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Numerical Analysis

Reduced-basis approximation of the viscous Burgers equation: rigorous a posteriori error bounds

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Abstract

We present rigorous, sharp, and inexpensive *a posteriori* error bounds for reduced-basis approximations of the viscosity-parametrized Burgers equation. There are two critical ingredients: the Brezzi, Rappaz and Raviart (Numer. Math. 36 (1980) 1–25) framework for analysis of approximations of nonlinear elliptic partial differential equations; and offline/online computational procedures for efficient calculation of the necessary continuity and stability constants, and of the dual norm of the residual. Numerical results confirm the performance of the error bounds. ***To cite this article: K. Veroy et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Approximation par base réduite de l'équation de Burgers visqueuse : bornes d'erreur a posteriori rigoureuses. Nous présentons des bornes d'erreur *a posteriori* rigoureuses, précises et peu coûteuses pour l'approximation par base réduite de l'équation de Burgers avec la viscosité comme paramètre. Il y a deux composantes essentielles : l'approche de Brezzi, Rappaz et Raviart (Numer. Math. 36 (1980) 1–25) pour l'analyse d'approximations d'équations aux dérivées partielles nonlinéaires elliptiques ; et une procédure hors-ligne/en-ligne pour le calcul efficace des constantes nécessaires de continuité et de stabilité, et de la norme duale du résidu. Les résultats numériques confirment les performances de ces bornes d'erreur. ***Pour citer cet article : K. Veroy et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

Considérons un espace $Y \equiv H_0^1(\Omega)$ auquel on associe un produit scalaire $(\cdot,\cdot)_Y$, une norme $\|\cdot\|_Y$ et un crochet de dualité $\langle\cdot,\cdot\rangle$. On définit l'application $G : Y \to Y'$ donnée par (2) dont on cherche le zéro $u(\nu)$ dans Y étant donné $\nu \in \mathcal{D} \equiv [\nu^{\min}, \nu^{\max}] \subset \mathbb{R}^1_+$, (1), on définit également sa dérivée de Frechet $DG(\cdot,\cdot)$ en un point $(z,\nu) \in Y \times \mathcal{D}$, (3). Finalement, on définit la constante κ, (4), qui est essentielle à la majoration de la constante de continuité de DG.

Concernant l'approximation par base réduite (BR), on introduit un échantillonage emboîté de paramètres $S_1 = \{\bar{\nu}^1 \in \mathcal{D}\} \subset \cdots \subset S_{N^{\max}} \equiv \{\bar{\nu}^1 \in \mathcal{D}, \ldots, \bar{\nu}^{N^{\max}} \in \mathcal{D}\}$ et on construit l'espace d'approximation BR $W_N \equiv$

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span$\{\zeta_n \equiv u(\bar{\nu}^n),\ 1 \leqslant n \leqslant N\}$. Notre approximation BR s'écrit alors sous la forme (5) où on suppose que $DG(u_{N^{\max}}(\nu);\nu)$ est un isomorphisme et on note $\beta_N(\nu) \equiv \|DG(u_N(\nu);\nu)^{-1}\|^{-1}_{Y',Y}$. La Tableau 1 présente l'erreur normalisée de l'approximation BR en fonction de N. On notera la convergence rapide de cette approximation.

Quant à l'estimateur d'erreur *a posteriori*, il requiert une borne supérieure (respectivement, inférieure) de κ (respectivement de $\beta_N(\nu)$). Une borne supérieure de κ, $\tilde{\kappa}$, peut être calculée *a priori* et on choisit $\tilde{\kappa} = \frac{1}{4}$. Pour la construction de la borne inférieure de $\beta_N(\nu)$, $\tilde{\beta}(\nu)$, on introduit un ensemble de paramètres $U_J \equiv \{\tilde{\nu}^1 \in \mathcal{D}, \ldots, \tilde{\nu}^J \in \mathcal{D}\}$, une distance $d(\nu,\tilde{\nu};N) \equiv 2\beta_{N^{\max}}(\tilde{\nu})^{-1}[|\nu-\tilde{\nu}| + \frac{1}{2}\|u_N(\nu) - u_{N^{\max}}(\tilde{\nu})\|_{L^2(\Omega)}]$, et une application $\mathcal{I}_N\nu \equiv \arg\min_{\tilde{\nu}\in U_J} d(\nu,\tilde{\nu};N)$; on définit finalement $\tilde{\beta}(\nu) \equiv \frac{1}{2}\beta_{N^{\max}}(\mathcal{I}_N\nu)$. Le Lemme 3.1 montre alors que $\tilde{\beta}(\nu)$ est bien une borne inférieure pour $\beta_N(\nu)$ pour tout ν dans $\widetilde{\mathcal{D}}_N(U_J) \equiv \{\nu \in \mathcal{D} \mid d(\nu,\mathcal{I}_N\nu;N) \leqslant 1\}$. En suite, on définit $\Delta_N(\nu) \equiv \frac{3}{2}\,\varepsilon_N(\nu)/\tilde{\beta}(\nu)$ et $\tau(\nu) \equiv \frac{3}{2}\,\tilde{\kappa}/\tilde{\beta}(\nu)$, où $\varepsilon_N(\nu) \equiv \|G(u_N(\nu);\nu)\|_{Y'}$ est la norme duale du résidu. Le Théorème 3.2 montre que, sous certaines hypothèses ($\Delta_N(\nu)\tau(\nu) \leqslant 1,\ \nu \in \widetilde{\mathcal{D}}_N(U_J)$) et en utilisant la théorie développée par Brezzi, Rappaz et Raviart [3,4], $\Delta_N(\nu)$ définit des bornes d'erreur rigoureuses et précises pour notre approximation BR.

Dans la pratique, on cherche à évaluer $s(\nu) = \langle L, u(\nu)\rangle$, où L est un élément donné de Y', *rapidement* et *sûrement* : *rapidement* en approchant $s(\nu)$ par $s_N(\nu) \equiv \langle L, u_N(\nu)\rangle$ et *sûrement* en fournissant la borne *a posteriori* $\Delta^s_N(\nu) \equiv \|L\|_{Y'}\,\Delta_N(\nu)$; sous les hypothèses du Théorème 3.2, on a $|s(\nu) - s_N(\nu)| \leqslant \Delta^s_N(\nu)$. À cette fin – l'évaluation rapide et fiable de $s(\nu)$, – le stratagème algorithmique opère une décomposition hors-ligne/en-ligne [2,6,11] : elle permet de s'affranchir de la dépendance en $\mathcal{N}$, la dimension de l'approximation élément fini « de référence » de Y, lors de l'étape en-ligne pour les différentes composantes de l'approximation BR ainsi que l'estimateur d'erreur.

Pour deux valeurs du paramètre ($\nu = 0{,}305$ et $\nu = 0{,}013$) et en fonction de N, la Tableau 2 confirme les résultats de notre méthodologie. Le choix des deux cas montre clairement deux scénarios de comportement de l'estimateur d'erreur. Dans le premier cas, on observe non seulement la convergence rapide de Δ_N mais aussi que les conditions $\Delta_N(\nu)\tau(\nu) \leqslant 1$ et $\nu \in \widetilde{\mathcal{D}}_N(U_J)$ du Théorème 3.2 sont bien vérifiées pour $N \geqslant 1$; tandis que pour le deuxième cas, elle ne sont vérifiées que pour $N \geqslant 10$. Dans les deux cas, on observe que les bornes sont très précises. La propriété de convergence rapide de l'approximation BR est véritablement essentielle non seulement pour obtenir efficacement la précision désirée, mais également pour vérifier, encore une fois efficacement, la condition $\Delta_N(\nu)\tau(\nu) \leqslant 1$: la précision étant requise pour la prédiction et la certification.

1. Problem statement

The reliability of very low-dimensional reduced-basis (RB) approximations [1,2,5–12] of parametrized partial differential equations can only be assured by rigorous *a posteriori* error estimation procedures. In our earlier work we develop rigorous *a posteriori* error bounds for coercive linear [11], noncoercive linear [12], and special "coercive" nonlinear [12], elliptic partial differential equations. In this Note we demonstrate that rigorous and sharp *a posteriori* error bounds can also be obtained for RB approximations of certain *noncoercive* nonlinear elliptic partial differential equations. We consider here a very simple model problem, the viscous Burgers equations in one dimension;[1] however, the method directly extends to the multidimensional incompressible Navier–Stokes equations.

We define $Y \equiv H^1_0(\Omega \equiv]0,1[\subset \mathbb{R}^1)$, where $H^1_0(\Omega) = \{v \in H^1(\Omega) \mid v(0) = v(1) = 0\}$ and $H^1(\Omega) = \{v, v_x \in L^2(\Omega)\}$; here $L^p(\Omega),\ 1 \leqslant p \leqslant \infty$, is the space of measurable functions for which $\|v\|_{L^p(\Omega)} \equiv (\int_0^1 |v|^p)^{1/p}$ is finite. We associate to Y the inner product and norm $(w,v)_Y = \int_0^1 w_x v_x + wv$ and $\|w\|_Y = (w,w)_Y^{1/2}$, respectively. We denote the dual space of Y as Y', with duality pairing $\langle\cdot,\cdot\rangle$.

Given a $\nu \in \mathcal{D} \equiv [\nu^{\min}, \nu^{\max}] \subset \mathbb{R}^1_+$, we look for $u(\nu) \in Y$ such that

$$G\big(u(\nu);\nu\big) = 0, \tag{1}$$

where $G : Y \to Y'$ is the ν-parametrized C^1 mapping given by

[1] The method we describe here is similar in form, but greatly improved in interpretation, to the approach presented in [12].

$$\big\langle G(w;\nu), v\big\rangle \equiv \nu \int_0^1 w_x v_x - \frac{1}{2}\int_0^1 w^2 v_x - \langle F, v\rangle, \quad \forall w, v \in Y, \tag{2}$$

and $F \in Y'$ is a given linear functional.

The Frechet derivative of G at $(z;\nu)$ is given by

$$\big\langle DG(z;\nu)w, v\big\rangle \equiv \nu \int_0^1 w_x v_x - \int_0^1 z w v_x, \quad \forall w, v \in Y, \tag{3}$$

for any $z \in Y$. We further define

$$\kappa = \sup_{z\in Y}\sup_{w\in Y}\sup_{v\in Y} \frac{\int_0^1 z w v_x}{\|z\|_Y \|w\|_Y \|v\|_Y}, \tag{4}$$

in terms of which we can bound the continuity constant $\gamma(z;\nu) \equiv \|DG(z;\nu)\|_{Y,Y'}$ as $\gamma(z;\nu) \leqslant \nu + \kappa\|z\|_Y$. Note that κ, and hence $\gamma(z;\nu)$, is finite thanks to the continuous embedding of $H^1(\Omega)$ in $L^4(\Omega)$.

2. Reduced-basis approximation

We first introduce a nested set of parameter samples $S_1 \equiv \{\bar\nu^1 \in \mathcal{D}\} \subset \cdots \subset S_{N^{\max}} \equiv \{\bar\nu^1 \in \mathcal{D}, \ldots, \bar\nu^{N^{\max}} \in \mathcal{D}\}$.[2] Then, for any $N \leqslant N^{\max}$, we define the RB approximation space as $W_N \equiv \operatorname{span}\{\zeta_n \equiv u(\bar\nu^n),\ 1 \leqslant n \leqslant N\}$. In practice, $u(\bar\nu^{\cdot})$ is replaced by a "truth" Galerkin approximation $u^{\mathcal{N}}(\bar\nu^{\cdot}) \in Y^{\mathcal{N}} \subset Y$. We assume that the dimension of $Y^{\mathcal{N}}$, $\mathcal{N}$, is sufficiently large that $u^{\mathcal{N}}(\nu)$ may be effectively equated with $u(\nu)$; as we shall see, the online complexity is *independent* of $\mathcal{N}$.

Our RB approximation is then: Given a $\nu \in \mathcal{D}$, find $u_N(\nu) \in W_N$ such that

$$\big\langle G\big(u_N(\nu);\nu\big), v\big\rangle = 0, \quad \forall v \in W_N. \tag{5}$$

We shall assume that $DG(u_{N^{\max}}(\nu);\nu)$ is an isomorphism; more quantitatively, we suppose that $\beta_{N^{\max}}(\nu) \geqslant \beta_0 > 0,\ \forall \nu \in \mathcal{D}$, where $\beta_N(\nu) \equiv \|DG(u_N(\nu);\nu)^{-1}\|^{-1}_{Y',Y}$.

Standard proofs demonstrate that both (1) and (5) admit unique solutions for ν sufficiently large; observations suggest that, in fact, both (1) and (5) admit unique solutions *for all positive* ν. We present in Table 1 $\max_{\nu\in\mathcal{D}}(\|e(\nu)\|_Y/\|u(\nu)\|_Y)$ as a function of N for $\mathcal{D} \equiv [0.01, 10.0]$. We observe that the error $e(\nu) \equiv u(\nu) - u_N(\nu)$ tends to zero (uniformly in ν) quite rapidly.

3. *A posteriori* error bounds

Our *a posteriori* error estimators require rigorous upper (respectively, lower) bounds for κ (respectively, $\beta_N(\nu)$). The upper bound on κ, $\tilde\kappa$, is readily derived *a priori*: from the continuous embedding $H^1(\Omega) \subset L^4(\Omega)$ we derive $\int_0^1 z w v_x \leqslant \|z\|_{L^4(\Omega)}\|w\|_{L^4(\Omega)}\|v\|_Y$; it further follows from the embedding $H^1(\Omega \equiv]0,1[) \subset L^\infty(\Omega \equiv]0,1[)$ that $\|w\|_{L^4(\Omega)} \leqslant \|w\|_{L^\infty(\Omega)} \leqslant \frac{1}{2}\|w\|_Y$. We thus choose $\tilde\kappa = \frac{1}{4}$.

Table 1
Normalized error in the reduced-basis approximation as a function of N

N	1	5	9	13	17
$\max_{\nu\in\mathcal{D}} \frac{\|e(\nu)\|_Y}{\|u(\nu)\|_Y}$	3.04×10^{0}	1.20×10^{-1}	7.30×10^{-3}	3.34×10^{-5}	3.80×10^{-6}

[2] The samples are constructed adaptively based on the inexpensive *a posteriori* error estimators of Section 3 [12].

To construct a lower bound for $\beta_N(\nu)$, $\tilde{\beta}(\nu)$, we introduce a parameter set $U_J \equiv \{\tilde{\nu}^1 \in \mathcal{D}, \ldots, \tilde{\nu}^J \in \mathcal{D}\}$, a distance $d(\nu, \tilde{\nu}; N) \equiv 2\beta_{N^{\max}}(\tilde{\nu})^{-1}[|\nu - \tilde{\nu}| + \frac{1}{2}\|u_N(\nu) - u_{N^{\max}}(\tilde{\nu})\|_{L^2(\Omega)}]$, and a mapping $\mathcal{I}_N\nu \equiv \arg\min_{\tilde{\nu}\in U_J} d(\nu, \tilde{\nu}; N)$. We then define $\tilde{\beta}(\nu) \equiv \frac{1}{2}\beta_{N^{\max}}(\mathcal{I}_N\nu)$, and prove

Lemma 3.1. *The inf-sup approximation* $\tilde{\beta}(\nu)$ *satisfies* $0 < \beta_0/2 \leqslant \tilde{\beta}(\nu) \leqslant \beta_N(\nu)$, $\forall \nu \in \widetilde{\mathcal{D}}_N(U_J)$, *where* $\widetilde{\mathcal{D}}_N(U_J) \equiv \{\nu \in \mathcal{D} \mid d(\nu, \mathcal{I}_N\nu; N) \leqslant 1\}$.

Proof. We can derive, from (3) and the embedding $H^1(\Omega \equiv]0,1[) \subset L^\infty(\Omega \equiv]0,1[)$, that

$$\begin{aligned}\big\langle \big(DG\big(u_N(\nu);\nu\big) - DG\big(u_{N^{\max}}(\tilde{\nu});\tilde{\nu}\big)\big)w, v\big\rangle &= (\nu - \tilde{\nu})\int_0^1 w_x v_x - \int_0^1 \big(u_N(\nu) - u_{N^{\max}}(\tilde{\nu})\big)wv_x \\ &\leqslant |\nu - \tilde{\nu}|\|w\|_Y\|v\|_Y + \frac{1}{2}\big\|u_N(\nu) - u_{N^{\max}}(\tilde{\nu})\big\|_{L^2(\Omega)}\|w\|_Y\|v\|_Y, \quad \forall \nu, \tilde{\nu} \in \mathcal{D};\end{aligned}$$

hence $\|DG(u_{N^{\max}}(\mathcal{I}_N\nu);\mathcal{I}_N\nu)^{-1}(DG(u_N(\nu);\nu) - DG(u_{N^{\max}}(\mathcal{I}_N\nu);\mathcal{I}_N\nu))\|_{Y,Y} \leqslant \frac{1}{2}$, $\forall \nu \in \widetilde{\mathcal{D}}_N(U_J)$. Thus, $DG(u_N(\nu);\nu)^{-1} = [DG(u_{N^{\max}}(\mathcal{I}_N\nu);\mathcal{I}_N\nu) + (DG(u_N(\nu);\nu) - DG(u_{N^{\max}}(\mathcal{I}_N\nu);\mathcal{I}_N\nu))]^{-1}$ exists, and $\beta_N(\nu)^{-1} \leqslant 2\beta_{N^{\max}}(\mathcal{I}_N\nu)^{-1}$ [4]. The lower bound on $\tilde{\beta}(\nu)$ follows from $\beta_{N^{\max}}(\nu) \geqslant \beta_0$, $\forall \nu \in \mathcal{D}$. □

We now define $\Delta_N(\nu) \equiv \frac{3}{2}\varepsilon_N(\nu)/\tilde{\beta}(\nu)$, $\tau(\nu) \equiv \frac{3}{2}\tilde{\kappa}/\tilde{\beta}(\nu)$, and $\rho_N(\nu) \equiv 3\,\gamma(u_N(\nu);\nu)/\tilde{\beta}(\nu)$, where $\varepsilon_N(\nu) \equiv \|G(u_N(\nu);\nu)\|_{Y'}$ is the dual norm of the residual. We can then prove

Theorem 3.2. *For* ν *in* $\widetilde{\mathcal{D}}_N(U_J)$, *and* $\Delta_N(\nu) \leqslant \tau(\nu)^{-1}$, (i) *there exists a unique solution of* (1), $u(\nu)$, *in* $\mathcal{B}(u_N(\nu), \frac{3}{2}\tau(\nu)^{-1})$ *for which* (ii) $\|u(\nu) - u_N(\nu)\|_Y \leqslant \Delta_N(\nu)$, *and* (iii) $\Delta_N(\nu) \leqslant \rho_N(\nu)\,\|u(\nu) - u_N(\nu)\|_Y$. *Here* $\mathcal{B}(z,r) = \{w \in Y \mid \|w - z\|_Y < r\}$.

Proof. The proofs of (i) and (ii) follow directly from Lemma 3.1 and the Brezzi, Rappaz and Raviart framework, in particular Theorem 3.1 of [4] (slightly specialized to the quadratic nonlinearity of interest here).

To derive the effectivity result, (iii), we note that $e(\nu) \equiv u(\nu) - u_N(\nu)$ satisfies

$$\big\langle DG\big(u_N(\nu);\nu\big)e(\nu), v\big\rangle = -\big\langle G\big(u_N(\nu);\nu\big), v\big\rangle + \frac{1}{2}\int_0^1 e(\nu)^2 v_x. \tag{6}$$

We now note from standard duality arguments that $\varepsilon_N(\nu) = \|\hat{e}(\nu)\|_Y$, where $\hat{e}(\nu) \in Y$ satisfies $(\hat{e}(\nu), v)_Y = -\langle G(u_N(\nu);\nu), v\rangle$, $\forall v \in Y$. We next choose $v = \hat{e}(\nu)$ in (6), apply continuity, and invoke $\|e(\nu)\|_Y \leqslant \Delta_N(\nu) \equiv \frac{3}{2}\frac{\varepsilon_N(\nu)}{\tilde{\beta}(\nu)}$ to obtain $\Delta_N(\nu) \leqslant \frac{\rho_N(\nu)}{2}\|e(\nu)\|_Y + (\Delta_N(\nu)\tau(\nu))\frac{\Delta_N(\nu)}{2}$; the result then follows from $\Delta_N(\nu) \leqslant \tau(\nu)^{-1}$.

4. Computational complexity: offline/online decomposition

In actual practice, our interest is not in $u(\nu)$ *per se*, but rather in a (say) linear-functional output, $s(\nu) \equiv \langle L, u(\nu)\rangle$, where L is a prescribed member of Y'. We wish to reliably evaluate $s(\nu)$ rapidly in the *limit of many queries* – as demanded in the (adaptive) design optimization and (real-time) control contexts. For "rapidly", we approximate $s(\nu)$ by $s_N(\nu) \equiv \langle L, u_N(\nu)\rangle$. For "reliably", we provide the *a posteriori* bound $\Delta_N^s(\nu) \equiv \|L\|_{Y'}\,\Delta_N(\nu)$; under the hypotheses of Theorem 3.2, $|s(\nu) - s_N(\nu)| \leqslant \Delta_N^s(\nu)$.

We now discuss the computational stratagem by which we efficiently evaluate $s_N(\nu)$ and $\Delta_N(\nu)$ (and hence $\Delta_N^s(\nu)$). The fundamental ingredient is an offline/online computational decomposition [2,6,11] that breaks the requisite calculations into two parts: an expensive *offline* stage performed once, and an inexpensive *online* stage performed many times – for each new evaluation $\nu \to s_N(\nu), \Delta_N(\nu)$. The complexity of the online stage will

depend on N, which is typically small (see Table 1), but *not* on $\mathcal{N}$, which is typically large; we will thus realize marginal *real-time* – and, thanks to $\Delta_N^s(\nu)$, *reliable* – response.

We begin with the calculation of $u_N(\nu)$ and $s_N(\nu) = \langle L, u_N(\nu)\rangle$. To obtain $u_N(\nu)$, we apply a Newton iterative scheme: given $u_N^k(\nu) \in W_N$, find $\delta_N^k(\nu) \equiv u_N^{k+1}(\nu) - u_N^k(\nu) \in W_N$ such that $\langle DG(u_N^k(\nu);\nu)\delta_N^k(\nu), v\rangle = -\langle G(u_N^k(\nu);\nu), v\rangle,\ \forall v \in W_N$. This can be expressed in terms of our RB expansions $u_N^k(\nu) = \sum_{n=1}^N u_{N\,n}^k(\nu)\zeta_n$ and $\delta_N^k(\nu) = \sum_{n=1}^N \delta_{N\,n}^k(\nu)\zeta_n$ as

$$\sum_{n=1}^{N}\left(\nu A_{N\,m,n} + 2\sum_{n'=1}^{N} u_{N\,n'}^k(\nu) B_{N\,m,n',n}\right)\delta_{N\,n}^k(\nu) = F_m - \sum_{n=1}^{N}\left(\nu A_{N\,m,n} + \sum_{n'=1}^{N} u_{N\,n'}^k(\nu) B_{N\,m,n',n}\right)u_{N\,n}^k(\nu), \quad 1 \leqslant m \leqslant N, \tag{7}$$

where $A_{N\,m,n} \equiv \int_0^1 \frac{\mathrm{d}\zeta_n}{\mathrm{d}x}\frac{\mathrm{d}\zeta_m}{\mathrm{d}x},\ 1 \leqslant n, m \leqslant N$, $B_{N\,m,n',n} \equiv -\frac{1}{2}\int_0^1 \zeta_n\zeta_{n'}\frac{\mathrm{d}\zeta_m}{\mathrm{d}x},\ 1 \leqslant n, n', m \leqslant N$, and $F_{N\,m} = \langle F, \zeta_m\rangle$, $1 \leqslant m \leqslant N$. Once the iteration has converged, we evaluate our output as $s_N(\nu) = \sum_{n=1}^N u_{N\,n}(\nu)L_{N\,n}$, where $L_{N\,n} = \langle L, \zeta_n\rangle,\ 1 \leqslant n \leqslant N$. The offline/online decomposition is clear.

In the *offline* stage, we form the parameter-independent quantities $A_{N^{\max}} \in \mathbb{R}^{(N^{\max})^2},\ B_{N^{\max}} \in \mathbb{R}^{(N^{\max})^3}$, and $F_{N^{\max}} \in \mathbb{R}^{N^{\max}},\ L_{N^{\max}} \in \mathbb{R}^{N^{\max}}$ – at cost $\mathrm{O}((N^{\max})^3\mathcal{N})$. In the *online* stage, for any given ν, we first construct and solve (7) – at cost (per Newton iteration) $\mathrm{O}(N^3)$; and then, upon convergence, we evaluate $s_N(\nu)$ – at cost $\mathrm{O}(N)$. In the online stage, the complexity is *independent of* $\mathcal{N}$.[3]

We now turn to the *a posteriori* error bound. The two critical computational tasks are the construction of $\tilde{\beta}(\nu)$, and the evaluation of $\varepsilon_N(\nu)$. The latter is developed in detail in [12] – the resulting offline and online complexity are $\mathrm{O}((N^{\max})^4\mathcal{N})$ and $N^4/4$, respectively. We thus only discuss the former: we must first find $\mathcal{I}_N\nu$ and verify $\nu \in \widetilde{\mathcal{D}}_N(U_J)$; we can then evaluate $\frac{1}{2}\beta_{N^{\max}}(\mathcal{I}_N\nu)$. To determine $\mathcal{I}_N\nu$ we need only compare $d(\nu, \tilde{\nu}; N)$ at a few points $\tilde{\nu} \in U_J$ in the vicinity of ν. Once $\tilde{\nu}^i = \mathcal{I}_N\nu$ is obtained, we simply evaluate $\frac{1}{2}\beta_{N^{\max}}(\tilde{\nu}^i)$. The offline/online decomposition is clear.

In the *offline* stage we compute the parameter-independent quantities at cost $\mathrm{O}(JN^3) + \mathrm{O}(J\mathcal{N}) + \mathrm{O}((N^{\max})^2\mathcal{N})$. In the *online* stage, for any given ν and $u_N(\nu)$, we evaluate $d(\nu, \tilde{\nu}^j; N)$ for $\tilde{\nu}^j \approx \nu$ to obtain $\tilde{\nu}^i = \mathcal{I}_N\nu$ – at cost $\mathrm{O}((N^{\max})^2)$; then, assuming $d(\nu, \tilde{\nu}^i; N) \leqslant 1$,[4] we "look up" $\beta_{N^{\max}}(\tilde{\nu}^i)$. In summary, we may compute not only $s_N(\nu)$, but also $\Delta_N(\nu)$, at *online* cost independent of $\mathcal{N}$.[5]

5. Numerical results

We present here results for the case $\mathcal{D} = [\nu^{\min} = 0.01, \nu^{\max} = 10.0]$ and $N^{\max} = 18$; the resolution of our (uniform) truth approximation is $\mathcal{N} = 1{,}000$. We find empirically that $\beta_{N^{\max}}(\nu^{\min}) \sim \frac{1}{2}(\nu^{\min})^{-1}$ and $\gamma(u_{N^{\max}};\nu) \sim$ Const as $\nu^{\min} \to 0$; it follows that $\rho_N(\nu^{\min}) \sim \mathrm{Const}(\nu^{\min})^{-1}$. (Note if our primary interest is the ν-dependence of L^2 output functionals, we may gainfully reduce the relative weight of the H^1-seminorm contribution to $\|\cdot\|_Y$.) We also observe that, to achieve $\widetilde{\mathcal{D}}_N(U_J) = \mathcal{D}$, J increases quite slowly with decreasing $\nu^{\min}$; for the particular $U_{J=28}$ results presented here, $\widetilde{\mathcal{D}}_N(U_J) = \mathcal{D}$ for all $N \geqslant 9$.

We present in Table 2 $\Delta_N(\nu)\tau(\nu)$, $\Delta_N(\nu)/\|u(\nu)\|_Y$ (the relative error), and $\Delta_N(\nu)/\|e(\nu)\|_Y$ (the effectivity) as a function of N for $\nu = 0.305$ and $\nu = 0.013$. For $\nu = 0.305$ we observe that, even for small N, $\Delta_N(\nu)\tau(\nu) \leqslant 1$

[3] For the Burgers equation in $\mathbb{R}^1$, the cost savings provided by the RB method — $\mathrm{O}(N^3)$ vs. $\mathrm{O}(\mathcal{N})$ – are not interesting; however, for the incompressible Navier–Stokes equations in $\mathbb{R}^{d=2,3}$, the cost savings will be very significant.

[4] To minimize the risk that $d(\nu, \tilde{\nu}^i; N) > 1$, we would like to choose U_J such that $\widetilde{\mathcal{D}}_N(U_J) = \mathcal{D}$ for some suitably large N ($< N^{\max}$). This can be very efficiently effected.

[5] The offline/online decomposition applies generally to any affine parameter (data, property, or geometry) dependence [11].

Table 2
Numerical results for $\nu = 0.305$ and $\nu = 0.013$; for $N = 1$ and $N = 5$, $0.013 \notin \widetilde{\mathcal{D}}_N(U_J)$

N	$\nu = 0.305$			$\nu = 0.013$		
	$\Delta_N(\nu)\tau(\nu)$	$\frac{\Delta_N(\nu)}{\|u(\nu)\|_Y}$	$\frac{\Delta_N(\nu)}{\|e(\nu)\|_Y}$	$\Delta_N(\nu)\tau(\nu)$	$\frac{\Delta_N(\nu)}{\|u(\nu)\|_Y}$	$\frac{\Delta_N(\nu)}{\|e(\nu)\|_Y}$
1	7.95×10^{-1}	3.07×10^{-1}	3.19	–	–	–
5	4.57×10^{-1}	1.76×10^{-1}	3.21	–	–	–
9	5.65×10^{-2}	2.18×10^{-2}	3.22	1.13×10^{0}	2.43×10^{-3}	7.10
13	1.59×10^{-5}	6.16×10^{-6}	3.21	6.69×10^{-3}	1.43×10^{-5}	4.96
17	2.76×10^{-6}	1.06×10^{-6}	3.21	1.92×10^{-4}	4.12×10^{-7}	4.63

– and hence we can always provide a definitive error bound; that the relative error bound $\Delta_N(\nu)/\|u(\nu)\|_Y$ tends to zero rapidly; and that the effectivity is very good. Note also that the "uniqueness radius", $\frac{3}{2}\tau(0.305)^{-1} = 0.571$, is comfortably large relative to $\|u(0.305)\|_Y = 0.986$.

For $\nu = 0.013$ we observe that, for $N < 9$, $0.013 \notin \widetilde{\mathcal{D}}_N(U_J)$, and for $N < 10$, $\Delta_N(\nu)\tau(\nu) > 1$ – and hence we can obtain rigorous error bounds only for very accurate RB approximations; that the relative error bound $\Delta_N(\nu)/\|u(\nu)\|_Y$ still tends to zero rapidly with N – the sample S_N is constructed to provide uniform convergence; and that the effectivity is much better than the theoretical upper bound. For this value of ν the "uniqueness radius", $\frac{3}{2}\tau(0.013)^{-1} = 0.026$, is small relative to $\|u(0.013)\|_Y = 8.113$.

The rapid convergence of the RB method is important not only in efficiently achieving the desired accuracy, but also in efficiently satisfying $\Delta_N(\nu)\tau(\nu) \leqslant 1$; accuracy is required both to predict *and* to certify.

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