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COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

The modified KdV equation on a finite interval

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Abstract

We analyse an initial-boundary value problem for the mKdV equation on a finite interval by expressing the solution in terms of the solution of an associated matrix Riemann–Hilbert problem in the complex k-plane. This Riemann–Hilbert problem has explicit (x,t)-dependence and it involves certain functions of k referred to as "spectral functions". Some of these functions are defined in terms of the initial condition $q(x,0)=q_0(x)$, while the remaining spectral functions are defined in terms of two sets of boundary values. We show that the spectral functions satisfy an algebraic "global relation" that characterize the boundary values in spectral terms. ***To cite this article: A. Boutet de Monvel, D. Shepelsky, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

L'équation mKdV sur un intervalle borné. Nous étudions un problème aux limites pour l'équation mKdV en exprimant la solution en termes de la solution d'un problème de Riemann–Hilbert matriciel associé dans le plan complexe du paramètre spectral k. Ce problème de Riemann–Hilbert dépend de façon explicite de x et t. Il est déterminé par des fonctions de k appelées « fonctions spectrales ». Certaines d'entre elles sont définies en termes des données de Cauchy $q(x,0)=q_0(x)$, tandis que les autres sont définies par deux ensembles de valeurs aux limites. Nous démontrons que ces fonctions spectrales vérifient une « relation globale » algébrique qui caractérise les valeurs aux limites en termes spectraux. ***Pour citer cet article : A. Boutet de Monvel, D. Shepelsky, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

Une méthode générale pour traiter des problèmes aux limites pour des équations aux dérivées partielles linéaires en dimension 2 et pour des edp non-linéaires intégrables, a été annoncée dans [3], puis développée dans [4,5]. Elle est fondée sur l'analyse spectrale simultanée des deux équations linéaires de la paire de Lax associée. Elle donne la solution en termes de la solution d'un problème de Riemann–Hilbert (RH) matriciel formulé dans le plan complexe du paramètre spectral. Les « fonctions spectrales » qui déterminent le problème de Riemann–Hilbert sont données en termes des valeurs initiales et des valeurs au bord de la solution. Ces valeurs aux limites sont en général liées.

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Le fait remarquable est que leur dépendance se traduit de façon simple au niveau des fonctions spectrales, par une « relation globale » algébrique que celles-ci doivent satisfaire.

L'application rigoureuse de cette méthode pour l'équation de Korteweg de Vries modifiée (mKdV) sur la demi-droite est réalisée dans [1]. Dans cette Note, nous l'appliquons à l'équation mKdV

$$q_t - q_{xxx} + 6\lambda q^2 q_x = 0, \quad \lambda = \pm 1,$$

sur un intervalle borné. Le problème analogue pour l'équation NLS est étudié dans [6].

L'étude du problème aux limites pour l'équation mKdV dans le domaine $\{0 < x < L,\ 0 < t < T\},\ L < \infty,\ T \leqslant \infty$, se fait en deux étapes.

- En supposant que l'équation mKdV a une solution $q(x,t)$, on cherche à l'exprimer en termes de la solution d'un certain problème RH matriciel associé. Pour cela :
  - (i) On définit des solutions appropriées de (2) holomorphes par morceaux et bornées dans $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
  - (ii) On définit des fonctions spectrales $s(k),\ S(k),\ S_1(k)$ qui aient les propriétés suivantes :
    - Elles déterminent un problème RH.
    - $s(k)$ est déterminée par les données de Cauchy $q(x,0) = q_0(x),\ 0 < x < L$.
    - $S(k)$ est déterminée par les valeurs aux limites $q(0,t) = g_0(t),\ q_x(0,t) = g_1(t),\ q_{xx}(0,t) = g_2(t),\ 0 < t < T$.
    - $S_1(k)$ est déterminée par les valeurs aux limites $q(L,t) = f_0(t),\ q_x(L,t) = f_1(t),\ q_{xx}(L,t) = f_2(t),\ 0 < t < T$.
    - Elles vérifient une « relation globale » algébrique exprimant le fait qu'on ne peut pas fixer arbitrairement $q_0(x),\ \{g_j(t)\}_{j=0}^2$ et $\{f_j(t)\}_{j=0}^2$ comme valeurs aux limites pour l'équation mKdV.
- Etant donné $s(k)$ et en supposant que $\{g_j(t)\}_{j=0}^2$ et $\{f_j(t)\}_{j=0}^2$ sont telles que les fonctions spectrales associées $S(k)$ et $S_1(k)$ vérifient, avec $s(k)$, la relation globale, démontrer que la solution du problème RH construit à partir de $s(k),\ S(k)$ et $S_1(k)$ donne effectivement la solution du problème aux limites pour l'équation mKdV avec données de Cauchy $q(x,0) = q_0(x)$ et valeurs aux limites $q(0,t) = g_0(t),\ q_x(0,t) = g_1(t),\ q_{xx}(0,t) = g_2(t),\ q(L,t) = f_0(t),\ q_x(L,t) = f_1(t),\ q_{xx}(L,t) = f_2(t)$ (Théorème 1).

Nous appelons « relation globale » la relation algébrique entre les fonctions spectrales, (8) ou (9), qui caractérise, en termes spectraux, la dépendance mutuelle des données de Cauchy et des valeurs aux limites d'une solution de l'équation mKdV.

1. Introduction

The general method for solving initial-boundary value problems for two-dimensional linear and integrable nonlinear PDEs announced in [3] and developed further in [4,5] is based on the simultaneous spectral analysis of the two eigenvalue equations of the associated Lax pair. It expresses the solution in terms of the solution of a matrix Riemann–Hilbert (RH) problem formulated in the complex plane of the spectral parameter. The spectral functions determining the RH problem are expressed in terms of the initial and boundary values of the solution. The fact that these values are in general related can be expressed in a simple way in terms of a global relation satisfied by the corresponding spectral functions.

The rigorous implementation of the method to the modified Korteweg–de Vries (mKdV) equation on the half-line is presented in [1]. Here, this methodology is applied to the mKdV equation on a finite interval. Details can be found in [2]. The similar problem for the nonlinear Schrödinger equation is studied in [6].

The modified Korteweg–de Vries equation

$$q_t - q_{xxx} + 6\lambda q^2 q_x = 0, \quad \lambda = \pm 1, \tag{1}$$

admits the Lax pair formulation

$$\mu_x - \mathrm{i}k\hat{\sigma}_3\mu = Q(x,t)\mu, \qquad \mu_t + 4\mathrm{i}k^3\hat{\sigma}_3\mu = \widetilde{Q}(x,t,k)\mu, \tag{2}$$

where $\sigma_3 = \mathrm{diag}\{1,-1\}$, $\hat{\sigma}_3 A := \sigma_3 A - A\sigma_3$, $\mathrm{e}^{\hat{\sigma}_3}A = \mathrm{e}^{\sigma_3}A\mathrm{e}^{-\sigma_3}$,

$$Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ \lambda q(x,t) & 0 \end{pmatrix}, \qquad \widetilde{Q}(x,t,k) = -4k^2 Q - 2\mathrm{i}k(Q^2 + Q_x)\sigma_3 - 2Q^3 + Q_{xx}.$$

We study the initial-boundary value problem for the mKdV equation in the domain $\{0 < x < L,\ 0 < t < T\}$, $L < \infty$, $T \leqslant \infty$ using the following steps.

- Assuming that the solution $q(x,t)$ of the mKdV equation exists, express it via the solution of a matrix Riemann–Hilbert problem. For this purpose:
  (i) Define proper solutions of (2) sectionally analytic and bounded in $k \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
  (ii) Define spectral functions $s(k)$, $S(k)$, and $S_1(k)$ such that:
  - They determine a Riemann–Hilbert problem.
  - $s(k)$ is determined by the initial conditions $q(x,0) = q_0(x)$, $0 < x < L$.
  - $S(k)$ is determined by the boundary values $q(0,t) = g_0(t)$, $q_x(0,t) = g_1(t)$, $q_{xx}(0,t) = g_2(t)$, $0 < t < T$.
  - $S_1(k)$ is determined by the boundary values $q(L,t) = f_0(t)$, $q_x(L,t) = f_1(t)$, $q_{xx}(L,t) = f_2(t)$, $0 < t < T$.
  - They satisfied an algebraic "global relation", expressing the fact that $q_0(x)$, $\{g_j(t)\}_{j=0}^2$, $\{f_j(t)\}_{j=0}^2$ being the initial and boundary conditions for the mKdV equation, cannot be chosen arbitrarily.
- Given $s(k)$ and assuming that $\{g_j(t)\}_{j=0}^2$ and $\{f_j(t)\}_{j=0}^2$ are such that the associated $S(k)$ and $S_1(k)$ together with $s(k)$ satisfy the global relation, prove that the solution of the Riemann–Hilbert problem constructed from $s(k)$, $S(k)$, and $S_1(k)$ generates the solution of the initial-boundary value problem for the mKdV equation with initial data $q(x,0) = q_0(x)$ and boundary values $q(0,t) = g_0(t)$, $q_x(0,t) = g_1(t)$, $q_{xx}(0,t) = g_2(t)$, $q(L,t) = f_0(t)$, $q_x(L,t) = f_1(t)$, $q_{xx}(L,t) = f_2(t)$.

2. Eigenfunctions and spectral functions

Assume that there exists a real-valued function $q(x,t)$ with sufficient smoothness and decay satisfying (1) in $\{0 < x < L,\ 0 < t < T\}$, $T \leqslant \infty$. Define the *eigenfunctions* $\mu_n(x,t,k)$, $n = 1,2,3,4$, as matrix-valued solutions of the integral equations

$$\mu_n(x,t,k) = I + \int_{(x_n,t_n)}^{(x,t)} \mathrm{e}^{\mathrm{i}(k(x-y)-4k^3(t-\tau))\hat{\sigma}_3}(Q\mu_n\,\mathrm{d}y + \widetilde{Q}\mu_n\,\mathrm{d}\tau), \tag{3}$$

where $(x_1,t_1) = (0,T)$, $(x_2,t_2) = (0,0)$, $(x_3,t_3) = (L,0)$, $(x_4,t_4) = (L,T)$, and the paths of integration are chosen to be parallel to the x and t axes:

$$\mu_1(x,t,k) = I + \int_0^x \mathrm{e}^{\mathrm{i}k(x-y)\hat{\sigma}_3}(Q\mu_1)(y,t,k)\,\mathrm{d}y - \mathrm{e}^{\mathrm{i}kx\hat{\sigma}_3}\int_t^T \mathrm{e}^{-4\mathrm{i}k^3(t-\tau)\hat{\sigma}_3}(\widetilde{Q}\mu_1)(0,\tau,k)\,\mathrm{d}\tau,$$

$$\mu_4(x,t,k) = I - \int_x^L \mathrm{e}^{\mathrm{i}k(x-y)\hat{\sigma}_3}(Q\mu_4)(y,t,k)\,\mathrm{d}y - \mathrm{e}^{\mathrm{i}k(x-L)\hat{\sigma}_3}\int_t^T \mathrm{e}^{-4\mathrm{i}k^3(t-\tau)\hat{\sigma}_3}(\widetilde{Q}\mu_4)(L,\tau,k)\,\mathrm{d}\tau;$$

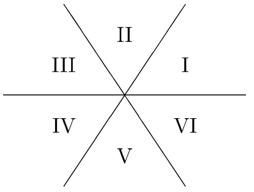

Fig. 1. Domains of boundedness of eigenfunctions.

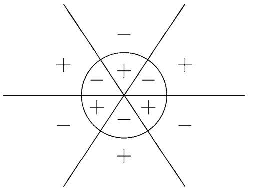

Fig. 2. Contour Σ and domains $\Omega_\pm$.

equations for μ_2 and μ_3 are similar to those for μ_1 and μ_4, respectively, with the integral term $\int_0^t$ instead of $-\int_t^T$. The columns of $\mu_n = (\mu_n^{(1)} \quad \mu_n^{(2)})$ are analytic and bounded in domains separated by the three lines $\{k \in \mathbb{C} \mid \operatorname{Im} k^3 = 0\}$, see Fig. 1:

$$\mu_1^{(1)},\ \mu_3^{(2)} \text{ in IV} \cup \text{VI}; \quad \mu_1^{(2)},\ \mu_3^{(1)} \text{ in I} \cup \text{III}; \quad \mu_2^{(1)},\ \mu_4^{(2)} \text{ in V}; \quad \mu_2^{(2)},\ \mu_4^{(1)} \text{ in II}.$$

Thus, in each domain I, ..., VI, one has a bounded 2×2 matrix-valued eigenfunction, consisting of the appropriate vectors $\mu_n^{(l)}$. The eigenfunctions μ_j are related by

$$\mu_3(x,t,k) = \mu_2(x,t,k)\, \mathrm{e}^{\mathrm{i}(kx-4k^3t)\hat\sigma_3} s(k), \tag{4}$$

$$\mu_1(x,t,k) = \mu_2(x,t,k)\, \mathrm{e}^{\mathrm{i}(kx-4k^3t)\hat\sigma_3} S(k), \tag{5}$$

$$\mu_4(x,t,k) = \mu_3(x,t,k)\, \mathrm{e}^{\mathrm{i}(kx-4k^3t)\hat\sigma_3}\, \mathrm{e}^{-\mathrm{i}kL\hat\sigma_3} S_1(k), \tag{6}$$

where the *spectral* (*matrix-valued*) *functions* are defined as follows:

$$s(k) \equiv \begin{pmatrix} \overline{a(\bar k)} & b(k) \\ \lambda \overline{b(\bar k)} & a(k) \end{pmatrix} := \mu_3(0,0,k),$$

$$S(k) \equiv \begin{pmatrix} \overline{A(\bar k)} & B(k) \\ \lambda \overline{B(\bar k)} & A(k) \end{pmatrix} := \mu_1(0,0,k), \qquad S_1(k) \equiv \begin{pmatrix} \overline{A_1(\bar k)} & B_1(k) \\ \lambda \overline{B_1(\bar k)} & A_1(k) \end{pmatrix} := \mu_4(L,0,k).$$

The direct and inverse spectral maps $\{q_0(x)\} \leftrightarrow \{a(k), b(k)\}$, $\{g_0(t), g_1(t), g_2(t)\} \leftrightarrow \{A(k), B(k)\}$, and $\{f_0(t), f_1(t), f_2(t)\} \leftrightarrow \{A_1(k), B_1(k)\}$ are well-defined [1]. They correspond to the separate spectral maps for the x-problem ($t = 0$) and t-problems ($x = 0$ and $x = L$) from the Lax pair (2).

3. Global relation

Evaluating Eqs. (4) and (6) at $x = 0$, $t = T$ and writing $\mu_3(0,0,k)$, $\mu_2(0,T,k)$, and $\mu_4(L,0,k)$ in terms of $s(k)$, $S(k)$, and $S_1(k)$, respectively, we obtain

$$S^{-1}(k)s(k)\big[\mathrm{e}^{-\mathrm{i}kL\hat\sigma_3} S_1(k)\big] = I - \mathrm{e}^{4\mathrm{i}k^3T\hat\sigma_3} \int_0^L \mathrm{e}^{-\mathrm{i}ky\hat\sigma_3}(Q\mu_4)(y,T,k)\,\mathrm{d}y. \tag{7}$$

- For $T < \infty$, the $(1,2)$ coefficient of (7) is ($k \in \mathbb{C}$)

$$\mathrm{e}^{-2\mathrm{i}kL}\big(\overline{a(\bar k)}A(k) - \lambda\overline{b(\bar k)}B(k)\big)B_1(k) - \big(a(k)B(k) - b(k)A(k)\big)A_1(k) = \mathrm{e}^{8\mathrm{i}k^3T}c(k), \tag{8}$$

  where $c(k) = \int_0^L \mathrm{e}^{-2\mathrm{i}ky}(Q\mu_4)_{12}(y,T,k)\,\mathrm{d}y$ is an entire function which is $\mathrm{O}((1+\mathrm{e}^{-2\mathrm{i}kL})/k)$ as $k \to \infty$.
- For $T = \infty$, the $(1,2)$ coefficient of (7) becomes

$$\mathrm{e}^{-2\mathrm{i}kL}\big(\overline{a(\bar k)}A(k) - \lambda\overline{b(\bar k)}B(k)\big)B_1(k) - \big(a(k)B(k) - b(k)A(k)\big)A_1(k) = 0, \tag{9}$$

  which is valid for $k \in \mathrm{I} \cup \mathrm{III} \cup \mathrm{V}$.

Eq. (8) for $T < \infty$, or (9) for $T = \infty$, is an algebraic relation between the spectral functions. We call it "global relation", because it express, in spectral terms, the relations between the initial and boundary values of a solution of the mKdV equation. The global relation can be used to characterize the unknown boundary values in a well-posed boundary value problem, say, $g_2(t)$, $f_1(t)$, and $f_2(t)$ in terms of the boundary conditions $\{q_0(x), g_0(t), g_1(t), f_0(t)\}$.

4. The Riemann–Hilbert problem

Define a sectionally holomorphic, matrix-valued function $M(x,t,k)$:

$$M = \begin{cases} \begin{pmatrix} \mu_3^{(1)} & \dfrac{\mu_1^{(2)}}{\overline{d(\bar{k})}} \end{pmatrix}, & k \in \mathrm{I} \cup \mathrm{III},\ |k| > R, \\ \begin{pmatrix} \dfrac{\mu_4^{(1)}\overline{a(\bar{k})}}{\overline{d_1(\bar{k})}} & \dfrac{\mu_2^{(2)}}{\overline{a(\bar{k})}} \end{pmatrix}, & k \in \mathrm{II},\ |k| > R, \\ \begin{pmatrix} \dfrac{\mu_1^{(1)}}{d(k)} & \mu_3^{(2)} \end{pmatrix}, & k \in \mathrm{IV} \cup \mathrm{VI},\ |k| > R, \\ \begin{pmatrix} \dfrac{\mu_2^{(1)}}{a(k)} & \dfrac{\mu_4^{(2)} a(k)}{d_1(k)} \end{pmatrix}, & k \in \mathrm{V},\ |k| > R, \\ \mu_2, & |k| < R, \end{cases} \tag{10}$$

where $d(k) = a(k)\overline{A(\bar{k})} - \lambda b(k)\overline{B(\bar{k})}$, $d_1(k) = a(k)A_1(k) + \lambda\, \mathrm{e}^{-2\mathrm{i}kL}\overline{b(\bar{k})}B_1(k)$, and R is large enough so that all possible zeros of $a(k)$, $d(k)$, and $d_1(k)$ in $\operatorname{Im} k \leqslant 0$ are in the disk $|k| < R$.

Denote by Σ the contour $\{k \mid \operatorname{Im} k^3 = 0\} \cup \{k \mid |k| = R\}$ (Fig. 2). Then the limit values $M_\pm(x,t,k)$ (as k approaches Σ from $\Omega_\pm$) of $M(x,t,k)$ are related on Σ by a jump matrix:

$$M_-(x,t,k) = M_+(x,t,k)\, \mathrm{e}^{(\mathrm{i}kx - 4\mathrm{i}k^3 t)\sigma_3} J_0(k)\, \mathrm{e}^{-(\mathrm{i}kx - 4\mathrm{i}k^3 t)\sigma_3}, \quad k \in \Sigma, \tag{11}$$

where

$$J_0(k) = \begin{cases} \begin{pmatrix} 1 & -\lambda\overline{\Gamma(\bar{k})} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \dfrac{1}{\lambda\overline{\Gamma_1(\bar{k})}} & 1 \end{pmatrix}, & \arg k = \dfrac{\pi}{3}, \dfrac{2\pi}{3},\ |k| > R, \\ \begin{pmatrix} 1 & -\lambda\overline{\Gamma(\bar{k})} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - \lambda|\gamma(k)|^2 & \gamma(k) \\ -\lambda\bar{\gamma}(k) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \Gamma(k) & 1 \end{pmatrix}, & \arg k = 0, \pi,\ |k| > R, \\ \begin{pmatrix} \dfrac{\overline{A(k)}}{\overline{d(\bar{k})}} & -\dfrac{\overline{B(k)}}{\overline{d(\bar{k})}} \\ -\lambda\overline{b(\bar{k})} & \overline{a(\bar{k})} \end{pmatrix}, & \arg k \in \left(0, \dfrac{\pi}{3}\right) \cup \left(\dfrac{2\pi}{3}, \pi\right),\ |k| = R, \\ \begin{pmatrix} \overline{a(\bar{k})} & 0 \\ \lambda\overline{\Gamma_2(\bar{k})} & \dfrac{1}{\overline{a(\bar{k})}} \end{pmatrix}, & \arg k \in \left(\dfrac{\pi}{3}, \dfrac{2\pi}{3}\right),\ |k| = R \end{cases} \tag{12}$$

for $k \in \Sigma$, $\operatorname{Im} k \geqslant 0$, $J_0(k) = \operatorname{diag}\{-1, \lambda\} J_0^*(\bar{k}) \operatorname{diag}\{-1, \lambda\}$ for $k \in \Sigma$, $\operatorname{Im} k < 0$, $J_0(k) = I$ for $k \in \Sigma$, $|k| < R$.

Here

$$\gamma(k) = \frac{b(k)}{\bar{a}(k)}, \qquad \Gamma(k) = \lambda \frac{\overline{B(\bar{k})}/\overline{A(\bar{k})}}{a(k)(a(k) - \lambda b(k)(\overline{B(\bar{k})}/\overline{A(\bar{k})}))},$$

$$\Gamma_1(k) = \frac{\mathrm{e}^{-2\mathrm{i}kL} a(k)(B_1(k)/A_1(k))}{a(k) + \lambda\, \mathrm{e}^{-2\mathrm{i}kL}\overline{b(\bar{k})}(B_1(k)/A_1(k))}, \qquad \Gamma_2(k) = a(k) \frac{\mathrm{e}^{-2\mathrm{i}kL}\overline{a(\bar{k})}(B_1(k)/A_1(k)) + b(k)}{a(k) + \lambda\, \mathrm{e}^{-2\mathrm{i}kL}\overline{b(\bar{k})}(B_1(k)/A_1(k))}. \tag{13}$$

Therefore, the jump data in (11) are determined by $a(k)$ and $b(k)$ for $k \in \mathbb{C}$, $|k| \geqslant R$ and by $B(k)/A(k)$ and $B_1(k)/A_1(k)$ for $k \in \mathrm{I} \cup \mathrm{III} \cup \mathrm{V}$, $|k| \geqslant R$.

Theorem 1. *Let $q_0(x) \in C^\infty([0, L])$. Let $\{g_j(t)\}_{0\leqslant j\leqslant 2}$ and $\{f_j(t)\}_{0\leqslant j\leqslant 2}$ be smooth functions such that:*

- $(\partial_x)^j q_0(0) = g_j(0)$, $(\partial_x)^j q_0(L) = f_j(0)$, $j = 0, 1, 2$;
- *the associated spectral functions $s(k)$, $S(k)$, and $S_1(k)$ satisfy the global relation* (8) *for $T < \infty$, or* (9) *for $T = \infty$, where $c(k)$ is an entire function such that $c(k) = \mathrm{O}((1 + \mathrm{e}^{-2\mathrm{i}kL})/k)$ as $|k| \to \infty$.*

Let $M(x, t, k)$ be a solution of the following 2×2 matrix RH problem:

- *M is sectionally holomorphic in $k \in \mathbb{C} \setminus \Sigma$.*
- *At $k \in \Sigma$, M satisfies the jump conditions* (11)*, where J_0 is defined in terms of the spectral functions a, b, A, B, A_1, and B_1 by Eqs.* (12), (13).
- *$M(x, t, k) = I + \mathrm{O}(1/k)$ as $k \to \infty$.*

Then:

(i) *$M(x, t, k)$ exists and is unique;*
(ii) *$q(x, t) := -2\mathrm{i} \lim_{k\to\infty} (kM(x, t, k))_{12}$ satisfies the mKdV equation* (1);
(iii) *$q(x, t)$ satisfies the initial condition $q(x, 0) = q_0(x)$ and boundary conditions $q(0, t) = g_0(t)$, $q_x(0, t) = g_1(t)$, $q_{xx}(0, t) = g_2(t)$, and $q(L, t) = f_0(t)$, $q_x(L, t) = f_1(t)$, $q_{xx}(L, t) = f_2(t)$.*

Sketch of proof. The unique solvability of the RH problem is a consequence of a "vanishing lemma" for the associated RH problem with vanishing condition at infinity $M = \mathrm{O}(1/k)$, $k \to \infty$.

The proof that the function $q(x, t)$ thus constructed solves the mKdV equation is straightforward and follows the proof in the case of the whole line problem.

The proof that q satisfies the initial condition $q(x, 0) = q_0(x)$ follows from the fact that it is possible to map the RH problem for $M(x, 0, k)$ to that for a sectionally holomorphic function $M^{(x)}(x, k)$ corresponding to the spectral problem for the x-part of the Lax pair (2): $M^{(x)}(x, k) = M(x, 0, k)P^{(x)}(x, k)$ where $P^{(x)}$ is sectionally holomorphic and $P^{(x)} = I + P^{(x)}_{\mathrm{off}}$, with $P^{(x)}_{\mathrm{off}}(x, k)$ off-diagonal and exponentially decaying as $k \to \infty$ for $\operatorname{Im} k \neq 0$.

The proof that q satisfies the boundary conditions is, in turn, based on the consideration of the maps $M(0, t, k) \mapsto M^{(t)}(t, k)$ and $M(L, t, k) \mapsto M^{(t)}_1(t, k)$, where $M^{(t)}(t, k)$ and $M^{(t)}_1(t, k)$ correspond to the spectral problems for the t-equation in the Lax pair (2) at $x = 0$ and $x = L$: $M^{(t)}(t, k) = M(0, t, k)P^{(t)}(t, k)$, $M^{(t)}_1(t, k) = M(L, t, k)P^{(t)}_1(t, k)$. In this case, it is the global relation (8), or (9), that guarantees that $P^{(t)} = P^{(t)}_{\mathrm{diag}} + P^{(t)}_{\mathrm{off}}$, where $P^{(t)}_{\mathrm{diag}}$ is diagonal, $P^{(t)}_{\mathrm{diag}} = I + \mathrm{O}(1/k)$, and $P^{(t)}_{\mathrm{off}}(t, k)$ is off-diagonal and exponentially decaying as $k \to \infty$, and similarly for $P^{(t)}_1(t, k)$. □

References

[1] A. Boutet de Monvel, A.S. Fokas, D. Shepelsky, The mKdV equation on the half-line, J. Inst. Math. Jussieu, in press.
[2] A. Boutet de Monvel, D. Shepelsky, Initial boundary value problem for the modified KdV equation on a finite interval, Preprint.
[3] A.S. Fokas, A unified transform method for solving linear and certain nonlinear PDEs, Proc. Roy. Soc. London Ser. A 453 (1997) 1411–1443.
[4] A.S. Fokas, On the integrability of linear and nonlinear partial differential equations, J. Math. Phys. 41 (2000) 4188–4237.
[5] A.S. Fokas, Integrable nonlinear evolution equations on the half-line, Comm. Math. Phys. 230 (2002) 1–39.
[6] A.S. Fokas, A.R. Its, The nonlinear Schrödinger equation on the interval, Preprint.