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Mathematical Problems in Mechanics

Equivariant cosymmetry and front solutions of the Dubreil–Jacotin–Long equation. Part 1: Boussinesq limit

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Abstract

The problem of two-dimensional internal travelling waves in a perfect fluid with smooth density being close to linear stratification is considered. Approximate front solutions connecting uniform flow with a conjugate shear flow of the first mode are constructed. It is demonstrated that the number of the front branches essentially depends on the fine-scale stratification for linear density background. *To cite this article: N. Makarenko, C. R. Acad. Sci. Paris, Ser. I 337 (2003).* © 2003 Académie des sciences. Published by Elsevier SAS. All rights reserved.

Résumé

Cosymétrie équivariante et solutions «fronts» de l'équation de Dubreil–Jacotin–Long. 1ère partie : la limite Boussinesq. On considère le problème d'écoulement bidimensionnel en ondes internes progressives dans un fluide parfait, à densité régulière voisine d'une stratification linéaire. On construit des solutions approchées de type « fronts » connectant un écoulement uniforme à un écoulement de cisaillement conjugué du premier mode. On montre que le nombre de branches de type « fronts » dépend essentiellement de l'échelle fine de la stratification de base. *Pour citer cet article : N. Makarenko, C. R. Acad. Sci. Paris, Ser. I 337 (2003).*

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1. Introduction

The fronts, or smooth bores, are nonlinear traveling waves which realize dissipationless transitions between conjugate stratified flows at opposite ends of the channel. Benjamin [2] developed a unified theory of conjugate flows including its subcritical and supercritical classification, as well as global existence results based on the topological fixed-point principles. Existence of fronts was proved by Amick and Turner [1], Makarenko [6] and Mielke [8] for a two-layer fluid and by James [4] for a continuously stratified fluid. Two-fluid Euler equations possess unique solution branch of conjugate flows for a given uniform flow. As a consequence, the center manifold has in this case a pair of heteroclinic orbits which give fronts of small amplitude. James [5] has proved recently the weak continuity of the center manifold in a singular limit when the regular stratification becomes to be piece-wise

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constant. In contrast, multiple branches of conjugate flows can exist for a fixed spectral mode when a continuous stratification is still close to linear or exponential profile, and these flows can be far away from the primary flow. In the present paper we discuss the existence of large amplitude fronts by means of the Lyapunov–Schmidt method. The Part 1 deals with perturbation scheme suggested by Benney and Ko [3] especially to treat large amplitude internal waves in a slightly stratified fluid. Here we use also some results on conjugate flows obtained in [7]. Looking for exact solutions, we consider in the Part 2 the Fredholm operator equation being equivalent to the front problem near the Boussinesq limit.

2. Statement of the problem

The Dubreil–Jacotin–Long equation for the stream function ψ of a steady flow of inhomogeneous fluid has the form

$$\rho(\psi)\Delta\psi + \rho'(\psi)\left(gy + \frac{1}{2}|\nabla\psi|^2\right) = \rho'(\psi)\left(\frac{g\psi}{c} + \frac{1}{2}c^2\right),$$

where ρ is fluid density, g is gravity acceleration and c is wave speed. We consider the flow between flat bottom and rigid lid, so the boundary conditions are $\psi = 0$ (y = 0) and $\psi = ch$ (y = h). In the frame moving with the front the flow tends to the uniform flow with known density profile: $\psi \to \psi_{\infty} = cy$, $\rho \to \rho_{\infty}(y)$ as $x \to -\infty$. Therefore we have $\rho(\psi) = \rho_{\infty}(\psi/c)$ due to the density conservation along the streamlines. Basic dimensionless numbers are the Boussinesq parameter $\sigma = N_0^2 h/(\pi g)$ where N_0 is the typical Brunt–Väisälä frequency, and inverse densimetric Froude number $\lambda = \sigma gh/(\pi c^2)$. We suppose that the dimensionless density $\rho(\bar{y}, \sigma) = \rho_{\infty}(y)/\rho_{\infty}(0)$ depends on the scaled variable $\bar{y} = \pi y/h$ as follows: $\rho(\bar{y}, \sigma) = 1 - \sigma \bar{y} - \sigma^2 \rho_1(\bar{y}, \sigma)$. Thus we take into account small perturbation of linear density by the terms of the order $O(\sigma^2)$ under following condition.

Condition A. The function $\rho_1(\bar{y}, \sigma) \in C^l([0, \pi] \times [0, \sigma_0])$, $l \ge 4$, is such that the inequalities $\rho > 0$, $\rho_{\bar{y}} < 0$ are valid by $(\bar{y}, \sigma) \in [0, \pi] \times (0, \sigma_0]$ with some $\sigma_0 > 0$.

Looking for the solution $\psi(x, y)/(ch) = \bar{y} + v(\bar{x}, \bar{y})$ versus scaled variables \bar{y} and $\bar{x} = \sqrt{\sigma \pi x}/h$ we formulate the problem for the parameter $\lambda > 0$ and the function v to satisfy the equations in the strip $\Omega = \mathbb{R} \times (0, \pi)$

$$F(v;\sigma,\lambda) \stackrel{\text{def}}{=} \sigma(\rho v_x)_x + (\rho v_y)_y - \lambda \rho' v - f(v,\nabla v, y;\sigma) = 0, \quad (x,y) \in \Omega,$$
(1)

(a)
$$v(x,0) = v(x,\pi) = 0;$$
 (b) $v \to v^{\pm}, \nabla v \to \nabla v^{\pm} \quad (x \to \pm \infty),$ (2)

where $\rho = \rho(y + v, \sigma)$, $\rho' = \sigma^{-1}\rho_y(y + v, \sigma)$, $f = (1/2)(\sigma^2 v_x^2 + \sigma v_y^2)\rho'$ with given $\sigma > 0$, bar is omitted everywhere. Conjugate pair v^{\pm} includes the upstream uniform flow $v^{-}(y) = 0$ and the downstream shear flow which is presented by the non-zero solution $v^{+}(y)$ of Eqs. (1), (2)(a). This pair should satisfy matching condition caused by the momentum conservation along the channel. The treatment uses variational formulation $F = \delta L/\delta v$ for the DJL operator with the Lagrangian

$$L = -\frac{1}{2}\rho(y+v,\sigma)\left(\sigma v_x^2 + v_y^2\right) + \sigma^{-1}\lambda \int_{y}^{y+v} \left(\rho(\psi,\sigma) - \rho(y+v,\sigma)\right) \mathrm{d}\psi.$$

Translation invariance of L in x implies by the Noether theorem the divergence relation

$$D_{x}(L - v_{x} L_{v_{x}}) + D_{y}(-v_{x} L_{v_{y}}) = v_{x} F(v; \sigma, \lambda).$$
(3)

As a consequence we obtain by integration (3) in y the flow force integral

$$\int_{0}^{\pi} \left(L + \sigma \rho v_x^2 \right) \mathrm{d}y = 0 \tag{4}$$

which must be satisfied for all $x \in \mathbb{R}$. Thus the conjugate state v^+ is admissible as the flow behind the front only if it lies at the same level of invariant functional (4) as the primary flow.

3. Supercritical conjugate flows and approximate front solutions

The operator *F* linearized to the primary solution v = 0 has the countable set of normal modes $e^{i\kappa x}\phi_n(y; \sigma, \kappa)$ (for a fixed value of the wave-number $\kappa \in \mathbb{R}$) where ϕ_n (n = 1, 2, ...) are the eigenfunctions of the Sturm–Liouville problem

$$\left(\rho(y,\sigma)\phi_y\right)_{y} - \left(\sigma\kappa^2\rho(y,\sigma) + \lambda\sigma^{-1}\rho_y(y,\sigma)\right)\phi = 0, \quad \phi(0) = \phi(\pi) = 0.$$

Under Condition A, all eigenvalues $\lambda_n(\sigma, \kappa)$ are real, positive and increase strictly monotone for increasing κ^2 , and also $\lambda_n \to +\infty$ as $\kappa^2 \to +\infty$. Therefore the normal modes with real κ exist for $\sigma \in (0, \sigma_0]$, $\lambda \ge \lambda_1(\sigma, 0)$ where λ_1 is minimal eigenvalue. The resolvent set is determined by the inequality $\lambda < \lambda_1(\sigma, 0)$ since in this case the equation $F'_v(0; \sigma, \lambda)u = f$ is unique solvable in Sobolev spaces $W_2^2(\Omega) \cap W_{2,0}^1(\Omega)$ for any given $f \in L_2(\Omega)$. This fact follows immediately from the below bounds $\lambda_n(\sigma, k) \ge (\rho_{\min}/\rho'_{\max})(n^2 + \sigma k^2)$ given by the Sturm comparison theorem with $\rho_{\min}(\sigma) = \min_y \rho(y, \sigma)$ and $\rho'_{\max}(\sigma) = \max_y(-\sigma^{-1}\rho_y(y, \sigma))$ for $y \in [0, \pi]$. For example, the parabola $\lambda = 1 + \sigma^2/4$ determines the spectrum boundary in the (σ, λ) -plane for DJL operator with exponential stratification $\rho = \exp(-\sigma y)$.

Conjugate flows v^+ bifurcate from simple eigenvalues $\lambda_n = n^2$ (n = 1, 2, ...) of one-dimensional DJL operator within the limit $\sigma = 0$. We seek $v^+(y) = b \sin ny + w(y)$ with $(w, \sin y)_{L_2[0,\pi]} = 0$ for small $\sigma > 0$, finite *b*, and λ being close to λ_n . The bound |b| < 1/n provides the absence of a return flow in the downstream. An unusual event is that the one-dimensional Lyapunov–Schmidt bifurcation equation must be coupled with the flow force integral (4). By this way we obtain the implicit system of two scalar equations $A_n(b) \mathbf{a} = q(\mathbf{a}, b)$ for the pair $\mathbf{a} = (\sigma, \lambda - \lambda_n)$ and amplitude *b*, with nonlinearity *q* being of the order $O(|\mathbf{a}|^2)$. The matrix A_n has the form

$$A_n(b) = \begin{pmatrix} s_n(b) & m_n(b) \\ s'_n(b) & m'_n(b) \end{pmatrix}$$

with the coefficients $m_n(b) = b^2/2$ and

$$s_n(b) = \frac{2n^2}{\pi} \int_0^{\pi} \int_{-y}^{y+b\sin ny} \left(\rho_0(y+b\sin ny) - \rho_0(\psi) \right) d\psi \, dy + \frac{\pi}{4} (nb)^2 + \frac{n}{3\pi} \left(1 - (-1)^n \right) b^3,$$

where $\rho_0(\psi) = \rho_1(\psi, 0)$. Note that A_n is the Wronski matrix because the DJL operator is the gradient of the functional (4) for v = v(y). Bifurcation occurs when the amplitude *b* is close to the roots of the Wronskian $\Delta_n(b) = -m_n^2(b)(s_n(b)/m_n(b))'$.

Theorem 3.1. Let $b_0 \neq 0$ ($|b_0| < 1/n$) be the simple root of the function $\Delta_n(b)$. Then for small $\sigma > 0$ there exists the unique branch of conjugate flows $(v_n^+, \lambda_n^+) \in C^l[0, \pi] \times \mathbb{R}$ such that $(v_n^+(y; \sigma), \lambda_n^+(\sigma)) \rightarrow (b_0 \sin ny, n^2)$ as $\sigma \rightarrow 0+$. The eigenvalues have the asymptotics $\lambda_n^+(\sigma) = n^2 - 2b_0^{-2}s_n(b_0)\sigma + O(\sigma^2)$.

It is easy to see that the branches of the *n*-th mode are generated by the extreme points of the function $\Lambda_n(b) = -2s_n(b)/b^2$. The number of branches depends on the fine-scale stratification ρ_0 presented here by the

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coefficient s_n . The value $\Lambda_1(0\pm) = -s_1''(0)$ determines the slope of the spectrum boundary $\lambda = \lambda_1(\sigma, 0)$ at the point $\lambda = 1$, $\sigma = 0$. Therefore the 1st mode conjugate flow is supercritical if $\Lambda_1(b_0) < \Lambda_1(0)$ is satisfied.

We construct approximate solution of the front problem (1), (2) taking the branch of conjugate flows $(\lambda_1^+(\sigma), v^+(y; \sigma))$ given by Theorem 3.1 for some simple root b_0 of the function Δ_1 . Looking for the formal power expansion $v = v_0 + \sigma v_1 + \cdots$ we obtain the set of equations $v_{jyy} + v_j = f_j$, $(x, y) \in \Omega$; $v_j = 0$ $(y = 0, y = \pi)$, where $f_0 = 0$ and

$$f_1 = -D_{\sigma}F(v_0; 0, 1) - A_1(b_0)D_{\lambda}F(v_0; 0, 1).$$
(5)

The lowest-order solution has the form $v_0 = a_0(x) \sin y$ where the unknown function a_0 should satisfy the vanishing condition $(f_1, \sin y)_{L_2[0,\pi]} = 0$ of the secular terms. Thus we obtain the equation

$$a_0'' + p'(a_0) = 0 (6)$$

with the function $p(a_0) = (1/2)a_0^2(\Lambda_1(b_0) - \Lambda_1(a_0))$ having double roots at $a_0 = 0$ and $a_0 = b_0$. The first integral $a'_0^2 + 2p(a_0) = 0$ gives the front-type solution when the function p is strictly negative on the interval $a_0 \in (0, b_0)$.

Condition B. The function $\rho_0(y)$ is such that the inequality $\Lambda_1(b_0) < \Lambda_1(b)$ is valid for all $b \in (0, b_0)$.

This additional condition can be fulfilled only for supercritical conjugate flows. So we obtain the solution $a_0(x) \in C^{l+1}$ which is monotone function in x. Therefore we can fix the shift in x at the lowest order in σ by the choice of origin to be at the centre of the front defined by usual square rule. Approximate solution has the asymptotics $|a_0(x)| \leq C \exp(\alpha_0 x)$ as $x \to -\infty$ and $|b_0 - a_0(x)| \leq C \exp(-\beta_0 x)$ as $x \to +\infty$ with the exponents $\alpha_0 > 0$, $\beta_0 > 0$ where $\alpha_0^2 = \Lambda_1(0) - \Lambda_1(b_0)$ and $\beta_0^2 = (1/2)b_0^2\Lambda_1''(b_0)$. Note that the function $\Lambda_1(b)$ has strong minimum at b_0 with $\Lambda_1''(b_0) > 0$ due to the Theorem 3.1 and Condition B.

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