

Available online at www.sciencedirect.com





C. R. Acad. Sci. Paris, Ser. I 337 (2003) 741-744

Probability Theory

Gross–Sobolev spaces on path manifolds: uniqueness and intertwining by Itô maps ☆

K. David Elworthy^a, Xue-Mei Li^b

^a Mathematics Institute, University of Warwick, Coventry CV4 7AL,UK ^b The Department of Computing and Mathematics, The Nottingham Trent University, Nottingham NG7 1AS, UK

Received 1 October 2003; accepted 6 October 2003

Presented by Jean-Michel Bismut

Abstract

Conditions are given under which the solution map \mathcal{I} of a stochastic differential equation on a Riemannian manifolds M intertwines the differentiation operator d on the path space of M and that of the canonical Wiener space, $d_{\Omega}\mathcal{I}^* = \mathcal{I}^* d_{C_{x_0}M}$. A uniqueness property of d on the path space follows. Results are also given for higher derivatives and covariant derivatives. *To cite this article: K.D. Elworthy, X.-M. Li, C. R. Acad. Sci. Paris, Ser. I 337 (2003).* © 2003 Académie des sciences. Published by Elsevier SAS. All rights reserved.

Résumé

Espaces de Gross–Sobolev sur les espaces des chemins : unicité et entrelacement par les applications d'Itô. Nous donnons des conditions sous lesquelles les applications d'Itô \mathcal{I} donnant la solution d'une équation différentielle stochastique sur une variété Riemannienne M entrelace l'opérateur de dérivation d sur l'espace de chemins de M, ainsi que celui de l'espace de Wiener canonique, de $d_{\Omega}\mathcal{I}^* = \mathcal{I}^* d_{C_{x_0}M}$. Nous en déduisons une propriété d'unicité de d sur l'espace de chemins. Des résultats sur les dérivées d'ordre supérieur ainsi que sur les dérivées covariantes sont également donnés. *Pour citer cet article : K.D. Elworthy, X.-M. Li, C. R. Acad. Sci. Paris, Ser. I 337 (2003).*

© 2003 Académie des sciences. Published by Elsevier SAS. All rights reserved.

1. Introduction

Let *M* be a compact C^{∞} Riemannian manifold with Levi-Civita connection ∇ . For x_0 a point in *M* and T > 0, let $C_{x_0}M$ be the C^{∞} manifold of continuous paths $\sigma : [0, T] \to M$ with $\sigma(0) = x_0$, equipped with Brownian motion measure μ_{x_0} . Its tangent space $T_{\sigma}C_{x_0}M$ at σ is the space of continuous vector fields on *M* along σ vanishing at 0. Let $//_s(\sigma): T_{x_0}M \to T_{\sigma_s}M$ be the stochastic parallel translation along σ defined almost surely. Denote by $W_t \equiv W_t(\sigma): T_{x_0}M \to T_{\sigma_t}M$ the damped parallel translation along σ defined by

$$\frac{\mathrm{d}}{\mathrm{d}t} / /_{t}^{-1}(\sigma) W_{t}(v) = -\frac{1}{2} / /_{t}^{-1}(\sigma) \operatorname{Ric}_{\sigma_{t}}^{\#} (W_{t}(v)), \quad W_{0}(v) = v, \quad v \in T_{x_{0}} M$$

* Research partially supported by NSF grant DMS 0072387 and EPSRC GR/NOO 845. *E-mail address:* xuemei.li@ntu.ac.uk (X.-M. Li).

1631-073X/\$ – see front matter © 2003 Académie des sciences. Published by Elsevier SAS. All rights reserved. doi:10.1016/j.crma.2003.10.004

for $\operatorname{Ric}_{x}^{\#}: T_{x}M \to T_{x}M$ given by the Ricci curvature. Set $\langle U, V \rangle_{\sigma} := \int_{0}^{T} \langle W_{t} \frac{d}{dt} W_{t}^{-1}(U_{t}), W_{t} \frac{d}{dt} W_{t}^{-1}(V_{t}) \rangle_{\sigma_{t}} dt$. Let \mathcal{H}_{σ} be the Hilbert space given for almost all σ by $\mathcal{H}_{\sigma} = \{V \in T_{\sigma}C_{x_{0}}M \mid W_{\cdot}^{-1}(V_{\cdot}) \text{ is absolutely continuous, } V_{0} = 0$ and $\|V\|_{\sigma}^{2} < \infty\}$ with the inner product $\langle, \rangle_{\sigma}$. Note that \mathcal{H}_{σ} is the same as the Bismut tangent space at σ , as used for example in [5], apart from the choice of inner product; and in the compact manifold and Brownian motion case considered here we could equally use either inner product (though Corollary 2.3(b) would need modification).

Choose a linear subspace $\text{Dom}(d_{\mathcal{H}})$ of $L^2(C_{x_0}M;\mathbb{R})$ such that

- (i) $\text{Dom}(d_{\mathcal{H}})$ contains smooth cylindrical functions on $C_{x_0}M$ and
- (ii) Each $f \in \text{Dom}(d_{\mathcal{H}})$ is Fréchet differentiable, bounded and with differential df bounded in the standard Finsler metric on $C_{x_0}M$.

Define $d_{\mathcal{H}}$: Dom $(d_{\mathcal{H}}) \subset L^2(C_{x_0}M; \mathbb{R}) \to L^2 \Gamma \mathcal{H}^*$, to be the restriction of the Fréchet derivative to \mathcal{H} . Denote by $d \equiv d_{C_{x_0}M}$ the closure of $d_{\mathcal{H}}$ and by $\mathbb{D}^{2,1}(C_{x_0}M; \mathbb{R})$ the domain of d with graph norm.

Consider the classical Wiener space $\Omega \equiv C_0([0, T]; \mathbb{R}^m)$ with Wiener measure \mathbb{P} , and a stochastic differential equation:

$$dx_t = X(x_t) \circ dB_t + A(x_t) dt \tag{1}$$

using canonical Brownian motion $\{B_t(\omega): 0 \le t \le T, \omega \in \Omega\}$, where $X: \mathbb{R}^m \times M \to TM$ is a \mathbb{C}^2 surjective bundle map and A a smooth vector field. Assume it is a Brownian motion on M and let \mathcal{F}^{x_0} be its filtration. The solution starting from x_0 shall be denoted by $x_t(\omega)$, defined for almost all $\omega \in \Omega$. Denote by $Y(x): T_x M \to \mathbb{R}^m$ the adjoint of X(x).

It is shown in [7] that such an sde determines a metric connection $\check{\nabla}$ on M, the LJW connection, and that this is the Levi-Civita connection if and only if X(x)(dY(v)) = 0 all $v \in T_x M$, $x \in M$. This holds if (1) is the gradient sde determined by an isometric immersion of M into \mathbb{R}^m .

There is the Itô map $\mathcal{I}: \Omega \to C_{x_0}M$,

$$\mathcal{I}_t(\omega) = x_t(\omega), \quad 0 \le t \le T, \tag{2}$$

which is measure preserving. Furthermore \mathcal{I}_t is differentiable at ω in the direction of the Cameron–Martin space $H = L_0^{2,1}(\mathbb{R}^m)$ in the sense of Malliavin calculus, giving a derivative of \mathcal{I} which we write as $T_\omega \mathcal{I}: H \to T_{\mathcal{I}(\omega)}C_{x_0}M$. It also gives $\mathcal{I}^*: L^2(C_{x_0}M; \mathbb{R}) \to L^2(\Omega; \mathbb{R})$ by $\mathcal{I}^*(f) = f \circ \mathcal{I}$.

On Ω we also have the closed operator $d \equiv d_{\Omega}$ on $L^2(\Omega; \mathbb{R})$ with associated space $\mathbb{D}^{2,1}(\Omega, \mathbb{R})$. Elements of $\mathbb{D}^{2,1}(C_{x_0}M; \mathbb{R})$ are characterised by a weak form of *H*-Gateaux differentiability in [14] and so $\mathbb{D}^{2,1}(C_{x_0}M; \mathbb{R})$ is independent of the choice of $\text{Dom}(d_{\mathcal{H}})$ provided it satisfies the analogue of (i) and (ii). For $C_{x_0}M$ this independence has not been clear and a particular consequence of the results announced here is that it does hold.

Here we discuss only the case of Brownian motion measure, and Levi-Civita connections for brevity, but the proofs extend easily to the case of non-degenerate diffusions with constant rank symbols, and are given in detail in this context in [9]. For a discussion of intertwining properties of the stochastic development map see [4] (but the intertwining there is different from that discussed here) and [11]. Denote by \mathcal{H} the 'vector bundle' with fibres \mathcal{H}_{σ} , by $L^2 \Gamma \mathcal{H}$ the space of L^2 sections of \mathcal{H} , and by $L^2 \Gamma \mathcal{H}^*$ the space of L^2 sections of the dual of \mathcal{H} .

This work draws on earlier work with S. Aida and Y. LeJan and was especially stimulated by our contacts with them, S. Fang and Z.-M. Ma. We are also grateful for comments by S. Fang on a preliminary version.

2. Main results

Theorem 2.1. Assume the LJW connection of (1) is the Levi-Civita connection. A real valued L^2 function f on $C_{x_0}M$ belongs to $\text{Dom}(d_{C_{x_0}M})$ if and only if $f \circ \mathcal{I} \in \text{Dom}(d_{\Omega})$. Consequently \mathcal{I}^* gives a topological linear isomorphism of $\mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})$ with the closed subspace of $\mathbb{D}^{2,1}(\Omega;\mathbb{R})$ consisting of \mathcal{F}^{x_0} measurable functions. Moreover $\mathbb{D}^{2,1}(\Omega,\mathbb{R})$ is mapped to itself by $\mathbb{E}\{-|\mathcal{F}^{x_0}\}$.

Idea of the proof. From [10], with a more general and corrected proof in [9], we know \mathcal{I}^* restricted to $\mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})$ has closed range in $\mathbb{D}^{2,1}(\Omega;\mathbb{R})$ and $\mathcal{I}^* d_{C_{x_0}M} \subset d_\Omega \mathcal{I}^*$. We can therefore prove the result by showing that if $\mathcal{I}^*(f) \in \mathbb{D}^{2,1}(\Omega;\mathbb{R})$, the domain of d on Ω , then there exists a sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})$ such that $\mathcal{I}^*(f_n) \to \mathcal{I}^*(f)$ in $\mathbb{D}^{2,1}(\Omega;\mathbb{R})$. To do this we shall use the characterisation of $\mathbb{D}^{2,1}(\Omega;\mathbb{R})$ in terms of the chaos expansion and write $\mathcal{I}^*(f) = \sum_{k=0}^{\infty} I_k(\alpha_k)$ with $I_k(\alpha_k)$ multiple stochastic integrals. Take $f_n(\sigma) = \mathbb{E}\{\sum_{k=0}^n \mathcal{I}_k(\alpha_k) \mid x_k = \sigma\}$ to be the conditional expectations of $\sum_{k=0}^n I_k(\alpha_k)$. In fact assuming $\mathbb{E}f = 0$ we have $\mathcal{I}^*(f_n) = \sum_{k=1}^n J_k(\alpha_k)$ for $J_k(\alpha_k)$ the iterated integral

$$J_k(\alpha_k) := k! \int_0^T \int_0^{t_k} \dots \int_0^{t_2} \langle \alpha_k(t_1, \dots, t_k), K^{\perp}(x_{t_1}) \, \mathrm{d}B_{t_1} \otimes \dots \otimes K^{\perp}(x_{t_k}) B_{t_k} \rangle_{\otimes \mathbb{R}^m}, \tag{3}$$

where $K^{\perp}(x) : \mathbb{R}^m \to [\ker X(x)]^{\perp}$ is the orthogonal projection Y(x)X(x), cf. [7,2]. Using the fact that $\mathbb{E}\{|\mathbf{d}(K^{\perp} \circ \mathcal{I}_t)|^2 | \mathcal{F}^{x_0}\}$ is in $L^{\infty}(\Omega; \mathbb{R})$ uniformly in t, as in [2] we obtain the estimate $\sum_k \|\mathbf{d}(J_k(\alpha_k))\|_{L^2(\Omega, H^*)}^2 \leq const. \sum_k kk! \|\alpha_k\|^2$ which is finite, by Proposition 1.2.1 of Nualart [12], if $\mathcal{I}^* f$ belongs to $\mathbb{D}^{2,1}(\Omega; \mathbb{R})$. This implies $\mathcal{I}^*(f_n) \to \mathcal{I}^*(f)$ in $\mathbb{D}^{2,1}(\Omega, \mathbb{R})$.

On the other hand take $\check{B}_t = \int_0^t //_s^{-1} X(x_s) dB_s$, the anti-development of x. to see

$$f_n(\sigma) = \sum_{k=1}^n k! \int_0^T \int_0^{t_k} \dots \int_0^{t_2} \langle \alpha_k(t_1, \dots, t_k), Y(x_{t_1}) / / _{t_1}(\sigma) \, \mathrm{d}\check{B}_{t_1} \otimes \dots \otimes Y(x_{t_k}) / / _{t_k}(\sigma) \, \mathrm{d}\check{B}_{t_k} \rangle_{\otimes \mathbb{R}^m}.$$

The fact that f_n is in $\text{Dom}(d_{C_{x_0}M})$ is essentially standard, e.g., see Cruzeiro–Malliavin [4] or the Appendix in Aida [1]. For a gradient stochastic differential equation (1) determined by an isometric $j: M \to \mathbb{R}^m$ it is especially clear since then $K^{\perp}(x_t) dB_t$ can be replaced by $d\tilde{x}_t - \frac{1}{2}\Delta j(x_t) dt$ for $\tilde{x}_t = j(x_t) \in \mathbb{R}^m$. \Box

Corollary 2.2. Dom $(d_{C_{x_0}M})$ is independent of the choice of Dom $(d_{\mathcal{H}})$ provided that it satisfies (i), (ii).

Corollary 2.3. (a) $\mathcal{I}^* d_{C_{x_0}M} = d_{\Omega}\mathcal{I}^*$.

(b) There is equality of the following two forms: $\int_{C_{x_0}M} |\mathbf{d}_{C_{x_0}M} f|^2 \, \mathrm{d}\mu_{x_0} = \int_{\Omega} |\mathbb{E}\{\mathbf{d}_{\Omega}\mathcal{I}^*(f)|\mathcal{F}^{x_0}\}|^2 \, \mathrm{d}P \text{ and there}$ is a constant c with $\int_{C_{x_0}M} |\mathbf{d}_{C_{x_0}M} f|^2 \, \mathrm{d}\mu_{x_0} \leq \int_{\Omega} |\mathbf{d}_{\Omega}\mathcal{I}^*f|^2 \, \mathrm{d}P \leq c \int_{C_{x_0}M} |\mathbf{d}_{C_{x_0}M}f|^2 \, \mathrm{d}\mu_{x_0}, f \in \mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R}),$ cf. [6,13].

Using the characterisation of Dom(div) for Ω in [12] (Proposition 1.3.1), Corollary 2.2 can be strengthened to:

Theorem 2.4. There is a unique closed operator d from $L^2(C_{x_0}M; \mathbb{R})$ to $L^2\Gamma\mathcal{H}^*$ such that (i) d agrees with $d_{\mathcal{H}}$ on smooth cylindrical functions; (ii) Dom(d^{*}) contains all smooth cylindrical one forms.

2.1. Higher derivatives and covariant derivatives

The main result extends to covariant differentiation using the damped Markovian connection introduced in [3], and to higher derivatives. Here we can only state some sample results. Details are in [9]. If *G* is a separable Hilbert space we define $d \equiv d^G$: Dom $(d^G) \subset L^2(C_{x_0}M; G) \to L^2\Gamma(\mathcal{H}^* \otimes G)$ to be the closure of the derivative naturally defined with domain the linear span of $\{F : C_{x_0}M \to \mathbb{R} \mid F(\sigma) = f(\sigma)g \text{ some } f \in \text{Dom}(d_{\mathcal{H}}), g \in G\}$. Then the canonical isometry of $L^2(C_{x_0}M; \mathbb{R}) \otimes G$ with $L^2(C_{x_0}M; G)$ maps Dom $(d) \otimes G$ onto Dom (d^G) so that Theorem 2.1 clearly holds for *G*-valued functions.

By an \mathcal{H} -1-form we mean a section of \mathcal{H}^*_{\cdot} . Define \mathbb{W} : Dom $(\mathbb{W}) \subset L^2 \Gamma \mathcal{H}^* \to L^2 \Gamma (\mathcal{H} \otimes \mathcal{H}^*)$ by

$$\Psi_{v}\phi(u) = d\left[\phi\left(W_{\cdot}\int_{0}^{\infty}W_{s}^{-1}X(ev_{s})(-)\,\mathrm{d}s\right)\right](v)\left(Y(\sigma_{\cdot})\left(\frac{\mathbb{D}}{\mathrm{d}\cdot}u\right)\right), \quad u,v\in\mathcal{H}_{\sigma}$$

K.D. Elworthy, X.-M. Li / C. R. Acad. Sci. Paris, Ser. I 337 (2003) 741-744

$$\operatorname{Dom}(\mathbb{W}) = \left\{ \phi \colon \phi \left(W \cdot \int_{0}^{1} W_{s}^{-1} X(\sigma(s)) \left(\frac{\mathrm{d}}{\mathrm{d}s} - \right) \mathrm{d}s \right) \text{ is in } \operatorname{Dom}(\mathrm{d}) \right\}.$$

With this domain \mathbb{W} is a closed operator, and is independent of the choice of the sde (1) provided it induces the Levi-Civita connection. From Theorem 2.1 and results in [8] on the conditional expectation of the *H*-derivative of an Itô map we have

Corollary 2.5. If $\phi \in L^2 \Gamma \mathcal{H}^*$ then $\phi \in \text{Dom}(\mathbb{W})$ if and only if $\mathcal{I}^*(\phi)$ has $\mathbb{E}\{\mathcal{I}^*(\phi) | \mathcal{F}^{x_0}\}$ in $\text{Dom}(d_{\Omega})$.

Here the pull back $\mathcal{I}^*(\phi) = \phi \circ T\mathcal{I}$ is defined as a limit in L^2 of $(\phi_n \circ T\mathcal{I})$ where the ϕ_n are cylindrical and converge to ϕ , [9,10]. It can be treated as a stochastic integral.

We can extend the definition of \mathbb{W} to other \mathcal{H} -tensors and define Sobolev spaces $\mathbb{D}^{2,k}(C_{x_0}M; G)$ for k = 2, 3, ... in the usual way. These depend only on the Riemannian structure of M.

Definition 2.1. Let $\mathbb{D}^{2,1}_{\mathcal{F}^{x_0}}(\Omega; G)$ be the subset of $\mathbb{D}^{2,1}(\Omega; G)$ whose elements are \mathcal{F}^{x_0} -measurable. Inductively $\mathbb{D}^{2,k}_{\mathcal{F}^{x_0}}(\Omega; G)$ consists of F such that (a) $F \in \mathbb{D}^{2,1}_{\mathcal{F}^{x_0}}(\Omega; G)$ and (b) $\mathbb{E}\{d_{\Omega}F|\mathcal{F}^{x_0}\}: \Omega \to H^* \otimes G$ is in $\mathbb{D}^{2,k-1}_{\mathcal{F}^{x_0}}(\Omega; H^* \otimes G)$, furnished with the norm $\|F\|_{\mathcal{F}^{x_0},2,k} := \left(\sum_{j=0}^k |(d_{\Omega} \circ \mathbb{E}\{-|\mathcal{F}^{x_0}\})^j F|^2_{L^2(\Omega; H^* \otimes G)}\right)^{1/2}$.

Corollary 2.6. An element of $f \in L^2(C_{x_0}M; \mathbb{R})$ is in $\mathbb{D}^{2,k}(C_{x_0}M; \mathbb{R})$ if and only if $f \circ \mathcal{I}$ is in the domain of the k-fold iterate of the operator $d_{\Omega} \circ \mathbb{E}\{-|\mathcal{F}^{x_0}\}$. Consequently \mathcal{I}^* restricts to give a linear isomorphism from $\mathbb{D}^{2,k}(C_{x_0}M; \mathbb{R})$ onto $\mathbb{D}^{2,k}_{\mathcal{F}^{x_0}}(\Omega; \mathbb{R})$.

References

- S. Aida, On the irreducibility of certain Dirichlet forms on loop spaces over compact homogeneous spaces, in: New Trends in Stochastic Analysis, World Scientific, 1997, pp. 3–42.
- [2] S. Aida, K.D. Elworthy, Differential calculus on path and loop spaces. 1. Logarithmic Sobolev inequalities on path spaces, C. R. Acad. Sci. Paris, Ser. I 321 (1995) 97–102.
- [3] A.B. Cruzeiro, S.-Z. Fang, An L² estimate for Riemannian anticipative stochastic integrals, J. Funct. Anal. 143 (2) (1997) 400-414.
- [4] A.B. Cruzeiro, P. Malliavin, Renormalized differential geometry on path spaces: structural equations, curvature, J. Funct. Anal. 139 (1996) 119–181.
- [5] B.K. Driver, A Cameron–Martin type quasi-invariance theorem for Brownian motion on a compact Riemannian manifold, J. Funct. Anal. 100 (1992) 272–377.
- [6] B.K. Driver, The non-equivalence of Dirichlet forms on path spaces, in: Stochastic Analysis on Infinite-Dimensional Spaces, Longman, 1994, pp. 75–87.
- [7] K.D. Elworthy, Y. Le Jan, X.-M. Li, Concerning the geometry of stochastic differential equations and stochastic flows, in: New Trends in Stochastic Analysis, World Scientific, 1997.
- [8] K.D. Elworthy, Y. LeJan, X.-M. Li, On the Geometry of Diffusion Operators and Stochastic Flows, in: Lecture Notes in Math., Vol. 1720, Springer, 1999.
- [9] K.D. Elworthy, X.-M. Li, Itô map and the chain rule in Malliavin calculus, in preparation.
- [10] K.D. Elworthy, X.-M. Li, Special Itô maps and an L² Hodge theory for one forms on path spaces, in: Stochastic Processes, Physics and Geometry: New Interplays, I, American Mathematical Society, 2000, pp. 145–162.
- [11] X.-D. Li, Sobolev spaces and capacities theory on path spaces over a compact Riemannian manifold, Probab. Theory Related Fields 125 (2003) 96–134.
- [12] D. Nualart, The Malliavin Calculus and Related Topics, Springer-Verlag, 1995.
- [13] I. Shigekawa, A quasihomeomorphism on the Wiener space, Proc. Sympos. Pure Math. 57 (1995) 473-486.
- [14] H. Sugita, On a characterization of the Sobolev spaces over an abstract wiener space, J. Math. Kyoto Univ. 25 (4) (1985) 717–725.

744