(x)| \leq C_1 (|x - x_0| \log |x - x_0| + 1).
\]

It follows that

\[
\frac{|\sigma^*(x_s)\eta_s|^2}{(\xi_s \log \xi_s + 1)^2} \leq C_1 \frac{\xi_s (\xi_s \log \xi_s + 1)}{(\xi_s \log \xi_s + 1)^2}
\]

which is dominated by a constant \(C_3\). According to (19), we get

\[
\int_0^{t \wedge T_R} \tilde{\Phi}''(\xi_s) |\sigma^*(x_s)\eta_s|^2 \, ds \leq C_3 \int_0^{t \wedge T_R} \Phi(\xi_s) \, ds.
\]

(20)

In the same way, for some constant \(C_4 > 0\), we have

\[
\|\langle \eta_s, b(x_s) \rangle\| + \|\sigma(x_s)\| \leq C_4 (\xi_s \log \xi_s + 1), \quad s > 0.
\]

(21)

Now using (18), (17), (21) and (20), we get

\[
E(\Phi(\xi_{t \wedge T_R})) \leq E(\tilde{\Phi}(\xi_{t \wedge T_R})) \leq 1 + C_5 \int_0^t E(\Phi(\xi_{s \wedge T_R})) \, ds,
\]

which implies that \(E(\Phi(\xi_{t \wedge T_R})) \leq e^{C_5 t}\). Letting \(R \to +\infty\), by Fatou lemma, we get

\[
E(\Phi(\xi_{t \wedge T_R})) \leq e^{C_5 t}.
\]

(22)

Now if \(P(\xi < +\infty) > 0\), then for some \(T > 0\), \(P(\xi \leq T) > 0\). Taking \(t = T\) in (22), we get \(E(1_{\xi \leq T}) \Phi(\xi_T)) \leq e^{C_5 T}\) which is impossible, because of \(\Phi(\xi_T) = +\infty\). □

References