

JACQUET MODULES OF LOCALLY ANALYTIC REPRESENTATIONS OF p -ADIC REDUCTIVE GROUPS I. CONSTRUCTION AND FIRST PROPERTIES

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ABSTRACT. – Let \mathbb{G} be a reductive group defined over a p -adic local field L , let \mathbb{P} be a parabolic subgroup of \mathbb{G} with Levi quotient \mathbb{M} , and write $G := \mathbb{G}(L)$, $P := \mathbb{P}(L)$, and $M := \mathbb{M}(L)$. In this paper we construct a functor J_P from the category of essentially admissible locally analytic G -representations to the category of essentially admissible locally analytic M -representations, which we call the Jacquet module functor attached to P , and which coincides with the usual Jacquet module functor of [Casselman W., Introduction to the theory of admissible representations of p -adic reductive groups, unpublished notes distributed by P. Sally, draft dated May 7, 1993. Available electronically at <http://www.math.ubc.ca/people/faculty/cass/research.html>. [5]] on the subcategory of admissible smooth G -representations. We establish several important properties of this functor.

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RÉSUMÉ. – Soit \mathbb{G} un groupe réductif défini sur un corps local p -adique L , soit \mathbb{P} un sous-groupe parabolique de \mathbb{G} avec quotient de Levi \mathbb{M} . On pose $G = \mathbb{G}(L)$, $P = \mathbb{P}(L)$, et $M = \mathbb{M}(L)$. On construit un foncteur J_P de la catégorie des G -représentations localement analytiques essentiellement admissibles dans la catégorie des M -représentations localement analytiques essentiellement admissibles, que l'on appelle le foncteur de module de Jacquet attaché à P , et qui coïncide avec le foncteur de module de Jacquet usuel de [Casselman W., Introduction to the theory of admissible representations of p -adic reductive groups, unpublished notes distributed by P. Sally, draft dated May 7, 1993. Available electronically at <http://www.math.ubc.ca/people/faculty/cass/research.html>. [5]] sur la sous-catégorie des G -représentations admissibles lisses. On établit plusieurs propriétés importantes de ce foncteur.

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The goal of this paper is to introduce in the theory of locally analytic representations of p -adic reductive groups an analogue of the Jacquet module functors that are defined in [5] for smooth representations of p -adic reductive groups.

We refer the reader to the papers [16–18,20] of Schneider and Teitelbaum for the foundations of locally analytic representation theory. Some additional contributions to the theory have been supplied by [8,12,14]. (The short article [10] provides a summary of some of these results, as well as a brief discussion of the Jacquet module functors that are the subject of this paper and its sequel [11].) We will rely in particular on the concept of essentially admissible locally analytic representation introduced in [8], which generalizes the concept of admissible locally analytic representation introduced in [20].

We now introduce some notation that will be in force throughout this introduction. Let L be a finite extension of \mathbb{Q}_p for some prime p , let \mathbb{G} be a connected reductive linear algebraic group over L , let \mathbb{P} denote a parabolic subgroup of \mathbb{G} , let \mathbb{N} denote the unipotent radical of \mathbb{P} , and write $\mathbb{M} := \mathbb{P}/\mathbb{N}$ (so that \mathbb{M} is the Levi quotient of \mathbb{P}). Choose an opposite parabolic $\overline{\mathbb{P}}$ to \mathbb{P} ; the intersection $\mathbb{P} \cap \overline{\mathbb{P}}$ then provides a lifting of \mathbb{M} to a Levi factor of \mathbb{P} (and indeed making a choice of $\overline{\mathbb{P}}$ is equivalent to making a choice of such a lifting of \mathbb{M}). Write $G := \mathbb{G}(L)$, $M := \mathbb{M}(L)$, $N := \mathbb{N}(L)$, $P := \mathbb{P}(L)$, and $\overline{P} := \overline{\mathbb{P}}(L)$. Fix an extension K of L , complete with respect to a discrete valuation extending the discrete valuation on L . All representations will be on K -vector spaces (whether or not this is explicitly stated).

We briefly recall the theory of Jacquet module functors for smooth representations [5], before turning to a discussion of the theory for locally analytic representations that is the subject of this paper.

Jacquet module functors for smooth representations. The Jacquet module of a smooth G -representation V with respect to P is defined to be the space V_N of N -coinvariants of V . The formation of Jacquet modules yields a functor from smooth G -representations to smooth M -representations, satisfying the following three properties: (i) it takes admissible smooth G -representations to admissible smooth M -representations [5, thm. 3.3.1]; (ii) it is exact [5, prop. 3.2.3]; (iii) if V is non-zero and irreducible, then its Jacquet module is non-zero if and only if V appears as a subrepresentation of a representation parabolically induced from a representation of M [5, thm. 3.2.4].

Property (iii) follows from the obvious Frobenius reciprocity formula

$$(0.1) \quad \mathrm{Hom}_G(V, (\mathrm{Ind}_P^G W)_{\mathrm{sm}}) \xrightarrow{\sim} \mathrm{Hom}_M(V_N, W),$$

where V is an arbitrary smooth G -representation, W is an arbitrary smooth M -representation W , and $(\mathrm{Ind}_P^G W)_{\mathrm{sm}}$ denotes the smooth induction of W from P to G . The proofs of properties (i) and (ii) are more subtle. They follow from a key observation of [5], which is that, although the Jacquet module of an admissible smooth representation V is at first defined as a space of coinvariants, and hence as a quotient of V , it admits an alternative description as a subspace of V . (This is the theory of the “canonical lifting” developed in [5, §4].) This alternative description is given in terms of the action of certain Hecke operators. It allows one to prove properties (i) and (ii), and also forms the basis of the Casselman duality theorem [5, cor. 4.2.5]. One way to phrase this latter result is as follows: for any admissible smooth M -representation U and admissible smooth G -representation V there is a canonical isomorphism

$$(0.2) \quad \mathrm{Hom}_G((\mathrm{Ind}_P^G U)_{\mathrm{sm}}, V) \xrightarrow{\sim} \mathrm{Hom}_M(U(\delta), V_N),$$

where $U(\delta)$ denotes the twist of U by the modulus character of P (which is trivial on N , and so may be thought of as a character of M). Note that in the source of this isomorphism, the parabolic induction appears “on the wrong side” (from the naive point of view of Frobenius reciprocity).

Jacquet module functors for locally analytic representations. For any locally L -analytic group H , denote by $\text{Rep}_{\text{la.c.}}(H)$ the category of locally analytic H -representations on compact type locally convex topological K -vector spaces. (See [17] for the precise definition of this category.) For any object V of $\text{Rep}_{\text{la.c.}}(G)$, one may certainly consider the space of Hausdorff N -coinvariants of V (i.e. the Hausdorff completion of the space of N -coinvariants of V —the quotient topology on this latter space might be non-Hausdorff). This construction gives rise to a functor from $\text{Rep}_{\text{la.c.}}(G)$ to $\text{Rep}_{\text{la.c.}}(M)$ which satisfies an obvious analogue of (0.1), with smooth parabolic induction being replaced by locally analytic parabolic induction, and the Hom-spaces being replaced by spaces of continuous homomorphisms. (See [12, §4] for a discussion of Frobenius reciprocity in the locally analytic context.) Unfortunately, it is not known (to this author at least) whether this functor preserves any of the natural admissibility conditions that one may impose on objects of $\text{Rep}_{\text{la.c.}}(G)$ and $\text{Rep}_{\text{la.c.}}(M)$ (such as strong admissibility, admissibility, or essential admissibility, as defined in [17,20,8] respectively).

In this paper we define a functor J_P on $\text{Rep}_{\text{la.c.}}(G)$ which agrees with the usual Jacquet module functor on admissible smooth representations, and which takes essentially admissible G -representations to essentially admissible M -representations. The construction of this functor is modelled on the approach to smooth Jacquet functors via Hecke operators that was discussed above. Just as in the smooth case, for any G -representation V , the construction of $J_P(V)$ will depend only on the structure of V as a P -representation, and so we initially define the functor J_P in the more general context of locally analytic P -representations.

In Section 3.1 we define a certain full subcategory $\text{Rep}_{\text{la.c.}}^z(M)$ of $\text{Rep}_{\text{la.c.}}(M)$. For U an object of $\text{Rep}_{\text{la.c.}}^z(M)$, let $\mathcal{C}_c^{\text{sm}}(N, U)$ denote the space of compactly supported locally constant U -valued functions on N . The formation of this space yields a functor $\mathcal{C}_c^{\text{sm}}(N, -) : \text{Rep}_{\text{la.c.}}^z(M) \rightarrow \text{Rep}_{\text{la.c.}}(P)$. (See Section 3.5.)

The following result follows from Theorem 3.5.6 below.

THEOREM 0.3. – *The functor $\mathcal{C}_c^{\text{sm}}(N, -)$ admits a right-adjoint.*

We define the Jacquet module functor $J_P : \text{Rep}_{\text{la.c.}}(P) \rightarrow \text{Rep}_{\text{la.c.}}^z(M)$ to be the right adjoint of $\mathcal{C}_c^{\text{sm}}(N, -)$, twisted by the modulus character δ of P . Thus for U in $\text{Rep}_{\text{la.c.}}^z(M)$ and V in $\text{Rep}_{\text{la.c.}}(P)$, there is a natural isomorphism

$$(0.4) \quad \mathcal{L}_P(\mathcal{C}_c^{\text{sm}}(N, U), V) \xrightarrow{\sim} \mathcal{L}_M(U(\delta), J_P(V)).$$

If we compose J_P with the forgetful functor $\text{Rep}_{\text{la.c.}}(G) \rightarrow \text{Rep}_{\text{la.c.}}(P)$ then we obtain a functor $\text{Rep}_{\text{la.c.}}(G) \rightarrow \text{Rep}_{\text{la.c.}}^z(M)$ (which we again denote by J_P). Note that for any object U of $\text{Rep}_{\text{la.c.}}^z(M)$, the space $\mathcal{C}_c^{\text{sm}}(N, U)$ embeds as a closed P -invariant subspace of the parabolic induction $\text{Ind}_P^G U$. (It can be identified with the subspace of locally constant functions in $\text{Ind}_P^G U$ whose support lies in the open cell of $\overline{P} \backslash G$.) Thus the adjointness formula (0.4) is a weak analogue of formula (0.2).

We have already mentioned the following fundamental result (proved as Theorem 4.2.32 below).

THEOREM 0.5. – *The functor J_P takes essentially admissible locally analytic G -representations to essentially admissible locally analytic M -representations.*

It is natural to ask whether the other basic properties of Jacquet modules of smooth representations extend to the locally analytic setting. It follows directly from its construction that the functor J_P is left exact and additive. On the other hand, simple examples show that it is not right exact in general. (This shows, incidentally, that J_P does not coincide with the functor of Hausdorff N -coinvariants in general.) As for the relation with parabolic induction, one might

hope to strengthen (0.4) so as to obtain an analogue of (0.2). In fact, this is possible; however, the details are a little involved, and so we will postpone them to the sequel [11]. Here is one of the results that we will establish there.

THEOREM 0.6. – *Suppose that G is quasi-split, and that P is a Borel subgroup of G (so that the Levi factor M is a maximal torus of G). If V is a topologically irreducible admissible locally analytic G -representation that admits a continuous injection into an admissible continuous representation of G on a K -Banach space, and if $J_P(V)$ is non-zero, then V is isomorphic to a subquotient of the locally analytic parabolic induction from P to G of a locally analytic character of M .*

Global motivations. Our original motivation for defining the functor J_P was not local, but global. In the paper [9], it is applied to the problem of p -adic interpolation of automorphic forms, and yields a generalization of the theory of the eigencurve developed in [7]. (If the reader recalls the important role played by the Hecke operator at p in the construction of [7], then they will get a hint of the relation between the problem of p -adic interpolation and the construction of this paper, in which Hecke operators also play a key role.)

Some properties of the functor J_P . We establish a number of additional properties of the functor J_P in this paper. Before describing them, we introduce some more notation. Let $\text{Rep}_{\text{es}}(G)$ (respectively $\text{Rep}_{\text{es}}(M)$) denote the category of essentially admissible locally analytic G -representations (respectively M -representations). Let \mathfrak{g} denote the Lie algebra of G , let Z_G denote the centre of G , and let $\mathfrak{z}(\mathfrak{g})$ denote the centre of the universal enveloping algebra of \mathfrak{g} . Similarly, let \mathfrak{m} denote the Lie algebra of M , let Z_M denote the centre of M , and let $\mathfrak{z}(\mathfrak{m})$ denote the centre of the universal enveloping algebra of \mathfrak{m} . Let \widehat{Z}_M denote the rigid analytic space that parameterizes the locally L -analytic characters of Z_M , and for any $\chi \in \widehat{Z}_M(K)$, and object V of $\text{Rep}_{\text{es}}(G)$, let $J_P^\chi(V)$ denote the χ -eigenspace for the Z_M -action on $J_P(V)$ (which is a closed admissible subrepresentation of $J_P(V)$ [8, cor. 6.4.14]).

We now summarize the main properties of J_P that are proved in this paper.

0.7. – The definition of the functor $\mathcal{C}_{\text{sm}}^c(N, -) : \text{Rep}_{\text{la.c}}^z(M) \rightarrow \text{Rep}_{\text{la.c}}(P)$ depends on the choice of $\overline{\mathbb{P}}$ (or equivalently, on the lifting of \mathbb{M} to a Levi factor of \mathbb{P}). However, we show that the functor J_P is independent, up to natural isomorphism, of the choice of $\overline{\mathbb{P}}$.

0.8. – Note that there are natural actions of $Z_G \times \mathfrak{z}(\mathfrak{g})$ on $\text{Rep}_{\text{la.c}}(G)$, and of $Z_M \times \mathfrak{z}(\mathfrak{m})$ on $\text{Rep}_{\text{la.c}}^z(M)$. Also, since M is the Levi quotient of P , there are natural injections $Z_G \rightarrow Z_M$ and $\mathfrak{z}(\mathfrak{g}) \rightarrow \mathfrak{z}(\mathfrak{m})$ (the latter is the “unnormalized Harish–Chandra homomorphism”, recalled in Section 1.3 below; it depends on the choice of P). These maps induce the upper horizontal arrow in the diagram

$$\begin{array}{ccc}
 Z_G \times \mathfrak{z}(\mathfrak{g}) & \xrightarrow{\hspace{10em}} & Z_M \times \mathfrak{z}(\mathfrak{m}) \\
 \downarrow & & \downarrow \\
 & & \text{Aut}(\text{Rep}_{\text{la.c}}^z(M)) \times \text{End}(\text{Rep}_{\text{la.c}}^z(M)) \\
 & & \downarrow \\
 \text{Aut}(\text{Rep}_{\text{la.c}}(G)) \times \text{End}(\text{Rep}_{\text{la.c}}(G)) & \xrightarrow{\hspace{10em}} & \text{Aut}(J_P) \times \text{End}(J_P)
 \end{array}$$

in which the left-hand vertical arrow and upper right-hand vertical arrow arise from the natural actions referred to above, and in which the lower horizontal arrow and the lower right-hand vertical arrow are induced by the functorial nature of J_P . We prove that this diagram commutes.

0.9. – If V is an object of $\text{Rep}_{\text{la.c}}(P)$, so that $J_P(V)$ is an object of $\text{Rep}_{\text{la.c}}^z(M)$, then regard $J_P(V)$ as a P -module, by having P act through its quotient M . For any choice of a compact open subgroup P_0 of P and of a lifting of M to a Levi factor of P , we construct an M_0N_0 -equivariant map

$$(0.10) \quad J_P(V) \rightarrow V,$$

functorial in V . (Here $M_0 = M \cap P_0$, $N_0 = N \cap P_0$.) Extending the terminology of [5, p. 40], we refer to this map as the “canonical lifting” determined by the given choice of P_0 and of a lifting of M .

0.11. – Suppose that \mathbb{G} is quasi-split over L , and take \mathbb{P} to be a Borel subgroup of \mathbb{G} . The Levi quotient \mathbb{M} is then a torus, and so $M = Z_M$. Passing to duals induces an anti-equivalence of categories between the category $\text{Rep}_{\text{es}}(M)$, and the category of coherent rigid analytic sheaves on \widehat{M} . Thus, for any object V in $\text{Rep}_{\text{es}}(G)$, its Jacquet module $J_P(V)$ gives rise to a coherent rigid analytic sheaf on \widehat{M} . The support of this sheaf is a Zariski closed subset of \widehat{M} , which we denote by $\text{Exp}(J_P(V))$. Passing to the derivative of a character induces a natural map $\widehat{M} \rightarrow \widehat{\mathfrak{m}}$, and hence (by restriction) a map

$$(0.12) \quad \text{Exp}(J_P(V)) \rightarrow \widehat{\mathfrak{m}}.$$

We prove that if V is an admissible locally analytic representation of G , then the map (0.12) has discrete fibres.

0.13. – Let W be a finite dimensional algebraic representation of G , write $B = \text{End}_G(W)$, let X be an admissible smooth representation of G over B , and set $V = X \otimes_B W$ (so that V is an admissible locally W -algebraic representation of G). We prove that $J_P(V)$ is naturally isomorphic as an M -representation to $X_N \otimes_B W^N$. In particular, $J_P(V)$ is an admissible locally W^N -algebraic representation of M . (Taking W to be the trivial representation, we find that J_P restricted to admissible smooth representations coincides with the Jacquet module functor of [5].)

0.14. – Assuming that $L = \mathbb{Q}_p$ and that G is split over K , we give a criterion for the functor J_P to commute with the passage to locally algebraic vectors. More precisely, if V is an essentially admissible locally analytic representation of G , and if W is a finite dimensional irreducible algebraic representation of G , then the closed embedding $V_{W\text{-lalg}} \rightarrow V$ induces a closed embedding

$$J_P(V_{W\text{-lalg}}) \rightarrow J_P(V)_{W^N\text{-lalg}}.$$

(Here the subscripts indicate subspaces of locally algebraic vectors, transforming locally according to W and W^N respectively.) If $\chi \in \widehat{Z}_M(K)$, then we may restrict to χ -eigenspaces, and so obtain a closed embedding

$$(0.15) \quad J_P^\chi(V_{W\text{-lalg}}) \rightarrow J_P^\chi(V)_{W^N\text{-lalg}}.$$

Let ψ denote the character through which Z_M acts on W^N . (There is such a ψ , since \mathbb{G} , and so Z_M , is assumed to be split over K .) If χ and ψ do not coincide locally, and so in particular if χ is not locally algebraic, then both the source and target of (0.15) vanish.

Suppose that χ and ψ do coincide locally. If V admits a G -invariant norm, and if χ is of “non-critical slope” with respect to the irreducible representation W^N (in the sense of Definition 4.4.3 below), then we prove that (0.15) is an isomorphism.

Admissibility vs. essential admissibility. The Jacquet module of an admissible locally analytic G -representation V , while certainly essentially admissible, need not be an admissible locally analytic M -representation (as the examples of [9] show). Nevertheless, for admissible locally analytic representation V that are topologically of finite length, one might hope that $J_P(V)$ is again admissible, of finite length. Since J_P is a left exact functor, it would suffice to prove this when V is topologically irreducible. For those irreducible representations that admit a continuous injection into an admissible continuous representation of G , this follows from Theorem 0.6.

The arrangement of the paper. The first two sections are preliminary in nature. Section 1 recalls various standard Lie-theoretic results that we will require. Section 2 is devoted to the development of a theory of compact operators for topological modules over certain topological K -algebras, generalizing the theory of [6]. The main results are Propositions 2.2.6 and 2.3.5. The key technical problem with which we must deal is that of analyzing compact operators on Banach modules over a Banach algebra that do not admit an orthonormal basis. The same problem has also been studied by Ash and Stevens [1]; their approach is quite different to ours.

Section 3 presents the construction of the Jacquet module functor. The order of development of the ideas is somewhat different to that given in this introduction: we define the Jacquet module directly before establishing its characterization as an adjoint functor. In the preliminary Section 3.1 we recall the definitions and some properties of various categories of locally analytic representations. The definition of the Jacquet module functor is the subject of Sections 3.2, 3.3, and 3.4. Properties 0.7, 0.8, and 0.9 follow easily from the construction, and are all established in Section 3.4. Finally, the adjointness formula (0.4) (and so also Theorem 0.3) is established in Section 3.5.

Section 4 establishes the deeper properties of the Jacquet module functor. After making some preliminary constructions in Section 4.1, in Section 4.2 we give the proof of Theorem 0.5, which relies heavily on the results of Section 2. As a byproduct of the argument, we obtain a proof of property 0.11 above. Section 4.3 establishes property 0.13, while property 0.14 is proved in Section 4.4 using Verma module techniques.

Notation and conventions. Throughout the paper, we fix a finite extension L of \mathbb{Q}_p (for some fixed prime p), an algebraic closure \bar{L} of L , as well as an extension K of L , complete with respect to a discrete valuation extending that on L . We let ord_L denote the discrete valuation on L (normalized so as to take the value 1 on a uniformizer of L), $\text{ord}_{\bar{L}}$ denote its extension to \bar{L} , and ord_K its extension to K . We let \mathcal{O}_L and \mathcal{O}_K denote the ring of integers in each of L and K respectively.

We follow closely the notational and terminological conventions introduced in [8], and will adhere to the conventions laid down in Section 0 of that paper. In particular, if G is a topological group (or more generally, a topological semigroup), then we will distinguish between a topological action of G on a topological K -vector space V (that is, an action of G by continuous endomorphisms of V) and a continuous action of G on V (that is, an action for which the action map $G \times V \rightarrow V$ is jointly continuous). Also, we will often write “convex K -vector space” as an abbreviation for “locally convex topological K -vector space”.

1. Lie-theoretic preliminaries

1.1. Let \mathbb{G} be a split connected reductive linear algebraic group defined over the field K . We will briefly recall the highest weight theory of representations of \mathbb{G} , and also of its Lie algebra \mathfrak{g} .

Since \mathbb{G} is split, we may find a split maximal torus \mathbb{T} in \mathbb{G} , and a Borel subgroup \mathbb{B} containing \mathbb{T} . We let \mathbb{N} denote the unipotent radical of \mathbb{B} . We let \mathfrak{g} , \mathfrak{t} , \mathfrak{b} , and \mathfrak{n} denote the Lie algebras of \mathbb{G} , \mathbb{T} , \mathbb{B} , and \mathbb{N} respectively. We let $X^\bullet(\mathbb{T})^+$ denote the cone of dominant weights

in $X^\bullet(\mathbb{T})$. (It consists of all weights that pair non-negatively with the coroots that are positive with respect to the chosen Borel \mathbb{B} .)

THEOREM 1.1.1. – *Let W be a finite dimensional irreducible algebraic representation of \mathbb{G} over K .*

- (i) *The subspace $W^{\mathbb{N}}$ of W is one-dimensional, and \mathbb{T} acts on this space through an element $\chi \in X^\bullet(\mathbb{T})^+$.*
- (ii) *Associating χ to W induces a bijection between the isomorphism classes of irreducible algebraic \mathbb{G} -representations over K and the elements of $X^\bullet(\mathbb{T})^+$.*

Proof. – This summarizes the standard highest weight theory for the irreducible representations of the split group \mathbb{G} . \square

In the situation of Theorem 1.1.1, the dominant weight χ is called the highest weight of W , and any basis element of $W^{\mathbb{N}}$ is called a highest weight vector of W .

If \mathfrak{t} denotes the Lie algebra of \mathbb{T} , then $\check{\mathfrak{t}}$ (the K -dual to \mathfrak{t}) is naturally isomorphic to $K \otimes_{\mathbb{Z}} X^\bullet(\mathbb{T})$. We say that an element of $\check{\mathfrak{t}}$ is infinitesimally integral if it assumes integral values on all positive coroots. We let $\check{\mathfrak{t}}^+$ denote the cone of infinitesimally integral elements of $\check{\mathfrak{t}}$ that pair non-negatively with all positive coroots.

THEOREM 1.1.2. – *Suppose that we are given an irreducible representation of \mathfrak{g} on a finite dimensional K -vector space W .*

- (i) *The subspace $W^{\mathfrak{n}}$ of W is one-dimensional, and \mathfrak{t} acts on this space through an element χ in $\check{\mathfrak{t}}^+$.*
- (ii) *Associating χ to W induces a bijection between the isomorphism classes of irreducible finite dimensional representations of \mathfrak{g} over K and the elements of $\check{\mathfrak{t}}^+$.*

Proof. – This summarizes the standard highest weight theory for the reductive Lie algebra \mathfrak{g} . \square

Just as for \mathbb{G} -representations, the dominant weight χ is called the highest weight of W , and any basis element of $W^{\mathfrak{n}}$ is called a highest weight vector of W .

The following results provide the link between the irreducible representations of \mathbb{G} and those of \mathfrak{g} .

THEOREM 1.1.3. – *If W is a finite dimensional algebraic representation of \mathbb{G} defined over K then the induced representation of \mathfrak{g} is also irreducible.*

Proof. – This follows from the fact that \mathbb{G} is connected. \square

COROLLARY 1.1.4. – *If W is a finite dimensional K -vector space equipped with an irreducible representation of \mathfrak{g} , then we may lift the \mathfrak{g} -action on W to a finite dimensional algebraic representation of \mathbb{G} if and only if the highest weight of W lies in $X^\bullet(\mathbb{T})$. In this case, there is a uniquely determined finite dimensional algebraic representation of \mathbb{G} on W lifting the given \mathfrak{g} -action.*

Proof. – This follows from Theorem 1.1.3, together with the fact that irreducible representations of both \mathbb{G} and \mathfrak{g} are determined by their highest weights. (We remark that since \mathbb{N} is connected, the inclusion $W^{\mathbb{N}} \subset W^{\mathfrak{n}}$ is an isomorphism for any representation W of \mathbb{G} , and so the highest weight spaces of W , thought of alternatively as a \mathbb{G} -representation or as a \mathfrak{g} -representation, coincide.) \square

Related to the highest weight theory of irreducible representations of \mathfrak{g} is the theory of Verma modules of \mathfrak{g} , which we now recall.

DEFINITION 1.1.5. – Let $\chi \in \check{\mathfrak{t}}$ be a character of \mathfrak{t} . Regarding χ as a character of \mathfrak{b} that is trivial on \mathfrak{n} , let $K(\chi)$ denote the field K regarded as a \mathfrak{b} -module via the character χ . We define the Verma module of highest weight χ to be the left $U(\mathfrak{g})$ -module $\text{Ver}(\chi) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} K(\chi)$.

We refer to the element $1 \otimes 1$ of $\text{Ver}(\chi)$ as the highest weight vector of $\text{Ver}(\chi)$, and denote it by $v(\chi)$. The lie algebra \mathfrak{b} acts on $v(\chi)$ through the character χ .

In addition to the action of \mathfrak{g} , the Verma module $\text{Ver}(\chi)$ is equipped with a certain ‘‘adjoint’’ action of the algebraic group \mathbb{B} , which we now explain. The adjoint action of the group \mathbb{G} on \mathfrak{g} induces a natural adjoint action of \mathbb{G} on $U(\mathfrak{g})$. Now $U(\mathfrak{b})$ is invariant under the restriction of this adjoint action to \mathbb{B} , as is the character χ . Thus $\text{Ver}(\chi)$ is naturally equipped with an adjoint action of \mathbb{B} .

We can differentiate the adjoint \mathbb{B} -action on $\text{Ver}(\chi)$ to obtain an adjoint \mathfrak{b} -action. Of course \mathfrak{b} also acts on $\text{Ver}(\chi)$ via restricting the left \mathfrak{g} -module structure on $\text{Ver}(\chi)$. The adjoint action and the left-module action of \mathfrak{b} are related in the following way: if $b \in \mathfrak{b}$ and $Y \in \text{Ver}(\chi)$ then $\text{ad}_b(Y) + \chi(b)Y = bY$.

We will require the following structure theorem for Verma modules whose highest weight lies in $\check{\mathfrak{t}}^+$. (In the statement of theorem, $\bar{\mathfrak{n}}$ denotes the opposite nilpotent to \mathfrak{n} determined by \mathfrak{t} .)

PROPOSITION 1.1.6. – *Let χ lie in $\check{\mathfrak{t}}^+$.*

- (i) *If α is a positive simple root of \mathbb{T} over K , if $X_{-\alpha} \in \bar{\mathfrak{n}}$ is a non-zero element of the (one-dimensional) root space in $\bar{\mathfrak{n}}$ corresponding to the negative simple root $-\alpha$, and if $m_\alpha = \langle \chi, \check{\alpha} \rangle$ (where $\check{\alpha}$ is the coroot corresponding to α), then $X_{-\alpha}^{m_\alpha+1}v(\chi)$ is annihilated by \mathfrak{n} , is fixed by the adjoint action of \mathbb{N} , and generates a $U(\mathfrak{g})$ -submodule of $\text{Ver}(\chi)$ isomorphic to $\text{Ver}(\chi - (m_\alpha + 1)\alpha)$.*
- (ii) *If for each positive simple root α as in (i) we let $\phi_\alpha: \text{Ver}(\chi - (m_\alpha + 1)\alpha) \rightarrow \text{Ver}(\chi)$ denote the homomorphism of $U(\mathfrak{g})$ -modules obtained by mapping $v(\chi - (m_\alpha + 1)\alpha)$ to $X_{-\alpha}^{m_\alpha+1}v(\chi)$, then the cokernel of the map*

$$\bigoplus_{\alpha} \text{Ver}(\chi - (m_\alpha + 1)\alpha) \xrightarrow{\bigoplus_{\alpha} \phi_\alpha} \text{Ver}(\chi)$$

is a finite dimensional irreducible representation of \mathfrak{g} of highest weight χ .

Proof. – This is well known. One reference is [13, thm. 4.37], together with its proof, in particular Lemma 4.40. For ease of comparison of our statements with those of Knapp, we note that the Knapp uses a different normalization to label the Verma modules: our $\text{Ver}(\chi)$ is his $V(\chi + \rho)$ (where ρ denotes one-half the sum of the positive roots). We also remark that if s_α denotes the simple reflection corresponding to the positive simple root α , then $\chi + \rho - (m_\alpha + 1)\alpha = \chi + \rho - \langle \chi + \rho, \check{\alpha} \rangle \alpha = s_\alpha(\chi + \rho)$. \square

1.2. We let \mathbb{G} be a split connected reductive linear algebraic group over K , let \mathbb{P} be a parabolic subgroup of \mathbb{G} , let \mathbb{N} denote the unipotent radical of \mathbb{P} , and let \mathbb{M} be a Levi factor of \mathbb{P} . Choose a Borel \mathbb{B} of \mathbb{G} that is contained in \mathbb{P} , and a maximal torus \mathbb{T} of \mathbb{B} that is contained in \mathbb{M} (so that \mathbb{T} is also a maximal torus of \mathbb{M}). We let \mathbb{N}' denote the unipotent radical of \mathbb{B} , let $\mathbb{Z}_{\mathbb{G}}$ denote the centre of \mathbb{G} , and let $\mathbb{Z}_{\mathbb{M}}$ denote the centre of \mathbb{M} . We also let $\mathfrak{g}, \mathfrak{p}, \mathfrak{n}, \mathfrak{m}, \mathfrak{b}, \mathfrak{n}'$, and \mathfrak{t} denote the Lie algebras of $\mathbb{G}, \mathbb{P}, \mathbb{N}, \mathbb{M}, \mathbb{B}, \mathbb{N}'$, and \mathbb{T} respectively. The intersection $\mathbb{M} \cap \mathbb{B}$ is a Borel subgroup of \mathbb{M} , with unipotent radical $\mathbb{M} \cap \mathbb{N}'$. Also $\mathbb{N}' = (\mathbb{M} \cap \mathbb{N}')\mathbb{N}$, and so

$$(1.2.1) \quad \mathfrak{n}' = (\mathfrak{m} \cap \mathfrak{n}') \oplus \mathfrak{n}.$$

Let $\Delta(\mathbb{G}, \mathbb{T}) \subset X^\bullet(\mathbb{T})$ denote the set of positive roots of \mathbb{T} with respect to \mathbb{G} (that is, the set of characters appearing in the adjoint action of \mathbb{T} on \mathfrak{n}'). We let $\Delta(\mathbb{G}, \mathbb{T})_s$ denote the subset of $\Delta(\mathbb{G}, \mathbb{T})$ consisting of simple positive roots. The set $\Delta(\mathbb{G}, \mathbb{T})_s$ is a basis for a finite index sublattice of $X^\bullet(\mathbb{T}/\mathbb{Z}_{\mathbb{G}})$.

Similarly, we let $\Delta(\mathbb{M}, \mathbb{T})$ denote the set of positive roots of \mathbb{T} with respect to \mathbb{M} (that is, the set of characters appearing in the adjoint action of \mathbb{T} on $\mathfrak{m} \cap \mathfrak{n}'$). If we let $\Delta(\mathbb{M}, \mathbb{T})_s$ denote the subset of $\Delta(\mathbb{M}, \mathbb{T})$ consisting of simple positive roots, then $\Delta(\mathbb{M}, \mathbb{T})_s$ is the basis of a finite index sublattice of $X^\bullet(\mathbb{T}/\mathbb{Z}_{\mathbb{M}})$.

LEMMA 1.2.2. – *There is an equality $\Delta(\mathbb{M}, \mathbb{T}) \cap \Delta(\mathbb{G}, \mathbb{T})_s = \Delta(\mathbb{M}, \mathbb{T})_s$.*

Proof. – Clearly the left-hand side is contained in the right-hand side. On the other hand, since \mathfrak{n} is an ideal of \mathfrak{p} , and so in particular of \mathfrak{n}' , we see that any root whose decomposition in terms of simple roots involves an element of $\Delta(\mathbb{G}, \mathbb{T})_s \setminus \Delta(\mathbb{M}, \mathbb{T})$ appears in \mathfrak{n} , and so not in $\mathfrak{m} \cap \mathfrak{n}'$. Consequently any element of $\Delta(\mathbb{M}, \mathbb{T})_s$ is simple even when regarded as an element of $\Delta(\mathbb{G}, \mathbb{T})$. This proves the lemma. \square

We let $\Delta(\mathbb{G}, \mathbb{Z}_{\mathbb{M}}) \subset X^\bullet(\mathbb{Z}_{\mathbb{M}})$ denote the set of positive restricted roots of $\mathbb{Z}_{\mathbb{M}}$ (that is, the set of characters appearing in the adjoint action of $\mathbb{Z}_{\mathbb{M}}$ on \mathfrak{n}). We let $\Delta(\mathbb{G}, \mathbb{Z}_{\mathbb{M}})_s$ denote the subset consisting of simple positive restricted roots. The decomposition (1.2.1) shows that the elements of $\Delta(\mathbb{G}, \mathbb{Z}_{\mathbb{M}})$ are the restrictions to $\mathbb{Z}_{\mathbb{M}}$ of the elements of $\Delta(\mathbb{G}, \mathbb{T}) \setminus \Delta(\mathbb{M}, \mathbb{T})$.

LEMMA 1.2.3. – *Restriction of characters from \mathbb{T} to $\mathbb{Z}_{\mathbb{M}}$ induces a bijection*

$$\Delta(\mathbb{G}, \mathbb{T})_s \setminus \Delta(\mathbb{M}, \mathbb{T})_s \xrightarrow{\sim} \Delta(\mathbb{G}, \mathbb{Z}_{\mathbb{M}})_s.$$

Proof. – Lemma 1.2.2 shows that $\Delta(\mathbb{G}, \mathbb{T})_s = \Delta(\mathbb{G}, \mathbb{T})_s \setminus \Delta(\mathbb{M}, \mathbb{T})_s \cup \Delta(\mathbb{M}, \mathbb{T})_s$. As already remarked, $\Delta(\mathbb{G}, \mathbb{T})_s$ is a basis for a finite index sublattice of $X^\bullet(\mathbb{T}/\mathbb{Z}_{\mathbb{G}})$, while $\Delta(\mathbb{M}, \mathbb{T})_s$ is a basis for a finite index sublattice of $X^\bullet(\mathbb{T}/\mathbb{Z}_{\mathbb{M}})$. It follows that the restriction of $\Delta(\mathbb{G}, \mathbb{T})_s \setminus \Delta(\mathbb{M}, \mathbb{T})_s$ to $\mathbb{Z}_{\mathbb{M}}$ is a basis for a finite index sublattice of $X^\bullet(\mathbb{Z}_{\mathbb{M}}/\mathbb{Z}_{\mathbb{G}})$. In particular, restriction to $\mathbb{Z}_{\mathbb{M}}$ yields an embedding

$$(1.2.4) \quad \Delta(\mathbb{G}, \mathbb{T})_s \setminus \Delta(\mathbb{M}, \mathbb{T})_s \hookrightarrow \Delta(\mathbb{G}, \mathbb{Z}_{\mathbb{M}})_s.$$

Any element of $\Delta(\mathbb{G}, \mathbb{Z}_{\mathbb{M}})_s$ is the restriction to $\mathbb{Z}_{\mathbb{M}}$ of some element $\alpha \in \Delta(\mathbb{G}, \mathbb{T}) \setminus \Delta(\mathbb{M}, \mathbb{T})$. We may write $\alpha = \sum_{\beta \in \Delta(\mathbb{G}, \mathbb{T})_s} n_\beta \beta$ with $n_\beta \geq 0$. Since any $\beta \in \Delta(\mathbb{M}, \mathbb{T})_s$ has trivial restriction to $\mathbb{Z}_{\mathbb{M}}$, we may omit these terms from the sum, and not alter the restriction to $\mathbb{Z}_{\mathbb{M}}$. Thus we assume that $\alpha = \sum_{\beta \in \Delta(\mathbb{G}, \mathbb{T})_s \setminus \Delta(\mathbb{M}, \mathbb{T})_s} n_\beta \beta$. As already observed, the simple roots β appearing in this sum have non-zero restrictions to $\mathbb{Z}_{\mathbb{M}}$. Thus, if the restriction of α to $\mathbb{Z}_{\mathbb{M}}$ is simple, then we see that α itself must be simple. Hence any element of $\Delta(\mathbb{G}, \mathbb{Z}_{\mathbb{M}})_s$ lies in the image of (1.2.4).

Now consider an arbitrary $\alpha \in \Delta(\mathbb{G}, \mathbb{T})_s \setminus \Delta(\mathbb{M}, \mathbb{T})_s$. The restriction $\alpha|_{\mathbb{Z}_{\mathbb{M}}}$ is a linear combination of elements of $\Delta(\mathbb{G}, \mathbb{Z}_{\mathbb{M}})_s$ with non-negative coefficients. By the result of the preceding paragraph, we may write $\alpha|_{\mathbb{Z}_{\mathbb{M}}} = \sum_{\beta \in \Delta(\mathbb{G}, \mathbb{T})_s \setminus \Delta(\mathbb{M}, \mathbb{T})_s} n_\beta \beta|_{\mathbb{Z}_{\mathbb{M}}}$, for some $n_\beta \geq 0$. Since (1.2.4) induces an injection of the span of its domain into $X^\bullet(\mathbb{Z}_{\mathbb{M}})$, we deduce that $\alpha = \sum_{\beta \in \Delta(\mathbb{G}, \mathbb{T})_s \setminus \Delta(\mathbb{M}, \mathbb{T})_s} n_\beta \beta$. Since α is simple, we conclude that $n_\beta = 0$ if $\beta \neq \alpha$, while $n_\alpha = 1$. Thus $\alpha|_{\mathbb{Z}_{\mathbb{M}}}$ is also simple, and so the image of (1.2.4) is contained in $\Delta(\mathbb{G}, \mathbb{Z}_{\mathbb{M}})_s$. \square

For any root α of \mathbb{T} in \mathbb{G} (that is, any element of $\Delta(\mathbb{G}, \mathbb{T}) \cup -\Delta(\mathbb{G}, \mathbb{T})$), we let s_α denote the corresponding simple reflection in the Weyl group $W(\mathbb{G}, \mathbb{T})$ of \mathbb{T} with respect to \mathbb{G} . Note that the Weyl group $W(\mathbb{M}, \mathbb{T})$ of \mathbb{T} with respect to \mathbb{M} naturally embeds as a subgroup of the Weyl group $W(\mathbb{G}, \mathbb{T})$. Indeed, it is identified with the centralizer of the subtorus $\mathbb{Z}_{\mathbb{M}}$ of \mathbb{T} in $W(\mathbb{G}, \mathbb{T})$.

LEMMA 1.2.5. – *If $w \in W(\mathbb{M}, \mathbb{T})$ and α is a root of \mathbb{T} in \mathbb{G} , then for any $\chi \in X^\bullet(\mathbb{T})$, the characters $s_\alpha(\chi)$ and $s_{w(\alpha)}(w(\chi))$ have the same restriction to $\mathbb{Z}_{\mathbb{M}}$.*

Proof. – Indeed, $w(s_\alpha(\chi)) = s_{w(\alpha)}(w(\chi))$, and w acts trivially on $\mathbb{Z}_{\mathbb{M}}$. \square

Let W be an irreducible algebraic \mathbb{G} -representation. Highest weight theory shows that $W^{\mathbb{N}}$ is an irreducible algebraic \mathbb{M} -representation, and hence $\mathbb{Z}_{\mathbb{M}}$ acts on $W^{\mathbb{N}}$ through a character $\psi \in X^\bullet(\mathbb{Z}_{\mathbb{M}})$. This extends to the highest weight character $\tilde{\psi} \in X^\bullet(\mathbb{T})$ of W . If α is any element of $\Delta(\mathbb{G}, \mathbb{Z}_{\mathbb{M}})_s$, then Lemma 1.2.3 shows that α lifts uniquely to $\tilde{\alpha} \in \Delta(\mathbb{G}, \mathbb{T})_s$. Note that ψ and α are both well-determined, independent of the choice of \mathbb{B} contained in \mathbb{P} , while $\tilde{\psi}$ and $\tilde{\alpha}$ in general depend on this choice.

Let $\rho(\mathbb{G}, \mathbb{T})$ denote one-half of the sum of the elements of $\Delta(\mathbb{G}, \mathbb{T})$.

LEMMA 1.2.6. – *The restriction of $s_{\tilde{\alpha}}(\tilde{\psi} + \rho(\mathbb{G}, \mathbb{T})) + \rho(\mathbb{G}, \mathbb{T})$ to $\mathbb{Z}_{\mathbb{M}}$ is independent of the choice of the Borel subgroup contained in \mathbb{P} .*

Proof. – If \mathbb{B}' is any Borel subgroup contained in \mathbb{P} , then we may choose the maximal torus \mathbb{T} to be contained in $\mathbb{B} \cap \mathbb{B}'$, and then may write $\mathbb{B}' = w\mathbb{B}w^{-1}$ for some element $w \in W(\mathbb{M}, \mathbb{T})$. The corresponding lift of α (respectively ψ) is thus equal to $w(\tilde{\alpha})$ (respectively $w(\tilde{\psi})$). Also $\rho(\mathbb{G}, \mathbb{T})$ is replaced by $w(\rho(\mathbb{G}, \mathbb{T}))$. The lemma is now seen to be a consequence of Lemma 1.2.5. \square

1.3. Let \mathbb{G} denote a connected reductive linear algebraic group over K , let \mathbb{P} denote a parabolic subgroup of \mathbb{G} , let \mathbb{N} denote the unipotent radical of \mathbb{P} , and write $\mathbb{M} = \mathbb{P}/\mathbb{N}$ (the Levi quotient of \mathbb{P}). Let \mathfrak{g} , \mathfrak{m} , and \mathfrak{p} denote the Lie algebras of \mathbb{G} , \mathbb{M} and \mathbb{P} respectively, and in addition let $\mathfrak{z}(\mathfrak{g})$ (respectively $\mathfrak{z}(\mathfrak{m})$) denote the centre of the universal enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} (respectively the centre of the universal enveloping algebra $U(\mathfrak{m})$ of \mathfrak{m}). In this subsection we recall the construction of the unnormalized Harish–Chandra homomorphism $\mathfrak{z}(\mathfrak{g}) \rightarrow \mathfrak{z}(\mathfrak{m})$.

We use the inclusion $\mathfrak{p} \subset \mathfrak{g}$ to regard the universal enveloping algebra $U(\mathfrak{p})$ as a subalgebra of $U(\mathfrak{g})$.

PROPOSITION 1.3.1. – *The sum $U(\mathfrak{p}) + U(\mathfrak{g})\mathfrak{n}$ is a subalgebra of $U(\mathfrak{g})$, which contains $U(\mathfrak{g})\mathfrak{n}$ as an ideal, and which contains the centre $\mathfrak{z}(\mathfrak{g})$ of $U(\mathfrak{g})$ as a subalgebra.*

Proof. – The first two claims follow from the fact that \mathfrak{n} is an ideal in \mathfrak{p} . To prove the final claim, choose an opposite parabolic $\overline{\mathbb{P}}$ to \mathbb{P} , with unipotent radical $\overline{\mathbb{N}}$; the intersection $\mathbb{P} \cap \overline{\mathbb{P}}$ provides a lift of \mathbb{M} to a Levi factor of \mathbb{P} . If $\overline{\mathfrak{n}}$ denotes the Lie algebra of $\overline{\mathbb{N}}$, then there is a direct sum decomposition $U(\mathfrak{g}) \cong \overline{\mathfrak{n}}U(\overline{\mathfrak{p}}) \oplus U(\mathfrak{m}) \oplus U(\mathfrak{g})\mathfrak{n}$. If \mathbb{A} is the maximal split torus in the centre of \mathbb{M} , then since the adjoint \mathbb{A} -action on $U(\mathfrak{g})$ preserves this decomposition, and since the \mathbb{A} -invariant subspace of $\overline{\mathfrak{n}}U(\overline{\mathfrak{p}})$ is trivial, we see that the elements of $\mathfrak{z}(\mathfrak{g})$ (which are necessarily invariant under the adjoint \mathbb{A} -action) must lie in the direct sum $U(\mathfrak{m}) \oplus U(\mathfrak{g})\mathfrak{n} = U(\mathfrak{p}) + U(\mathfrak{g})\mathfrak{n}$. \square

We define the “unnormalized Harish–Chandra homomorphism” $\gamma' : \mathfrak{z}(\mathfrak{g}) \rightarrow U(\mathfrak{m})$ to be the composite of the inclusion $\mathfrak{z}(\mathfrak{g}) \subset U(\mathfrak{p}) + U(\mathfrak{g})\mathfrak{n}$ of the preceding proposition with the projection $U(\mathfrak{p}) + U(\mathfrak{g})\mathfrak{n} = U(\mathfrak{m}) \oplus U(\mathfrak{g})\mathfrak{n} \rightarrow U(\mathfrak{m})$.

PROPOSITION 1.3.2. – *The map γ' is injective with image lying in $\mathfrak{z}(\mathfrak{m}) \subset U(\mathfrak{m})$.*

Proof. – Since $\mathfrak{z}(\mathfrak{g})$ lies in the centre of $U(\mathfrak{p}) + U(\mathfrak{g})\mathfrak{n}$, its image under γ' lies in $\mathfrak{z}(\mathfrak{m})$. By construction the kernel of γ' is contained in $U(\mathfrak{g})\mathfrak{n}$. If W is any finite dimensional representation of \mathbb{G} , then $W^{\mathbb{N}}$ is annihilated by $U(\mathfrak{g})\mathfrak{n}$. Since W is generated as a \mathbb{G} -representation by $W^{\mathbb{N}}$ (apply highest weight theory over an extension of K that splits \mathbb{G}), and since $\mathfrak{z}(\mathfrak{g})$ commutes with the \mathbb{G} -action on W , we see that the kernel of γ' annihilates W . An element of $U(\mathfrak{g})$

that annihilates every finite-dimensional representation of \mathbb{G} necessarily vanishes, and thus γ' is injective. \square

1.4. We recall some basic structure theory of tori over the local field L . If \mathbb{T} is a torus defined over L , let \mathbb{S} denote the maximal split subtorus of \mathbb{T} . Let X_\bullet (respectively X^\bullet) denote the cocharacter lattice (respectively the character lattice) of \mathbb{T} over \bar{L} , and similarly let Y_\bullet (respectively Y^\bullet) denote the cocharacter lattice (respectively the character lattice) of \mathbb{S} . Each of X^\bullet and X_\bullet is equipped with an action of $\text{Gal}(\bar{L}/L)$, compatible with respect to the canonical pairing between the two of them. The natural embedding $Y_\bullet \rightarrow X_\bullet$ identifies Y_\bullet with the sublattice of Galois invariants in X_\bullet . The natural surjection $X^\bullet \rightarrow Y^\bullet$ identifies Y^\bullet with the quotient by its torsion subgroup of the group of Galois coinvariants of X^\bullet .

Write $S := \mathbb{S}(L)$ and $T := \mathbb{T}(L)$, and let S^0 (respectively T^0) denote the maximal compact subgroup of S (respectively T). If $t \in T$, then the element of $\text{Hom}(X^\bullet, \mathbb{Q})$ defined by $\chi \mapsto \text{ord}_{\bar{L}}(\chi(t))$ is Galois invariant, and so defines a map $T \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} Y_\bullet$. Denote the image by Y'_\bullet . Restricted to S , this yields a surjection $S \rightarrow Y'_\bullet$, with kernel S^0 . Altogether, we obtain a diagram of short exact sequences

$$(1.4.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & S^0 & \longrightarrow & S & \longrightarrow & Y_\bullet \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T^0 & \longrightarrow & T & \longrightarrow & Y'_\bullet \longrightarrow 0 \end{array}$$

Both Y_\bullet and Y'_\bullet are free \mathbb{Z} -modules of finite rank, and so each of the short exact sequences appearing in this diagram is split. The cokernel of the inclusion $Y_\bullet \rightarrow Y'_\bullet$ is finite. (All this follows from the discussion of [4, §3.2] applied to \mathbb{T} .)

We let \hat{T} denote the rigid analytic variety of locally L -analytic characters of T (as discussed in [8, §6.4]). If χ is an element of $\hat{T}(K)$, then the function $t \mapsto \text{ord}_K(\chi(t))$ is trivial on T^0 , and so factors to yield a morphism $Y'_\bullet \rightarrow \mathbb{Q}$.

DEFINITION 1.4.2. – If $\chi \in \hat{T}(K)$, then we define $\text{slope}(\chi)$ to be the element of $\mathbb{Q} \otimes_{\mathbb{Z}} Y^\bullet$ corresponding to the morphism $Y'_\bullet \rightarrow \mathbb{Q}$ described in the preceding paragraph.

We remark that if χ in fact happens to be an element of X^\bullet (that is, an algebraic character of T that can be defined over K), then $\text{slope}(\chi)$ is equal to the image of χ in Y^\bullet . (This follows immediately from the construction of the diagram (1.4.1).)

2. Functional-analytic preliminaries

2.1. In this subsection we develop some operator theory. If X is a set, then $c_0(X, K)$ denotes the K -Banach space of functions $f: X \rightarrow K$ whose values become arbitrarily small outside sufficiently large finite subsets of X (topologized via the sup norm). We embed X into $c_0(X, K)$ by identifying a point $x \in X$ with the function 1_x that is equal to 1 at the point x and vanishes on $X \setminus \{x\}$. For any K -vector space V , we let $\mathcal{F}(X, V)$ denote the K -vector space of V -valued functions on X .

PROPOSITION 2.1.1. – *If V is a Hausdorff convex K -vector space, then the natural map $\mathcal{L}(c_0(X, K), V) \rightarrow \mathcal{F}(X, V)$, given by restricting maps to the subset X of $c_0(X, K)$, is injective, and its image contains the set of maps $X \rightarrow V$ for which the image of X is contained in a bounded complete subset of V .*

Proof. – The discussion of [15, pp. 59–60] shows that X topologically spans $c_0(X, K)$, implying the claimed injectivity. It is also proved there that if V is quasi-complete then the map under consideration has image equal to the set of maps $X \rightarrow V$ with bounded image. If V is arbitrary, and if $X \rightarrow V$ has image contained in a bounded complete subset of V , then this map factors as a composite $X \rightarrow U \rightarrow V$ in which the second arrow is continuous and K -linear, with source equal to a Banach space (and so in particular quasi-complete). Thus the first arrow extends to a continuous K -linear map $c_0(X, K) \rightarrow U$, which when composed with the second arrow yields the required extension of the given map $X \rightarrow V$. \square

PROPOSITION 2.1.2. – *Let V and W be convex K -vector spaces, with W Hausdorff and quasi-complete, and let X be a collection of continuous K -linear maps from V to W . The following are equivalent:*

- (i) *The collection of maps X is equicontinuous.*
- (ii) *The map $X \times V \rightarrow W$ that describes the action of the maps in X extends (in a necessarily unique fashion) to a jointly continuous map $c_0(X, K) \times V \rightarrow W$.*

Proof. – Suppose that (i) holds. By assumption we are given a map $X \rightarrow \mathcal{L}_s(V, W)$ with equicontinuous image. We infer from [15, lem. 6.8, cor. 6.11, prop. 7.13] that the closure of the image is equicontinuous, bounded and complete. By Proposition 2.1.1, this map uniquely determines a continuous K -linear map $c_0(X, K) \rightarrow \mathcal{L}_s(V, W)$, and thus a separately continuous K -bilinear map

$$(2.1.3) \quad c_0(X, K) \times V \rightarrow W.$$

Since the closed \mathcal{O}_K -lattice spanned by X is the unit ball of $c_0(X, K)$ (with respect to the sup norm on this space), the unit ball of $c_0(X, K)$ maps to an equicontinuous subset of $\mathcal{L}(V, W)$. It follows directly that (2.1.3) is jointly continuous.

Conversely, if we are given a jointly continuous map $c_0(X, K) \times V \rightarrow W$, then we see that the entire unit ball of $c_0(X, K)$ (and so in particular the set X) induces an equicontinuous set of mappings $V \rightarrow W$. \square

We next recall some properties of equicontinuous collections of maps.

LEMMA 2.1.4. –

- (i) *If X and Y are equicontinuous collections of K -linear maps from T to V and from U to W respectively, then the collection $X \otimes Y := \{x \otimes y \mid x \in X, y \in Y\}$ of maps $T \otimes_{K, \pi} U \rightarrow V \otimes_{K, \pi} W$ is again equicontinuous.*
- (ii) *If X and Y are two equicontinuous collections of K -linear maps $U \rightarrow V$ then so is the union $X \cup Y$.*
- (iii) *If X is an equicontinuous collection of K -linear maps $U \rightarrow V$, and if $\alpha: T \rightarrow U$ and $\beta: V \rightarrow W$ are continuous K -linear maps, then the set $\beta \circ X \circ \alpha := \{\beta \circ x \circ \alpha \mid x \in X\}$ is an equicontinuous collection of maps $T \rightarrow W$.*
- (iv) *If X is a collection of K -linear maps $U \rightarrow V$, and $\alpha: T \rightarrow U$ is a continuous K -linear strict surjection, then X is an equicontinuous collection of maps if and only if $X \circ \alpha := \{x \circ \alpha \mid x \in X\}$ is an equicontinuous collection of maps $T \rightarrow V$.*

Proof. – All four parts of the lemma are immediate. \square

DEFINITION 2.1.5. – Let ϕ be a continuous K -linear endomorphism of a convex K -vector space V . We say that ϕ is power-bounded if the set $\{\phi^n \mid n \geq 0\}$ of endomorphisms of V is equicontinuous.

For any element x in K^\times , denote by $K\langle xt \rangle$ the Tate algebra over K of power series in t that converge on the closed disk $|t| \leq |x|^{-1}$. (If $x = 1$, then we omit it from the notation, and write

simply $K\langle t \rangle$.) Similarly denote by $K\langle xt, xt^{-1} \rangle$ the Tate algebra over K of power series in t and t^{-1} that converge on the closed annulus $|x| \leq |t| \leq |x|^{-1}$. (If $|x| > 1$ then set $K\langle xt, xt^{-1} \rangle = 0$.) For any K -Banach algebra A , denote by $A\langle xt \rangle$ the completed tensor product $A \widehat{\otimes}_K K\langle xt \rangle$ (and similarly for $A\langle xt, x^{-1}t \rangle$).

PROPOSITION 2.1.6. – *A continuous K -linear endomorphism ϕ of a quasi-complete Hausdorff convex K -vector space V is power-bounded if and only if V admits the structure of a topological $K\langle t \rangle$ -module with respect to which t acts via ϕ , and this module structure is unique if it exists.*

Proof. – Since the map $1_n \mapsto t^n$ induces a topological isomorphism $c_0(\mathbb{N}, K) \xrightarrow{\sim} K\langle t \rangle$, Proposition 2.1.2 shows that ϕ is power-bounded if and only if there is a jointly continuous map

$$(2.1.7) \quad K\langle t \rangle \times V \rightarrow V$$

such that $(t^n, v) \mapsto \phi^n(v)$ for every $v \in V$, and that ϕ uniquely determines (2.1.7) (assuming that it exists). Since $K[t]$ is dense in $K\langle t \rangle$, we see immediately that (2.1.7) (if it exists) gives V the structure of a $K\langle t \rangle$ -module. \square

PROPOSITION 2.1.8. –

- (i) *If ϕ is a power-bounded endomorphism of a convex K -vector space V , and if U is a second convex K -vector space, then $\text{id}_U \otimes \phi$ is a power-bounded endomorphism of $U \otimes_{K, \pi} V$.*
- (ii) *Let $\alpha: U \rightarrow V$ and $\beta: V \rightarrow U$ be a pair of maps between quasi-complete Hausdorff convex K -vector spaces. Let $\phi := \beta \circ \alpha$ and $\psi := \alpha \circ \beta$. Then ϕ is a power-bounded endomorphism of U if and only if ψ is a power-bounded endomorphism of V . If these equivalent conditions hold, and if U and V are equipped with the $K\langle t \rangle$ -module structure of Proposition 2.1.6, then α and β are $K\langle t \rangle$ -linear maps.*
- (iii) *Let $U \rightarrow V$ be a strict surjection of Hausdorff quasi-complete convex K -vector spaces, and let ψ and ϕ be endomorphisms of U and V respectively that fit into a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{\psi} & U \\ \downarrow & & \downarrow \\ V & \xrightarrow{\phi} & V \end{array}$$

If ψ is power-bounded, then so is ϕ , and the surjection $U \rightarrow V$ is furthermore a homomorphism of $K\langle t \rangle$ -modules.

Proof. – Part (i) follows immediately from part (i) of Lemma 2.1.4.

As for part (ii), write $X := \{\phi^n \mid n \geq 0\}$ and $Y := \{\psi^n \mid n \geq 0\}$. From the definition of ϕ and ψ we see that $Y = \alpha \circ X \circ \beta \cup \{\text{id}_V\}$, and that $X = \beta \circ Y \circ \alpha \cup \{\text{id}_U\}$, and so Lemma 2.1.4(ii) and (iii) show that X is equicontinuous if and only if Y is so. Assuming that they are equicontinuous, and noting that $\alpha \circ \phi = \psi \circ \alpha$ and that $\beta \circ \psi = \phi \circ \beta$, we see that α and β are $K[t]$ -linear, and hence $K\langle t \rangle$ -linear, by continuity.

Part (iii) follows from Lemma 2.1.4(iii) and (iv) and the definition of the $K\langle t \rangle$ -module structure on U and V in terms of ψ and ϕ respectively. \square

PROPOSITION 2.1.9. – *Let R be a locally convex topological K -algebra, let U and V be locally convex topological R -modules, and let $\alpha: U \rightarrow V$ and $\beta: V \rightarrow U$ be continuous R -linear maps such that the composites $\phi = \beta \circ \alpha$ and $\psi = \alpha \circ \beta$ are each equal to an element $t \in R$. (In*

the situation of Proposition 2.1.8(ii), if ϕ and ψ are power-bounded, then these hypotheses are satisfied for any R admitting a continuous injection into $K\langle\langle t \rangle\rangle$ whose image contains $K[t]$.) If S is a topological R -algebra in which the element t becomes invertible, then the natural S -linear map $S \otimes_{R,\pi} U \rightarrow S \otimes_{R,\pi} V$ induced by α is a topological isomorphism.

Proof. – The endomorphisms $\text{id}_S \otimes \phi$ of $S \otimes_{R,\pi} U$ and $\text{id}_S \otimes \psi$ of $S \otimes_{R,\pi} V$ are both topological isomorphisms, since t is a unit in S . The proposition thus follows from the formulas $\beta \circ \alpha = \phi$ and $\alpha \circ \beta = \psi$. \square

PROPOSITION 2.1.10. – *If ϕ is a continuous endomorphism of a K -Banach space V , then $x\phi$ is power-bounded if $x \in K$ has sufficiently large valuation.*

Proof. – Fix a norm defining the topology of V ; this determines a corresponding operator norm on $\mathcal{L}_b(V, V)$. For any $x \in K$ for which $x\phi$ has operator norm at most one, all powers of $x\phi$ again have operator norm at most one, and so $x\phi$ is power bounded. \square

2.2. In this subsection we extend the notion of a compact operator on a convex K -vector space to the context of Banach modules over a Banach algebra A , and we prove a spectral theorem for such operators (Proposition 2.2.6 below) which has the merit of applying to A -Banach modules that are not necessarily orthonormalizable. (Note that since we are assuming that K is discretely valued, any K -Banach space is isomorphic to $c_0(X, K)$ for some set X [15, prop. 10.1]. Thus the orthonormalizable A -Banach modules are precisely those of the form $A \widehat{\otimes}_K V$, for some K -Banach space V .) We fix the K -Banach algebra A for the duration of this subsection.

LEMMA 2.2.1. – *Let M and N be A -Banach modules, and let V be a K -Banach space. If we are given A -linear continuous maps $A \widehat{\otimes}_K V \rightarrow M$ and $N \rightarrow M$ with the image of the first map being contained in the image of the second then we may find an A -linear map $A \widehat{\otimes}_K V \rightarrow N$ that gives a commutative diagram*

$$\begin{array}{ccc} A \widehat{\otimes}_K V & & \\ \downarrow & \searrow & \\ N & \longrightarrow & M \end{array}$$

Proof. – Let P denote the image of the map $N \rightarrow M$, equipped with the topology that it inherits as a subspace of M , and let P_1 denote the same abstract vector space, but equipped with the (Banach space) topology that it inherits as a quotient of N . By assumption the given map $V \rightarrow M$ factors through the inclusion $P \subset M$, and the closed graph theorem shows that the induced map $V \rightarrow P_1$ is again continuous [8, prop. 1.1.2]. By [15, prop. 10.5(a)] (and its proof) we may find a continuous K -linear splitting of the surjection $N \rightarrow P_1$. Composing such a splitting with the continuous map $V \rightarrow P_1$ induces a map of K -Banach spaces $V \rightarrow N$, which in turn induces a map of A -Banach modules $A \widehat{\otimes}_K V \rightarrow N$. This is the required map. \square

LEMMA 2.2.2. – *If V is a K -Banach space and M is an A -Banach module, then a continuous A -linear map $A \widehat{\otimes}_K V \rightarrow M$ has its image contained in a finitely generated A -submodule of M if and only if it may be factored as a composite of continuous A -linear maps $A \widehat{\otimes}_K V \rightarrow A^r \rightarrow M$ for some natural number r .*

Proof. – The if direction is immediate. Its converse follows from Lemma 2.2.1. \square

DEFINITION 2.2.3. –

- (i) If V and W are K -Banach spaces, then an A -linear morphism $\phi: A \widehat{\otimes}_K V \rightarrow A \widehat{\otimes}_K W$ is called A -compact if we may write it as the limit in $\mathcal{L}_b(A \widehat{\otimes}_K V, A \widehat{\otimes}_K W)$ of a sequence $\{\phi_n\}$ of A -linear maps $A \widehat{\otimes}_K V \rightarrow A \widehat{\otimes}_K W$, each of whose image is contained in a finitely generated A -submodule of $A \widehat{\otimes}_K W$.
- (ii) If M and N are A -Banach modules, then a continuous A -linear map $\phi: M \rightarrow N$ is called A -compact if there exist K -Banach spaces V and W and a commutative diagram of continuous A -linear maps

$$\begin{array}{ccc} A \widehat{\otimes}_K V & \longrightarrow & A \widehat{\otimes}_K W \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi} & N \end{array}$$

in which the left-hand vertical arrow is surjective, and the upper horizontal arrow is A -compact in the sense of (i).

The compatibility of parts (i) and (ii) of the preceding definition, in the case of orthonormalizable A -Banach modules, is a consequence of the following lemma.

LEMMA 2.2.4. – *Let $\phi: M \rightarrow N$ be A -compact in the sense of Definition 2.2.3(ii). If V and W are K -Banach spaces, and if we are given continuous A -linear morphisms $A \widehat{\otimes}_K V \rightarrow M$ and $A \widehat{\otimes}_K W \rightarrow N$, the latter being surjective, then we may lift ϕ to a morphism $A \widehat{\otimes}_K V \rightarrow A \widehat{\otimes}_K W$ that is A -compact in the sense of Definition 2.2.3(i).*

Proof. – Follow Definition 2.2.3(ii) and form a commutative diagram

$$\begin{array}{ccc} A \widehat{\otimes}_K V_1 & \longrightarrow & A \widehat{\otimes}_K W_1 \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi} & N \end{array}$$

in which V_1 and W_1 are K -Banach spaces, the left-hand vertical arrow is surjective, and the upper horizontal arrow is A -compact in the sense of Definition 2.2.3(i). Use Lemma 2.2.1 to lift the maps $A \widehat{\otimes}_K V \rightarrow M$ and $A \widehat{\otimes}_K W_1 \rightarrow N$ to A -linear maps $A \widehat{\otimes}_K V \rightarrow A \widehat{\otimes}_K V_1$ and $A \widehat{\otimes}_K W_1 \rightarrow A \widehat{\otimes}_K W$ respectively. Composing these lifts with the upper horizontal map of the commutative diagram yields a map $A \widehat{\otimes}_K V \rightarrow A \widehat{\otimes}_K W$ which lifts ϕ and satisfies the conditions of Definition 2.2.3(i). \square

In [6, §A.1] the notion of a completely continuous A -homomorphism between A -Banach modules is introduced, but although its definition is given in the context of arbitrary Banach modules, most of the results of this reference are proved just for orthonormalizable Banach modules. (In comparing our discussion with that of [6, §A], the reader should note that in that reference all Banach spaces are considered to being equipped with a particular choice of norm, while in this paper we follow [15] and hence do not regard our Banach spaces as being so equipped.) Lemma 2.2.4 shows that any map between orthonormalizable A -Banach modules that satisfies the condition of part (ii) of Definition 2.2.3 in fact satisfies the condition of part (i) of that definition, and so our definition of an A -compact morphism between orthonormalizable A -Banach modules agrees with the notion of a completely continuous A -linear map introduced in [6]. However, if M and N are not orthonormalizable, the two notions are distinct (as far as we know). The class of A -compact maps may be characterized as being the smallest class of continuous A -linear maps between A -Banach modules that contains the completely continuous

A -linear maps (as defined in [6]) between orthonormalizable A -Banach modules, and that satisfies conditions (i) and (iii) of the following proposition.

PROPOSITION 2.2.5. –

- (i) If $M_1 \rightarrow M$, $M \rightarrow N$ and $N \rightarrow N_1$ are continuous A -linear maps of A -Banach modules, the second being A -compact, then the composite $M_1 \rightarrow M \rightarrow N \rightarrow N_1$ is A -compact.
- (ii) If $A \rightarrow B$ is a continuous homomorphism of K -Banach algebras, and if $M \rightarrow N$ is an A -compact map of A -Banach modules, then the induced map $B \widehat{\otimes}_A M \rightarrow B \widehat{\otimes}_A N$ is B -compact.
- (iii) If we are given a commutative diagram of continuous A -linear maps between A -Banach modules

$$\begin{array}{ccc} M & \longrightarrow & N \\ \downarrow & & \downarrow \\ P & \longrightarrow & Q \end{array}$$

in which the upper horizontal arrow is A -compact and the left-hand vertical arrow is surjective, then the bottom horizontal arrow is also A -compact.

Proof. – For part (i), note that the maps $M_1 \rightarrow M$ and $N \rightarrow N_1$ induce the top horizontal arrows of the commutative diagram

$$\begin{array}{ccccccc} A \widehat{\otimes}_K M_1 & \longrightarrow & A \widehat{\otimes}_K M & & A \widehat{\otimes}_K N & \longrightarrow & A \widehat{\otimes}_K N_1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M_1 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & N_1 \end{array}$$

in which the vertical arrows are the natural maps induced by the A -module structures on each of M , M_1 , N , and N_1 respectively. Lemma 2.2.4 shows that we may lift the A -compact map $M \rightarrow N$ to a map $\phi: A \widehat{\otimes}_K M \rightarrow A \widehat{\otimes}_K N$ satisfying Definition 2.2.3(i). If we fill in the above diagram with ϕ , then the composite of the upper horizontal arrows yields a map $A \widehat{\otimes}_K M_1 \rightarrow A \widehat{\otimes}_K N_1$ that again satisfies Definition 2.2.3(i). Thus the composite of the lower horizontal arrows of this diagram satisfies condition (ii) of that definition, and so is A -compact, as claimed.

Part (ii) follows from the definition, once we note that for a K -Banach space V there is a natural isomorphism $B \widehat{\otimes}_A (A \widehat{\otimes}_K V) \xrightarrow{\sim} B \widehat{\otimes}_K V$, and take into account Lemma 2.2.2, while part (iii) follows directly from the definition. \square

We can now prove the main result of this subsection.

PROPOSITION 2.2.6. – Let ϕ be an A -compact endomorphism of an A -Banach module M . If $x \in K^\times$ has sufficiently large valuation then:

- (i) The endomorphism $x\phi$ of M is power-bounded, in the sense of Definition 2.1.5.
- (ii) If we regard M as a topological $K\langle\langle xt \rangle\rangle$ -module (with t acting via ϕ on M), then $K\langle\langle xt, xt^{-1} \rangle\rangle \widehat{\otimes}_{K\langle\langle xt \rangle\rangle} M$ is finitely generated as an $A\langle\langle xt, xt^{-1} \rangle\rangle$ -module.

Proof. – Choose a continuous A -linear surjection $A \widehat{\otimes}_K V \rightarrow M$ for some K -Banach space V (e.g. take $V = M$). Lemma 2.2.4 shows that we may lift ϕ to an A -linear endomorphism ψ of $A \widehat{\otimes}_K V$ that satisfies Definition 2.2.3(i).

Proposition 2.1.10 shows that $x\psi$ is power-bounded if $\text{ord}_K(x)$ is large enough. Proposition 2.1.8(iii) implies that $x\phi$ is then also power-bounded, and that the surjection $A \widehat{\otimes}_K V \rightarrow M$

is $K\langle xt \rangle$ -linear. Since it is also A -linear, it is an $A\langle xt \rangle$ -linear surjection. It thus induces an $A\langle xt, xt^{-1} \rangle$ -linear surjection

$$K\langle xt, xt^{-1} \rangle \widehat{\otimes}_{K\langle xt \rangle} A \widehat{\otimes}_K V \rightarrow K\langle xt, xt^{-1} \rangle \widehat{\otimes}_{K\langle xt \rangle} M.$$

Hence, if the source of this map is finitely generated as an $A\langle xt, xt^{-1} \rangle$ -module, then the same will be true of its target. In summary, we see that (i) holds, and that in proving (ii), we may replace M with $A \widehat{\otimes}_K V$ and ϕ with ψ .

We thus suppose that M is an orthonormalizable A -Banach module, equipped with an A -linear endomorphism ϕ that may be written as the limit in $\mathcal{L}_b(M, M)$ of a sequence of A -linear maps ϕ_n , each having image contained in a finitely generated A -submodule of M . We will prove that if $x \in K^\times$ is such that $x\phi$ is power-bounded, then $K\langle xt, xt^{-1} \rangle \widehat{\otimes}_{K\langle xt \rangle} M$ is a finitely generated $A\langle xt, xt^{-1} \rangle$ -module.

Choose n sufficiently large so that the element $x^{-1}(\phi - \phi_n)$ of $\mathcal{L}(M, M)$ is topologically nilpotent. Applying Lemma 2.2.2, we factor ϕ_n as $M \xrightarrow{\phi'_n} A^r \xrightarrow{\phi''_n} M$ for some $r \geq 0$. Define a continuous A -linear map $\xi: M \rightarrow A\langle xt, xt^{-1} \rangle^r$ via the formula $\xi(m) = \sum_{i=0}^\infty x^i t^{-i-1} \phi'_n(x^{-i}(\phi - \phi_n)^i(m))$. (This expression is well-defined, since $x^{-1}(\phi - \phi_n)$ is topologically nilpotent.) The map ξ induces a continuous $A\langle xt, xt^{-1} \rangle$ -linear map $\xi': K\langle xt, xt^{-1} \rangle \widehat{\otimes}_K M \rightarrow A\langle xt, xt^{-1} \rangle^r$, while ϕ''_n induces a continuous $A\langle xt, xt^{-1} \rangle$ -linear map $\xi'': A\langle xt, xt^{-1} \rangle^r \rightarrow K\langle xt, xt^{-1} \rangle \widehat{\otimes}_{K\langle xt \rangle} M$. One computes that $\xi'' \circ \xi'$ coincides with the quotient map $K\langle xt, xt^{-1} \rangle \widehat{\otimes}_K M \rightarrow K\langle xt, xt^{-1} \rangle \widehat{\otimes}_{K\langle xt \rangle} M$. Thus ξ'' is a surjection, and so $K\langle xt, xt^{-1} \rangle \widehat{\otimes}_{K\langle xt \rangle} M$ is finitely generated over $A\langle xt, xt^{-1} \rangle$. \square

To conclude this subsection, let us note the following strengthened version of the preceding proposition, in the special case where $A = K$.

PROPOSITION 2.2.7. – *If ϕ is a compact endomorphism of a K -Banach space V , then for $x \in K^\times$ of sufficiently large valuation we have:*

- (i) *The endomorphism $x\phi$ of V is power-bounded, in the sense of Definition 2.1.5.*
- (ii) *If we regard V as a topological $K\langle xt \rangle$ -module (with t acting via ϕ on V), then $K\langle xt, xt^{-1} \rangle \widehat{\otimes}_{K\langle xt \rangle} V$ is a finite dimensional K -vector space.*

Proof. – Part (i) is Proposition 2.2.6(i) in the case $A = K$, while (ii) is a restatement of the usual theory of compact operators on K -Banach spaces [21]. \square

2.3. This subsection extends the theory of compact operators to topological modules of compact type over a certain class of topological K -algebras.

DEFINITION 2.3.1. – A compact type topological algebra over K is a locally convex topological K -algebra that is isomorphic to a locally convex inductive limit $A \xrightarrow{\sim} \varinjlim_n A_n$, where $\{A_n\}_{n \geq 0}$ is a directed sequence of K -Banach algebras, with transition maps that are injective and of compact type as maps of convex K -vector spaces.

In particular, a compact type topological K -algebra is of compact type as a topological K -vector space.

When we write that a topological module over a topological K -algebra is locally convex, or of compact type, we mean simply that it is so when regarded as a topological K -vector space.

LEMMA 2.3.2. – *If $A \rightarrow B$ is a continuous homomorphism of compact type topological K -algebras, and if M is a compact type topological A -module, then the completed tensor product $B \widehat{\otimes}_A M$ is a compact type topological B -module.*

Proof. – By definition, $B \widehat{\otimes}_A M$ is a Hausdorff quotient of the completed tensor product $B \widehat{\otimes}_K M$. The latter space is of compact type [8, prop. 1.1.32], and thus so is the former. \square

Fix a compact type topological K -algebra A . The following definition, while a little *ad hoc*, will serve our purposes.

DEFINITION 2.3.3. – A continuous A -linear morphism $\phi: M \rightarrow N$ between compact type topological A -modules is called A -compact if it may be factored as a composite of continuous A -linear maps $M \rightarrow N_1 \rightarrow N$, where N_1 is a compact type topological A -module satisfying the following condition: there is a continuous K -linear map of convex K -vector spaces $V \rightarrow N_1$, with V a convex K -vector space of compact type, such that the natural map $A \widehat{\otimes}_K V \rightarrow N_1$ is surjective, and such that the composite $V \rightarrow N_1 \rightarrow N$ is a compact map of convex K -vector spaces.

Note that when $A = K$, the notion introduced in the preceding definition reduces to the usual notion of a compact map (between compact type K -vector spaces).

We next establish some stability properties of the notion of A -compact maps.

LEMMA 2.3.4. –

- (i) If $M \rightarrow N$ is an A -compact map of compact type topological A -modules, and if $M_1 \rightarrow M$ and $N \rightarrow N_1$ are any two continuous A -linear maps of compact type topological A -modules, then the composite $M_1 \rightarrow M \rightarrow N \rightarrow N_1$ is A -compact.
- (ii) If $A \rightarrow B$ is a continuous K -linear homomorphism of compact type topological K -algebras, and if $M \rightarrow N$ is an A -compact map of compact type A -module, then the induced map $B \widehat{\otimes}_A M \rightarrow B \widehat{\otimes}_A N$ is B -compact.
- (iii) If we are given a commutative diagram of continuous A -linear maps between compact type topological A -modules

$$\begin{array}{ccc} M & \longrightarrow & N \\ \downarrow & & \downarrow \\ P & \longrightarrow & Q \end{array}$$

in which the upper horizontal arrow is A -compact and the left-hand vertical arrow is surjective, then the bottom horizontal arrow is also A -compact.

Proof. – Property (i) follows immediately from the definition of an A -compact map, as does part (ii), once one takes into account Lemma 2.3.2.

To prove (iii), let us first factor the top horizontal map (which is assumed to be A -compact) as $M \rightarrow N_1 \rightarrow N$, as in Definition 2.2.8. Thus we assume that this factorization is A -linear, that N_1 is a topological A -module of compact type, and that there is a map $V \rightarrow N_1$ with V of compact type for which the induced map $A \widehat{\otimes}_K V \rightarrow N_1$ is surjective, and for which the composite $V \rightarrow N_1 \rightarrow N$ is compact. Let Q_1 denote the compact type A -module obtained as the quotient of N_1 by the kernel of the composite $N_1 \rightarrow N \rightarrow Q$. The kernel of the surjection $M \rightarrow P$ is contained in the kernel of the composite $M \rightarrow N_1 \rightarrow Q_1$. Since P is a strict quotient of M (both M and P being of compact type), we see that this latter composite factors through P . Thus the given map $P \rightarrow Q$ factors as $P \rightarrow Q_1 \rightarrow Q$. The surjection $A \widehat{\otimes}_K V \rightarrow N_1$ induces a surjection $A \widehat{\otimes}_K V \rightarrow Q_1$, and the composite $V \rightarrow N_1 \rightarrow Q_1$ is compact, since the first arrow is. This shows that the map $P \rightarrow Q$ satisfies the requirements of Definition 2.3.3, and hence is A -compact. \square

We now establish the analogue of Proposition 2.2.6 in the setting of compact type algebras and modules.

Fix a strictly decreasing sequence $\{r_n\}_{n \geq 0}$ of real numbers converging to 1 from above. Letting $R_n(x)$ denote the Tate algebra of rigid analytic functions defined over K that converge on the disk $|t| \leq r_n|x|^{-1}$, define $K\langle xt \rangle^\dagger := \varinjlim_n R_n(x)$. Each $R_n(x)$ is naturally a Banach space, and if we equip $K\langle xt \rangle^\dagger$ with its locally convex inductive limit topology, it becomes a compact type topological K -algebra. Letting $S_n(x)$ denote the Tate algebra of rigid analytic functions over A on the annulus $r_n^{-1}|x| \leq |t| \leq r_n|x|^{-1}$, we may similarly define the compact type algebra $K\langle xt, xt^{-1} \rangle^\dagger = \varinjlim_n S_n(x)$. Both $K\langle xt \rangle^\dagger$ and $K\langle xt, xt^{-1} \rangle^\dagger$ are independent (up to canonical isomorphism) of the choice of sequence $\{r_n\}$.

If $A = \varinjlim_n A_n$ is a compact type topological K -algebra (written as the locally convex inductive limit, with compact and injective transition maps, of the K -Banach algebras A_n), then we write $A\langle xt \rangle^\dagger := A \widehat{\otimes}_K K\langle xt \rangle^\dagger \xrightarrow{\sim} \varinjlim_n A_n \widehat{\otimes}_K R_n(x)$ and $A\langle xt, xt^{-1} \rangle^\dagger := A \widehat{\otimes}_K K\langle xt, xt^{-1} \rangle^\dagger \xrightarrow{\sim} \varinjlim_n A_n \widehat{\otimes}_K S_n$ (the isomorphisms following from [8, prop. 1.1.32(i)]).

PROPOSITION 2.3.5. – *Let ϕ be an A -compact endomorphism of a compact type topological A -module M . If $x \in K^\times$ has sufficiently large valuation then:*

- (i) *The endomorphism $x\phi$ of M is power-bounded endomorphism, in the sense of Definition 2.1.5.*
- (ii) *If we regard M as a topological $K\langle xt \rangle^\dagger$ -module, with t acting via ϕ on M , then $K\langle xt, xt^{-1} \rangle^\dagger \widehat{\otimes}_{K\langle xt \rangle^\dagger} M$ is finitely generated as an $A\langle xt, xt^{-1} \rangle^\dagger$ -module.*

Proof. – By assumption, we may factor the map ϕ in the form $M \rightarrow M_1 \rightarrow M$, where M_1 is a compact type A -module for which there is a surjection $\pi: A \widehat{\otimes}_K V \rightarrow M_1$ for some compact type K -vector space V , with the additional property that the composite $V \rightarrow M_1 \rightarrow M$ is a compact K -linear map. Applying Propositions 2.1.8(ii) and 2.1.9 (taking the algebras R and S of Proposition 2.1.9 to be $K\langle xt \rangle^\dagger$ and $K\langle xt, xt^{-1} \rangle^\dagger$), we see that it suffices to prove the proposition with M_1 in place of M , and with the composite $M_1 \rightarrow M \rightarrow M_1$ (which we denote by ϕ_1) in place of ϕ .

Write $A \xrightarrow{\sim} \varinjlim_n A_n$ as in Definition 2.3.1, and write $V \xrightarrow{\sim} \varinjlim_n V_n$ as the locally convex inductive limit of a sequence of Banach spaces with compact injective transition maps. By [8, prop. 1.1.32], the natural map $\varinjlim_n A_n \widehat{\otimes}_K V_n \rightarrow A \widehat{\otimes}_K V$ is a topological isomorphism. For each natural number n , let W_n denote the K -Banach space obtained as the quotient of $A_n \widehat{\otimes}_K V_n$ by the kernel of the composite $A_n \widehat{\otimes}_K V_n \rightarrow A \widehat{\otimes}_K V \rightarrow M_1$. For each value of n there is a continuous injection $\iota_n: W_n \rightarrow M_1$, and the induced map $\varinjlim_n W_n \rightarrow M_1$ is a topological isomorphism.

Since the composite $V \rightarrow M_1 \xrightarrow{\phi_1} M_1$ is compact, it factors through a continuous map $\alpha_m: V \rightarrow W_m$, for some m [3, prop. 1, p. I.20]. At the expense of replacing m by $m + 1$ if necessary, we may assume that α_m is a compact map (since the map $W_m \rightarrow W_{m+1}$ is compact). If U denotes the image of W_m in M_1 , then (since it is A -linear, and so in particular A_m -linear) the map ϕ_1 restricts to an A_m -linear endomorphism of U . The closed graph theorem then implies that this endomorphism of U lifts to a continuous A_m -linear endomorphism ψ_m of W_m [8, prop. 1.1.2], which by Proposition 2.2.5 is A_m -compact (in the sense of Definition 2.2.3), since α_m is compact.

Proposition 2.1.10 implies that $x\psi_m$ is power-bounded if x has sufficiently large valuation. Thus the sequence of maps $\{x^n \psi_m^n\}_{n \geq 0}$ is equicontinuous, and thus so is the sequence of maps $\{\iota_m \circ x^n \psi_m^n \circ \alpha_m\}_{n \geq 0}$ from V to M_1 , by Lemma 2.1.4(iii). Extending these sequence of maps by A -linearity to a sequence of maps $A \widehat{\otimes}_K V \rightarrow M_1$, we obtain precisely the sequence of composites $\{x^n \phi_1^{n+1} \circ \pi\}_{n \geq 0}$, which is thus also equicontinuous (by Lemma 2.1.4(i) and (iii)). Finally, since π is a surjection of spaces of compact type, it is strict, and so Lemma 2.1.4(iv)

implies that the sequence of map $\{x^n \phi_1^{n+1}\}_{n \geq 0}$ is equicontinuous, and hence that $x\phi_1$ is power bounded.

If $m' \geq m$, then ϕ_1 equally well induces a compact map $\psi_{m'} : W_{m'} \rightarrow W_{m'}$. Since ϕ_1 is A -linear (and so in particular $A_{m'}$ -linear), since $A_{m'} \widehat{\otimes}_K V_{m'}$ surjects onto $W_{m'}$, and since $\phi_1(V_{m'}) \subset W_m$, we find that $\psi_{m'}$ admits a factorization into continuous $A_{m'}$ -linear maps $W_{m'} \xrightarrow{h_m} A_{m'} \widehat{\otimes}_{A_m} W_m \xrightarrow{j_m} W_{m'}$, where j_m is induced by the natural map $W_m \rightarrow W_{m'}$, and h_m is defined so that $\psi_{m'} = j_m \circ h_m$. (We have also appealed again to [8, prop. 1.1.2].) The composite $h_m \circ j_m$ is then equal to $\text{id}_{A_{m'}} \widehat{\otimes} \psi_m$. It thus follows from Proposition 2.1.8(i) and (iii) that if $x\psi_m$ is power-bounded, then the same is true of $x\psi_{m'}$.

For each natural number n , let $r_n, R_n(x)$ and $S_n(x)$ be as defined in the discussion preceding the statement of the proposition. Proposition 2.2.6 implies that $S_m(x) \widehat{\otimes}_{R_m(x)} W_m$ (respectively $S_{m'}(x) \widehat{\otimes}_{R_{m'}(x)} W_{m'}$) is a finitely generated $A_m \widehat{\otimes}_K S_m(x)$ module (respectively $A_{m'} \widehat{\otimes}_K S_{m'}(x)$ module). Proposition 2.1.9 implies that the continuous $A_{m'}$ -linear map

$$(A_{m'} \widehat{\otimes}_K S_{m'}(x)) \widehat{\otimes}_{A_m \widehat{\otimes}_K S_m(x)} S_m(x) \widehat{\otimes}_{R_m(x)} W_m \rightarrow S_{m'}(x) \widehat{\otimes}_{R_{m'}(x)} W_{m'}$$

induced by j_m is an isomorphism (of finitely generated $A_{m'} \widehat{\otimes}_K S_{m'}(x)$ -modules). Passing to the locally convex inductive limit in m now shows that the completed tensor product $K \langle\langle xt, xt^{-1} \rangle\rangle^\dagger \widehat{\otimes}_{K \langle\langle xt \rangle\rangle^\dagger} M_1$ is a finitely generated module over $A \langle\langle xt, xt^{-1} \rangle\rangle^\dagger$, as required. \square

It is convenient to note that in the situation of part (ii) of the preceding proposition, the natural map

$$K \langle\langle xt, xt^{-1} \rangle\rangle^\dagger \widehat{\otimes}_{K[t]} M \rightarrow K \langle\langle xt, xt^{-1} \rangle\rangle^\dagger \widehat{\otimes}_{K \langle\langle xt \rangle\rangle^\dagger} M$$

is an isomorphism. Indeed, this follows immediately from the fact that $K[t]$ is dense in $K \langle\langle xt \rangle\rangle^\dagger$.

PROPOSITION 2.3.6. – *Let V be a compact type K -vector space equipped with a compact endomorphism ϕ . If $x \in K^\times$ has sufficiently large valuation, then:*

- (i) *The endomorphism $x\phi$ of V is power-bounded, in the sense of Definition 2.1.5.*
- (ii) *If we regard M as a topological $K \langle\langle xt \rangle\rangle^\dagger$ -module (with t acting via ϕ on M), then $K \langle\langle xt, xt^{-1} \rangle\rangle^\dagger \widehat{\otimes}_{K \langle\langle xt \rangle\rangle^\dagger} V$ is a finite dimensional K -vector space.*

Proof. – This follows from Proposition 2.2.7 in the same way that Proposition 2.3.5 follows from Proposition 2.2.6. \square

3. Construction of the Jacquet module functor

3.1. If G is a topological group (or semigroup), we let $\text{Rep}_{\text{top.c}}(G)$ denote the category whose objects are Hausdorff locally convex K -vector spaces of compact type, equipped with a topological action of G , and whose morphisms are continuous G -equivariant K -linear maps. (Recall that by a “topological action” of G on a topological vector space V we mean an action of G on V by continuous endomorphisms.)

LEMMA 3.1.1. –

- (i) *If U and V are two objects of $\text{Rep}_{\text{top.c}}(G)$, then the direct product $U \times V$, with the diagonal G -action, is again an object of $\text{Rep}_{\text{top.c}}(G)$.*
- (ii) *If V is an object of $\text{Rep}_{\text{top.c}}(G)$, and if U is a closed G -invariant subspace of V , then U is again an object of the category $\text{Rep}_{\text{top.c}}(G)$, as is the quotient V/U .*

- (iii) If $\{V_n\}_{n \geq 0}$ is an inductive sequence of objects in $\text{Rep}_{\text{top.c}}(G)$, having injective transition maps, then the locally convex inductive limit $\varinjlim_n V_n$ again lies in $\text{Rep}_{\text{top.c}}(G)$.
- (iv) If U and V are two objects $\text{Rep}_{\text{top.c}}(G)$, then $U \widehat{\otimes}_K V$ (with the diagonal action of G) is again an object of $\text{Rep}_{\text{top.c}}(G)$.

Proof. – This is immediate, once one recalls the following facts: a finite direct product of spaces of compact type is again of compact type; any closed subspace or Hausdorff quotient of a convex K -vector space of compact type is again of compact type; the locally convex inductive limit of any inductive sequence of convex K -vector spaces of compact type having injective transition maps is again of compact type; the completed tensor product of two convex K -vector spaces of compact type is again of compact type [8, prop. 1.1.32(i)]. \square

The preceding lemma implies that $\text{Rep}_{\text{top.c}}(G)$ is an additive category, admitting kernels, cokernels, images and coimages. More precisely, if $\phi: U \rightarrow V$ is a morphism in $\text{Rep}_{\text{top.c}}(G)$, then the kernel of ϕ is the usual kernel of ϕ as a map of vector spaces, regarded as a closed subspace of U ; the coimage of ϕ is $\phi(U)$, regarded as a quotient of U ; the image of ϕ is the closure of $\phi(U)$, regarded as a subspace of V ; and the cokernel of ϕ is the quotient of V by the closure of $\phi(U)$.

Suppose now that G is a locally L -analytic group. We let $\text{Rep}_{\text{la.c}}(G)$ denote the full subcategory of $\text{Rep}_{\text{top.c}}(G)$ consisting of locally analytic representations of G on convex K -vector spaces of compact type. (The notion of a locally analytic representation of G is defined in [17, p. 12]; see also [8, def. 3.6.9].)

LEMMA 3.1.2. – *Let G be a locally L -analytic group.*

- (i) *If U and V are two objects of $\text{Rep}_{\text{la.c}}(G)$, then the direct product $U \times V$, with the diagonal G -action, is again an object of $\text{Rep}_{\text{la.c}}(G)$.*
- (ii) *If V is an object of $\text{Rep}_{\text{la.c}}(G)$, and if U is a closed G -invariant subspace of V , then U is again an object of the category $\text{Rep}_{\text{la.c}}(G)$, as is the quotient V/U .*
- (iii) *If $\{V_n\}_{n \geq 0}$ is an inductive sequence of objects in $\text{Rep}_{\text{la.c}}(G)$, having injective transition maps, then the locally convex inductive limit $\varinjlim_n V_n$ again lies in $\text{Rep}_{\text{top.c}}(G)$.*
- (iv) *If U and V are two objects $\text{Rep}_{\text{la.c}}(G)$, then $U \widehat{\otimes}_K V$ (with the diagonal action of G) is again an object of $\text{Rep}_{\text{la.c}}(G)$.*

Proof. – This follows from standard properties of convex K -vector spaces of compact type, together with Propositions 3.5.5 and 3.5.15 and Theorem 3.6.12 of [8]. \square

In particular, we see that $\text{Rep}_{\text{la.c}}(G)$ is an additive category that admits kernels, cokernels, images and coimages. Furthermore, these coincide with the corresponding constructions in the larger category $\text{Rep}_{\text{top.c}}(G)$.

Continuing to suppose that G is locally L -analytic, and also supposing that the centre Z_G of G is topologically finitely generated, we let $\text{Rep}_{\text{es}}(G)$ denote the full subcategory of $\text{Rep}_{\text{la.c}}(G)$ consisting of essentially admissible locally analytic representations of G (as defined in [8, def. 6.4.9]). We let $\text{Rep}_{\text{ad}}(G)$ denote the full subcategory of $\text{Rep}_{\text{es}}(G)$ consisting of admissible locally analytic representations of G (as defined in [20]; see also [8, def. 6.1.1]).

THEOREM 3.1.3. – *Each of $\text{Rep}_{\text{es}}(G)$ and $\text{Rep}_{\text{ad}}(G)$ is an Abelian category. Furthermore, the construction of direct products, kernels, cokernels, images and coimages coincide with their construction in $\text{Rep}_{\text{top.c}}(G)$.*

Proof. – In the case of admissible locally analytic representations, this is proved in [20] (see also [8, cor. 6.1.23]). In the case of essentially admissible locally analytic representations, it is proved in [8, prop. 6.4.11]. \square

A key point in Theorem 3.1.3 is that, since images and coimages agree in an Abelian category, any morphism in $\text{Rep}_{\text{es}}(G)$ is necessarily strict, with closed image.

There is one last full subcategory of $\text{Rep}_{\text{la.c}}(G)$ that we will consider, to be denoted by $\text{Rep}_{\text{la.c}}^z(G)$. The objects of this category are convex K -spaces V of compact type, equipped with a locally analytic G -representation, that may be written as the union of an increasing sequence of Z_G -invariant BH -subspaces.

As in [8, §6.4], let \widehat{Z}_G denote the rigid analytic space of locally L -analytic characters on Z_G , and let $\mathcal{C}^{\text{an}}(\widehat{Z}_G, K)$ denote the nuclear Fréchet algebra of K -valued rigid analytic functions on \widehat{Z}_G . Evaluation at points of Z_G induces a natural map $Z_G \rightarrow \mathcal{C}^{\text{an}}(\widehat{Z}_G, K)$, with image lying in the group of invertible elements of $\mathcal{C}^{\text{an}}(\widehat{Z}_G, K)$. It follows from [8, prop. 6.4.7] that $\text{Rep}_{\text{la.c}}^z(G)$ consists of those objects V in $\text{Rep}_{\text{la.c}}(G)$ for which the Z_G -action on V extends to a $\mathcal{C}^{\text{an}}(\widehat{Z}_G, K)$ -module structure on V , with the map $\mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \times V \rightarrow V$ that defines this module structure being separately continuous.

LEMMA 3.1.4. – *Let G be a locally L -analytic group.*

- (i) *If U and V are two objects of $\text{Rep}_{\text{la.c}}^z(G)$, then the direct product $U \times V$, with the diagonal G -action, is again an object of $\text{Rep}_{\text{la.c}}^z(G)$.*
- (ii) *If V is an object of $\text{Rep}_{\text{la.c}}^z(G)$, and if U is a closed G -invariant subspace of V , then U is again an object of the category $\text{Rep}_{\text{la.c}}^z(G)$, as is the quotient V/U .*
- (iii) *If $\{V_n\}_{n \geq 0}$ is an inductive sequence of objects in $\text{Rep}_{\text{top.c}}(G)$, having injective transition maps, then the locally convex inductive limit $\varinjlim_n V_n$ again lies in $\text{Rep}_{\text{la.c}}^z(G)$.*
- (iv) *If U and V are two objects $\text{Rep}_{\text{la.c}}^z(G)$, then $U \widehat{\otimes}_K V$ (with the diagonal action of G) is again an object of $\text{Rep}_{\text{la.c}}^z(G)$.*

Proof. – This follows from Lemma 3.1.2, together with standard properties of BH -spaces (such as those recalled in [8, §1.1]). (For part (iv), we also refer to [8, prop. 1.1.32(i)].) \square

By definition any object of $\text{Rep}_{\text{es}}(G)$ is an object of $\text{Rep}_{\text{la.c}}^z(G)$.

3.2. If Z is an Abelian group and Z^+ is a submonoid of Z , then the group algebra $K[Z]$ contains the monoid algebra $K[Z^+]$ as a subalgebra, and the functor $\text{Hom}_{Z^+}(K[Z], -)$ is right adjoint to the forgetful functor from the category of Z -modules to the category of Z^+ -modules. More precisely, if M is a Z^+ -module, then the natural map $\text{Hom}_{Z^+}(K[Z], M) \rightarrow M$ given by evaluation at the identity element of $K[Z]$ realizes the Z -module $\text{Hom}_{Z^+}(K[Z], M)$ as the final object in the category of Z -modules equipped with a Z^+ -equivariant map to M .

In this subsection we will explain a topological analogue of the preceding construction. We fix a topologically finitely generated Abelian locally L -analytic group Z . (See [8, prop. 6.4.1] for some alternative characterizations of such locally L -analytic groups.) We also fix a topological submonoid Z^+ of Z , such that Z^+ generates Z as a group, and contains a compact open subgroup of Z . Recall the categories $\text{Rep}_{\text{la.c}}^z(Z)$ and $\text{Rep}_{\text{top.c}}(Z^+)$, defined in Section 3.1. There is an obvious forgetful functor $\text{Rep}_{\text{la.c}}^z(Z) \rightarrow \text{Rep}_{\text{top.c}}(Z^+)$. We will construct a right adjoint to this functor.

DEFINITION 3.2.1. – If V is any object of $\text{Rep}_{\text{top.c}}(Z^+)$, then we write

$$V_{\text{fs}} := \mathcal{L}_{b, Z^+}(\mathcal{C}^{\text{an}}(\widehat{Z}, K), V).$$

(The right-hand side denotes the subspace of $\mathcal{L}_b(\mathcal{C}^{\text{an}}(\widehat{Z}, K), V)$ consisting of Z^+ -equivariant maps; the inclusion of Z into the group of units of $\mathcal{C}^{\text{an}}(\widehat{Z}, K)$ allows us to regard $\mathcal{C}^{\text{an}}(\widehat{Z}, K)$ as a Z -module, and so in particular as a Z^+ -module.) Motivated by [7], we refer to V_{fs} as the “finite slope part” of V .

If V is an object of $\text{Rep}_{\text{top.c}}(Z^+)$, then since the elements of Z^+ act on $\mathcal{C}^{\text{an}}(\widehat{Z}, K)$ and V via continuous operators, and since V is Hausdorff, the space V_{fs} is a closed subspace of the Hausdorff space $\mathcal{L}_b(\mathcal{C}^{\text{an}}(\widehat{Z}, K), V)$. The natural action of Z on $\mathcal{C}^{\text{an}}(\widehat{Z}, K)$ induces an action of Z on $\mathcal{L}_b(\mathcal{C}^{\text{an}}(\widehat{Z}, K), V)$, and since Z is Abelian, V_{fs} is obviously Z -invariant. Thus V_{fs} is equipped with a natural Z -action.

The formation of V_{fs} is obviously functorial in V . Evaluation of linear maps at the identity of $\mathcal{C}^{\text{an}}(\widehat{Z}, K)$ induces a natural Z^+ -linear map

$$(3.2.2) \quad V_{\text{fs}} \rightarrow V.$$

The following lemma provides a description of the strong dual of V_{fs} .

LEMMA 3.2.3. – *If V is an object of $\text{Rep}_{\text{top.c}}(Z^+)$, then there is a natural isomorphism $(V_{\text{fs}})'_b \xrightarrow{\sim} \mathcal{C}^{\text{an}}(\widehat{Z}, K) \widehat{\otimes}_{K[Z^+]} V'_b$.*

Proof. – It follows from [15, cor. 18.8] and [8, prop. 1.1.32] that $\mathcal{L}_b(\mathcal{C}^{\text{an}}(\widehat{Z}, K), V)$ is a compact type space whose strong dual is isomorphic to $\mathcal{C}^{\text{an}}(\widehat{Z}, K) \widehat{\otimes}_K V'_b$. Since V_{fs} is defined to be the closed subspace of $\mathcal{L}_b(\mathcal{C}^{\text{an}}(\widehat{Z}, K), V)$ consisting of Z^+ -equivariant maps, it follows that its strong dual $(V_{\text{fs}})'_b$ is naturally isomorphic to the Hausdorff quotient $\mathcal{C}^{\text{an}}(\widehat{Z}, K) \widehat{\otimes}_{K[Z^+]} V'_b$ of $\mathcal{C}^{\text{an}}(\widehat{Z}, K) \widehat{\otimes}_K V'_b$. \square

PROPOSITION 3.2.4. –

- (i) *If V is an object of $\text{Rep}_{\text{top.c}}(Z^+)$, then V_{fs} is an object of $\text{Rep}_{\text{la.c}}^z(Z)$. Thus $V \mapsto V_{\text{fs}}$ yields a functor $\text{Rep}_{\text{top.c}}(Z^+) \rightarrow \text{Rep}_{\text{la.c}}^z(Z)$.*
- (ii) *For any objects W and V of $\text{Rep}_{\text{la.c}}^z(Z)$ and $\text{Rep}_{\text{top.c}}(Z^+)$ respectively, the map (3.2.2) induces a natural topological isomorphism*

$$(3.2.5) \quad \mathcal{L}_{b,Z}(W, V_{\text{fs}}) \xrightarrow{\sim} \mathcal{L}_{b,Z^+}(W, V).$$

Proof. – To prove (i), note that the isomorphism of Lemma 3.2.3 shows that $(V_{\text{fs}})'_b$ is a topological $\mathcal{C}^{\text{an}}(\widehat{Z}, K)$ -module. It follows from [8, 6.4.7] that V_{fs} lies in $\text{Rep}_{\text{la.c}}^z(Z)$.

Now let W be as in (ii). (Thus W is of compact type, and hence is barrellled.) As recalled in Section 3.1, the Z -action on W endows W with the structure of a $\mathcal{C}^{\text{an}}(\widehat{Z}, K)$ -module, for which the induced map $\mathcal{C}^{\text{an}}(\widehat{Z}, K) \otimes_{K,\iota} W \rightarrow W$ is continuous. Thus we obtain natural maps

$$\mathcal{L}_b(W, V) \longrightarrow \mathcal{L}_b(\mathcal{C}^{\text{an}}(\widehat{Z}, K) \otimes_{K,\iota} W, V) \rightarrow \mathcal{L}_b(W, \mathcal{L}_b(\mathcal{C}^{\text{an}}(\widehat{Z}, K), V)),$$

the second map being the continuous bijection of [8, prop. 1.1.35]. One easily checks that the composite of these maps takes any Z^+ -equivariant map in the source to a Z^+ -equivariant map in the target. Since W and V_{fs} are both Z -modules, the inclusion $\mathcal{L}_{b,Z}(W, V_{\text{fs}}) \subset \mathcal{L}_{b,Z^+}(W, V_{\text{fs}})$ is an equality, and so we obtain a map $\mathcal{L}_{b,Z^+}(W, V) \rightarrow \mathcal{L}_{b,Z}(W, V_{\text{fs}})$. It is easily checked to provide an inverse to (3.2.5). \square

The preceding result implies in particular that $V \mapsto V_{\text{fs}}$ is a right adjoint to the forgetful functor $\text{Rep}_{\text{la.c}}^z(Z) \rightarrow \text{Rep}_{\text{top.c}}(Z^+)$.

PROPOSITION 3.2.6. –

- (i) *The functor $V \mapsto V_{\text{fs}}$ is additive.*
- (ii) *If $U \rightarrow V$ is an injective map in the category $\text{Rep}_{\text{top.c}}(Z^+)$, then the morphism $U_{\text{fs}} \rightarrow V_{\text{fs}}$ induced by functoriality is again injective.*

- (iii) If $U \rightarrow V$ is a closed embedding in the category $\text{Rep}_{\text{top.c}}(Z^+)$, then the morphism $U_{\text{fs}} \rightarrow V_{\text{fs}}$ induced by functoriality is again a closed embedding. Furthermore, in this situation, the morphism $V_{\text{fs}}/U_{\text{fs}} \rightarrow (V/U)_{\text{fs}}$ induced by functoriality is a continuous injection.
- (iv) If $\{V_n\}_{n \geq 0}$ is an inductive sequence in the category $\text{Rep}_{\text{top.c}}(Z^+)$ having injective transition maps, and if we form the locally convex inductive limit $V = \varinjlim_n V_n$ (an object of $\text{Rep}_{\text{top.c}}(Z^+)$, by Lemma 3.1.1), then the natural map $\varinjlim_n (V_n)_{\text{fs}} \rightarrow V_{\text{fs}}$ is a topological isomorphism.

Proof. – Parts (i) and (ii) are immediate from the construction of U_{fs} and V_{fs} .

Let $U \rightarrow V$ be a closed embedding as in (iii). Applying $\mathcal{L}_{b,Z^+}(\mathcal{C}^{\text{an}}(\widehat{Z}, K), -)$ to the short exact sequence $0 \rightarrow U \rightarrow V \rightarrow (V/U) \rightarrow 0$ yields the sequence of continuous Z -equivariant maps

$$0 \rightarrow \mathcal{L}_{b,Z^+}(\mathcal{C}^{\text{an}}(\widehat{Z}, K), U) \rightarrow \mathcal{L}_{b,Z^+}(\mathcal{C}^{\text{an}}(\widehat{Z}, K), V) \rightarrow \mathcal{L}_{b,Z^+}(\mathcal{C}^{\text{an}}(\widehat{Z}, K), V/U).$$

Since U embeds as a closed subspace of V , this is exact as a sequence of abstract vector spaces and the second arrow is a closed embedding. This proves part (iii).

Since $\mathcal{C}^{\text{an}}(\widehat{Z}, K)$ is a Fréchet space, if $V = \varinjlim_n V_n$ as in (iv), then the natural map $\varinjlim_n \mathcal{L}_{b,Z^+}(\mathcal{C}^{\text{an}}(\widehat{Z}, K), V_n) \rightarrow \mathcal{L}_{b,Z^+}(\mathcal{C}^{\text{an}}(\widehat{Z}, K), V)$ is a continuous bijection. Since its source and target are of compact type, it is a topological isomorphism. \square

LEMMA 3.2.7. – Let H be a locally L -analytic group H , and let the object V of $\text{Rep}_{\text{top.c}}(Z^+)$ be equipped with a locally analytic H -representation that commutes with the Z^+ -action on V .

- (i) The H -action on V_{fs} induced by functoriality is also locally analytic.
- (ii) If \mathfrak{h} denotes the Lie algebra of H , then the action of $U(\mathfrak{h})$ on V_{fs} , induced by functoriality from the $U(\mathfrak{h})$ -action on the locally analytic H -representation V , coincides with the $U(\mathfrak{h})$ -action on V_{fs} , obtained by regarding it as a locally analytic H -representation via (i).

Proof. – Lemma 3.2.3 yields an isomorphism $\mathcal{C}^{\text{an}}(\widehat{Z}, K) \widehat{\otimes}_{K[Z^+]} V'_b \xrightarrow{\sim} (V_{\text{fs}})'_b$. It follows from [17, cor. 3.3] that the H -action induced on the source of this isomorphism by the H -action on V extends to a topological $\mathcal{D}^{\text{la}}(H, K)$ -module structure. Since the isomorphism is natural, it intertwines the H -action on the source with the H -action on the target induced by the H -action on V , and thus this latter action also extends to a topological $\mathcal{D}^{\text{la}}(H, K)$ -module structure. Thus (again by [17, cor. 3.3]) the H -action on V_{fs} induced by the H -action on V is locally analytic. This proves (i). Part (ii) is clear. \square

LEMMA 3.2.8. – For any object V of $\text{Rep}_{\text{la.c}}^{\tilde{z}}(Z)$, regarded via the forgetful functor as an object of $\text{Rep}_{\text{top.c}}(Z^+)$, the natural map (3.2.2) is a topological isomorphism.

Proof. – This follows from the universal property satisfied by V_{fs} , together with the fact that for any objects V and W of $\text{Rep}_{\text{la.c}}^{\tilde{z}}(Z)$, the natural map $\mathcal{L}_{Z^+}(W, V) \rightarrow \mathcal{L}_Z(W, V)$ is an isomorphism. \square

PROPOSITION 3.2.9. – If V and W are objects of $\text{Rep}_{\text{top.c}}(Z^+)$ and $\text{Rep}_{\text{la.c}}^{\tilde{z}}(Z)$ respectively, then there is a natural isomorphism $V_{\text{fs}} \widehat{\otimes}_K W \xrightarrow{\sim} (V \widehat{\otimes}_K W)_{\text{fs}}$.

Proof. – Since W is of compact type, it follows from [8, prop. 1.1.35] and [15, cor. 18.8] that for any compact type spaces U_1 and U_2 , there is an isomorphism

$$(3.2.10) \quad \mathcal{L}(U_1 \widehat{\otimes}_K W'_b, U_2) \xrightarrow{\sim} \mathcal{L}(U_1, U_2 \widehat{\otimes}_K W).$$

Now $V \widehat{\otimes}_K W$ is an object of $\text{Rep}_{\text{top.c}}(Z^+)$, while if U is an object of $\text{Rep}_{\text{la.c}}^z(Z)$ then so is $U \widehat{\otimes}_K W$ (as follows from part (iv) of Lemmas 3.1.2 and 3.1.4 respectively). Taking into account the universal property of V_{fs} and $(V \otimes_K W)_{\text{fs}}$, and applying (3.2.10), we obtain natural isomorphisms

$$\begin{aligned} \mathcal{L}_Z(U, V_{\text{fs}} \widehat{\otimes}_K W) &\xrightarrow{\sim} \mathcal{L}_Z(U \widehat{\otimes}_K W'_b, V_{\text{fs}}) = \mathcal{L}_{Z^+}(U \widehat{\otimes}_K W'_b, V) \\ &\xrightarrow{\sim} \mathcal{L}_{Z^+}(U, V \widehat{\otimes}_K W) \xrightarrow{\sim} \mathcal{L}_Z(U, (V \widehat{\otimes}_K W)_{\text{fs}}). \end{aligned}$$

Thus we obtain the required natural isomorphism $V_{\text{fs}} \widehat{\otimes}_K W \xrightarrow{\sim} (V \widehat{\otimes}_K W)_{\text{fs}}$. \square

PROPOSITION 3.2.11. – *If V is an object of $\text{Rep}_{\text{top.c}}(Z^+)$ and if X is any collection of continuous Z^+ -equivariant endomorphisms of V , then the closed embedding $V^X \rightarrow V$ induces an isomorphism $(V^X)_{\text{fs}} \xrightarrow{\sim} (V_{\text{fs}})^X$. (Here the superscript “ X ” denotes the closed subspace consisting of vectors annihilated by each of the elements of X .)*

Proof. – This is immediate from the definition of the finite slope parts and the evident isomorphism $\mathcal{L}_{b,Z^+}(\mathcal{C}^{\text{an}}(\widehat{Z}, K), V^X) \xrightarrow{\sim} \mathcal{L}_{b,Z^+}(\mathcal{C}^{\text{an}}(\widehat{Z}, K), V)^X$. \square

If χ is a locally L -analytic K -valued character of Z (i.e. a point of $\widehat{Z}(K)$) and if V is an object of $\text{Rep}_{\text{top.c}}(Z^+)$, then we define

$$V^\chi = \{v \in V \mid zv = \chi(z)v \text{ for all } z \in Z^+\}.$$

We regard V^χ as a locally analytic representation of Z , by having Z act through the character χ . Note that since V , and hence its closed subspace V^χ , is of compact type, and since Z acts on V^χ through a scalar, the representation V^χ is certainly an object of $\text{Rep}_{\text{la.c}}^z(Z)$. Note also that the preceding construction and notation applies in particular to objects of $\text{Rep}_{\text{la.c}}^z(Z)$, which may be regarded as objects of $\text{Rep}_{\text{top.c}}(Z^+)$, by restricting the Z -action to a Z^+ -action.

PROPOSITION 3.2.12. – *If $\chi \in \widehat{Z}(K)$ and if V is an object of $\text{Rep}_{\text{top.c}}(Z^+)$, then the map (3.2.2) induces an isomorphism $(V_{\text{fs}})^\chi \xrightarrow{\sim} V^\chi$.*

Proof. – The universal property of V_{fs} , applied to the map $V^\chi \rightarrow V$, yields a map $V^\chi \rightarrow V_{\text{fs}}$, and hence a map $V^\chi \rightarrow (V_{\text{fs}})^\chi$. On the other hand, the map (3.2.2) certainly induces a map $(V_{\text{fs}})^\chi \rightarrow V^\chi$. The two maps constructed are immediately checked to be inverse to one another. \square

We may regard the monoid Z^+ as a directed set, directed under the relationship of divisibility. (That is, for a pair of elements $z_1, z_2 \in Z^+$, we have $z_1 < z_2$ if there exists $z \in Z^+$ such that $z_2 = zz_1$.) If $z \in Z^+$ and V is an object of $\text{Rep}_{\text{top.c}}(Z^+)$, then following the notation introduced in the statement of Proposition 3.2.11, we let V^z denote the Z^+ -invariant closed subspace of V consisting of vectors annihilated by the element $z \in Z^+$. If $z_1, z_2 \in Z^+$ such that $z_1 < z_2$, then $V^{z_1} \subset V^{z_2}$.

DEFINITION 3.2.13. – If V is an object of $\text{Rep}_{\text{top.c}}(Z^+)$, then we define $V_{\text{null}} := \varinjlim_{z \in Z^+} V^z$, the locally convex inductive limit being indexed by the monoid Z^+ , directed, as above, under the relation of divisibility.

Let V be as in the preceding definition. For each $z \in Z^+$, we have a closed embedding $V^z \rightarrow V$. Since Z^+ is commutative, each of the closed subspaces V^z is also a Z^+ -invariant

subspace of V . Passing to the inductive limit, we obtain a topological Z^+ -action on V_{null} , and a continuous Z^+ -equivariant injection

$$(3.2.14) \quad V_{\text{null}} \rightarrow V.$$

PROPOSITION 3.2.15. –

- (i) *If V is an object of $\text{Rep}_{\text{top.c}}(Z^+)$, then V_{null} is also an object of $\text{Rep}_{\text{top.c}}(Z^+)$.*
- (ii) *Suppose that $\{V_n\}_{n \geq 0}$ is a Z^+ -equivariant inductive sequence object of $\text{Rep}_{\text{top.c}}(Z^+)$, having injective transition maps. If V denotes the locally convex inductive limit $V = \varinjlim_n V_n$ (also an object of $\text{Rep}_{\text{top.c}}(Z^+)$, by Lemma 3.1.1), then the natural map $\varinjlim_n (V_n)_{\text{null}} \rightarrow V_{\text{null}}$ is a topological isomorphism.*

Proof. – By assumption, Z^+ contains a compact open subgroup Z_0 of Z , and the inductive limit in Definition 3.2.13 may be regarded as being indexed by Z^+/Z_0 , rather than by Z^+ . The quotient Z^+/Z_0 is a subsemigroup of the finitely generated group Z/Z_0 , and hence is countable, and thus the inductive limit appearing in Definition 3.2.13 can be replaced by a countable inductive limit.

If V is an object of $\text{Rep}_{\text{top.c}}(Z^+)$ and $z \in Z^+$, then V^z is a closed Z^+ -invariant subspace of V , which, by Lemma 3.1.1(ii), is again an object of $\text{Rep}_{\text{top.c}}(Z^+)$. Thus part (iii) of the same lemma, when combined with the observation of the preceding paragraph, shows that V_{null} is again an object of $\text{Rep}_{\text{top.c}}(Z^+)$. This proves (i).

To prove (ii), suppose that $V = \varinjlim_n V_n$ is the locally convex inductive limit of an inductive sequence of objects of $\text{Rep}_{\text{top.c}}(Z^+)$ having injective transition maps. Clearly the natural map $\varinjlim_n (V_n)_{\text{null}} \rightarrow V_{\text{null}}$ is a continuous bijection. Since source and target are both of compact type, it is necessarily a topological isomorphism. \square

DEFINITION 3.2.16. – If V is an object of $\text{Rep}_{\text{top.c}}(Z^+)$, we say that V is null if the natural map $V_{\text{null}} \rightarrow V$ is a continuous bijection (or equivalently, since source and target are spaces of compact type, a topological isomorphism).

LEMMA 3.2.17. – *If V is a null object of $\text{Rep}_{\text{top.c}}(Z^+)$, then $V_{\text{fs}} = 0$.*

Proof. – Proposition 3.2.6(iv), combined with the observation in the first paragraph of the proof of Proposition 3.2.15, shows that for any object V of $\text{Rep}_{\text{top.c}}(Z^+)$, the natural map $\varinjlim_{z \in Z^+} (V^z)_{\text{fs}} \rightarrow (V_{\text{null}})_{\text{fs}}$ is an isomorphism. Thus it suffices to show that $(V^z)_{\text{fs}} = 0$ for any object V of $\text{Rep}_{\text{top.c}}(Z^+)$. Proposition 3.2.11 gives an isomorphism $(V^z)_{\text{fs}} \xrightarrow{\sim} (V_{\text{fs}})^z$. The target of this isomorphism is clearly trivial, since the z -action on V_{fs} arises from an action of the group Z . \square

LEMMA 3.2.18. – *If V is a finite dimensional object of $\text{Rep}_{\text{top.c}}(Z^+)$, then the natural map $V_{\text{null}} \oplus V_{\text{fs}} \rightarrow V$ (induced by (3.2.2) and (3.2.14)) is an isomorphism.*

Proof. – This is an immediate consequence of the elementary spectral theory of commuting linear operators on finite dimensional vector spaces. \square

LEMMA 3.2.19. – *Let Y be a subsemigroup of Z^+ that generates Z as a group. If V is an object of $\text{Rep}_{\text{top.c}}(Z^+)$, then the natural map $V_{\text{fs}} = \mathcal{L}_{b,Z^+}(\mathcal{C}^{\text{an}}(\widehat{Z}, K), V) \rightarrow \mathcal{L}_{b,Y}(\mathcal{C}^{\text{an}}(\widehat{Z}, K), V)$ is a topological isomorphism.*

Proof. – The map in question is an inclusion of closed subspaces of $\mathcal{L}_b(\mathcal{C}^{\text{an}}(\widehat{Z}, K), V)$, and we must show that it is an equality, that is, that any Y -equivariant map $\phi: \mathcal{C}^{\text{an}}(\widehat{Z}, K) \rightarrow V$ is necessarily Z^+ -equivariant. By assumption, if $z \in Z^+$, then we may find $y, y' \in Y$ such that

$zy = y'$. If $f \in \mathcal{C}^{\text{an}}(\widehat{Z}, K)$, then (taking into account that ϕ is Y -equivariant) we compute

$$\phi(zf) = \phi(y'y^{-1}f) = y'\phi(y^{-1}f) = zy\phi(y^{-1}f) = z\phi(f).$$

Thus ϕ is automatically Z^+ -equivariant, as required. \square

PROPOSITION 3.2.20. – *Let U and V be two objects of $\text{Rep}_{\text{top.c}}(Z^+)$, and let $\phi: U \rightarrow V$, $\psi: U \rightarrow V$ be a pair of continuous K -linear maps satisfying the following conditions:*

- (i) *There is a subsemigroup Y of Z^+ that generates Z as a group, such that ϕ and ψ are both Y -equivariant.*
- (ii) *There is an element $z \in Z^+$, such that $\psi \circ \phi: U \rightarrow U$ (respectively $\phi \circ \psi: V \rightarrow V$) is equal to the endomorphism of U (respectively V) induced by the action of z .*

Then the map $\mathcal{L}_b(\mathcal{C}^{\text{an}}(\widehat{Z}, K), U) \rightarrow \mathcal{L}_b(\mathcal{C}^{\text{an}}(\widehat{Z}, K), V)$ induced by ϕ restricts to a Z -equivariant topological isomorphism $U_{\text{fs}} \xrightarrow{\sim} V_{\text{fs}}$.

Proof. – The map ϕ induces a map $\tilde{\phi}: \mathcal{L}_{b,Y}(\mathcal{C}^{\text{an}}(\widehat{Z}, K), U) \rightarrow \mathcal{L}_{b,Y}(\mathcal{C}^{\text{an}}(\widehat{Z}, K), V)$ which is evidently Z -equivariant, and whose source and target are equal to U_{fs} and V_{fs} respectively, by Lemma 3.2.19. To show that $\tilde{\phi}$ is an isomorphism, it suffices to show that it is a bijection, since its source and target are of compact type.

Assumption (ii) implies that $\ker \phi \subset U^z$. Thus $\ker \tilde{\phi} \subset \mathcal{L}_{b,Y}(\mathcal{C}^{\text{an}}(\widehat{Z}, K), U^z) = (U^z)_{\text{fs}} = 0$ (the two equalities following from Lemmas 3.2.19 and 3.2.17 respectively). Thus $\tilde{\phi}$ is injective. If $f: \mathcal{C}^{\text{an}}(\widehat{Z}, K) \rightarrow V$ lies in V_{fs} , then define $\tilde{f}: \mathcal{C}^{\text{an}}(\widehat{Z}, K) \rightarrow U$ via $\tilde{f}(\alpha) = \psi \circ f(z^{-1}\alpha)$. Since ψ is Y -equivariant, one sees that \tilde{f} is Y -equivariant, and hence lies in U_{fs} . Furthermore, the composite $\phi \circ \tilde{f}$ is equal to f . Thus $\tilde{\phi}$ is surjective. \square

Now let A be a K -Fréchet algebra, written as a projective limit $A \xrightarrow{\sim} \varprojlim_n A_n$, where each A_n is a compact type topological K -algebra, each of the transition maps $A_{n+1} \rightarrow A_n$ is compact, and the map $A \rightarrow A_n$ has dense image for each n . Suppose also that the projective system A_n is cofinal with a projective system of K -Banach algebras, so that A is a nuclear Fréchet algebra, in the sense of [8, def. 1.2.12].

Choose an exhaustive increasing sequence $\{\widehat{Z}_n\}_{n \geq 0}$ of admissible open affinoid subdomains of \widehat{Z} , with the property that each inclusion $Z_n \subset Z_{n+1}$ is relatively compact. There is then an isomorphism $\mathcal{C}^{\text{an}}(\widehat{Z}, K) \xrightarrow{\sim} \varprojlim_n \mathcal{C}^{\text{an}}(\widehat{Z}_n, K)^\dagger$ (where $\mathcal{C}^{\text{an}}(\widehat{Z}_n, K)^\dagger$ denotes the space of overconvergent (in \widehat{Z}) rigid analytic functions on \widehat{Z}_n). In fact, as already recalled, $\mathcal{C}^{\text{an}}(\widehat{Z}, K)$ is a nuclear Fréchet algebra. (The projective sequence $\{\mathcal{C}^{\text{an}}(\widehat{Z}_n, K)\}_{n \geq 0}$ of K -Banach algebras is cofinal with the projective sequence $\{\mathcal{C}^{\text{an}}(\widehat{Z}_n, K)^\dagger\}_{n \geq 0}$, and satisfies the conditions of [8, def. 1.2.12].) By [8, lem. 1.2.13] the tensor product $\mathcal{C}^{\text{an}}(\widehat{Z}, K) \widehat{\otimes}_K A$ is again a nuclear Fréchet algebra, and the proof of that lemma yields an isomorphism

$$(3.2.21) \quad \mathcal{C}^{\text{an}}(\widehat{Z}, K) \widehat{\otimes}_K A \rightarrow \varprojlim_n (\mathcal{C}^{\text{an}}(\widehat{Z}_n, K)^\dagger \widehat{\otimes}_K A_n).$$

(Here we are using the fact that the projective sequences $\{\mathcal{C}^{\text{an}}(\widehat{Z}, K)^\dagger\}_{n \geq 0}$ and $\{A_n\}_{n \geq 0}$ are both cofinal with projective sequences of Banach spaces.)

LEMMA 3.2.22. – *Let V be a convex K -vector space of compact type, equipped with an A -module structure for which the multiplication map $A \times V \rightarrow V$ is separately continuous. Suppose also that V is equipped with a topological Z^+ -action commuting with the given A -action, so that V is an object of $\text{Rep}_{\text{top.c}}(Z^+)$. Then the A -module structure on V_{fs} induced by functoriality makes V_{fs} an A -module, and the multiplication map $A \times V_{\text{fs}} \rightarrow V_{\text{fs}}$ is again separately continuous.*

Proof. – By [8, prop. 1.1.35], the action of A on V induces a continuous map $A \rightarrow \mathcal{L}_{b,Z^+}(V, V)$, which when composed with (3.2.2) yields a continuous map $A \rightarrow \mathcal{L}_{b,Z^+}(V_{\text{fs}}, V)$. Composing this map with the inverse of the isomorphism (3.2.5) gives a continuous map $A \rightarrow \mathcal{L}_{b,Z}(V_{\text{fs}}, V_{\text{fs}})$. The A -module structure induced on V_{fs} by this map is precisely the A -module structure on V induced by the functoriality of the construction of finite slope parts, and so the multiplication map $A \times V_{\text{fs}} \rightarrow V_{\text{fs}}$ is separately continuous, as claimed. \square

In the context of the preceding lemma, the A -module structure on V_{fs} induces a topological A -module structure on its strong dual $(V_{\text{fs}})'_b$ [8, prop. 1.2.14]. Since V_{fs} is an object of $\text{Rep}_{\text{la.c}}^z(Z)$, the same reference shows that $(V_{\text{fs}})'_b$ is also a topological $\mathcal{C}^{\text{an}}(\widehat{Z}, K)$ -module. Thus $(V_{\text{fs}})'_b$ is in fact a topological $\mathcal{C}^{\text{an}}(\widehat{Z}, K) \widehat{\otimes}_K A$ -module.

PROPOSITION 3.2.23. – *In the above situation, suppose given the following data:*

- (i) *For each $n \geq 0$ a compact type topological A_n -module U_n , equipped with an A_n -linear action of Z^+ , as well as an $A_{n+1}[Z^+]$ -linear transition map $U_{n+1} \rightarrow U_n$, such that the induced $A_n[Z^+]$ -linear map $A_n \widehat{\otimes}_{A_{n+1}} U_{n+1} \rightarrow U_n$ is A_n -compact (in the sense of Definition 2.3.3);*
- (ii) *An element $z \in Z^+$, such that for each $n \geq 0$, the map $U_n \rightarrow U_n$ induced by z factors through the transition map $A_n \widehat{\otimes}_{A_{n+1}} U_{n+1} \rightarrow U_n$, so as to give a commutative diagram*

$$\begin{array}{ccc} A_n \widehat{\otimes}_{A_{n+1}} U_{n+1} & \longrightarrow & U_n \\ \text{id} \widehat{\otimes} z \downarrow & \swarrow & \downarrow z \\ A_n \widehat{\otimes}_{A_{n+1}} U_{n+1} & \longrightarrow & U_n \end{array}$$

- (iii) *An $A[Z^+]$ -equivariant isomorphism $V'_b \xrightarrow{\sim} \varprojlim_n U_n$.*

Then we may conclude that $(V_{\text{fs}})'_b$ is a coadmissible $\mathcal{C}^{\text{an}}(\widehat{Z}, K) \widehat{\otimes}_K A$ -module (in the sense of [8, def. 1.2.8]).

Proof. – The isomorphisms $V'_b \xrightarrow{\sim} \varprojlim_n U_n$ and $\mathcal{C}^{\text{an}}(\widehat{Z}, K) \xrightarrow{\sim} \varprojlim_n \mathcal{C}^{\text{an}}(\widehat{Z}_n, K)^\dagger$, together with [8, prop. 1.1.29] and Lemma 3.2.3, yield isomorphisms

$$(V_{\text{fs}})'_b \xrightarrow{\sim} \mathcal{C}^{\text{an}}(\widehat{Z}, K) \widehat{\otimes}_{K[Z^+]} V'_b \xrightarrow{\sim} \varprojlim_n \mathcal{C}^{\text{an}}(\widehat{Z}_n, K)^\dagger \widehat{\otimes}_{K[Z^+]} U_n.$$

Thus to prove the proposition, it suffices to prove that:

- (a) For each $n \geq 0$, the natural map

$$(3.2.24) \quad \mathcal{C}^{\text{an}}(\widehat{Z}_n, K)^\dagger \widehat{\otimes}_K A_n \widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{Z}_{n+1}, K)^\dagger \widehat{\otimes}_K A_{n+1}} \mathcal{C}^{\text{an}}(\widehat{Z}_{n+1}, K)^\dagger \widehat{\otimes}_{K[Z^+]} U_{n+1} \rightarrow \mathcal{C}^{\text{an}}(\widehat{Z}_n, K)^\dagger \widehat{\otimes}_{K[Z^+]} U_n$$

is an isomorphism;

- (b) $\mathcal{C}^{\text{an}}(\widehat{Z}_n, K)^\dagger \widehat{\otimes}_{K[Z^+]} U_n$ is a finitely generated $\mathcal{C}^{\text{an}}(\widehat{Z}_n, K)^\dagger \widehat{\otimes}_K A_n$ -module, for any n .

To prove (a), note that the map (3.2.24) may be written more simply as

$$(3.2.25) \quad \mathcal{C}^{\text{an}}(\widehat{Z}_n, K)^\dagger \widehat{\otimes}_{K[Z^+]} (A_n \widehat{\otimes}_{A_{n+1}} U_{n+1}) \rightarrow \mathcal{C}^{\text{an}}(\widehat{Z}_n, K)^\dagger \widehat{\otimes}_{K[Z^+]} U_n.$$

Let z be as in hypothesis (ii). Since the image of z in $\mathcal{C}^{\text{an}}(\widehat{Z}_n, K)^\dagger$ is invertible, Proposition 2.1.9 implies that (3.2.25), and hence (3.2.24), is an isomorphism.

To prove (b), let $z \in Z^+$ be as in hypothesis (ii). Since each \widehat{Z}_n is an affinoid domain, the rigid analytic functions on \widehat{Z}_n induced by each of z and z^{-1} are bounded, and so for $x \in K^\times$ with $\text{ord}_K(x)$ sufficiently large, the map $K[z, z^{-1}] \rightarrow \mathcal{C}^{\text{an}}(\widehat{Z}_n, K)^\dagger$ extends to a map $K\langle xz, xz^{-1} \rangle^\dagger \rightarrow \mathcal{C}^{\text{an}}(\widehat{Z}_n, K)^\dagger$. Hypotheses (i) and (ii) together imply that z acts as an A_n -compact operator on U_n , and thus Proposition 2.3.5(i) implies that (perhaps after increasing the lower bound on $\text{ord}_K(x)$) the operator xz is a power-bounded endomorphism of U_n , and hence that U_n is naturally a $K\langle xz \rangle^\dagger$ -module. For such values of x we obtain a surjection

$$\begin{aligned} \mathcal{C}^{\text{an}}(\widehat{Z}_n, K)^\dagger & \widehat{\otimes}_{K\langle xz, xz^{-1} \rangle^\dagger} K\langle xz, xz^{-1} \rangle^\dagger \widehat{\otimes}_{K\langle xz \rangle^\dagger} U_n \\ & \xrightarrow{\sim} \mathcal{C}^{\text{an}}(\widehat{Z}_n, K)^\dagger \widehat{\otimes}_{K\langle xz \rangle^\dagger} U_n \\ & \xrightarrow{\sim} \mathcal{C}^{\text{an}}(\widehat{Z}_n, K)^\dagger \widehat{\otimes}_{K[z]} U_n \rightarrow \mathcal{C}^{\text{an}}(\widehat{Z}_n, K)^\dagger \widehat{\otimes}_{K[Z^+]} U_n. \end{aligned}$$

(The second isomorphism follows from the remark following the proof of Proposition 2.3.5.) By Proposition 2.3.5(ii), together with [8, prop. 1.2.5], the source of this surjection is a finitely generated $\mathcal{C}^{\text{an}}(\widehat{Z}_n, K)^\dagger \widehat{\otimes}_K A_n$ -module (again after increasing the lower bound on $\text{ord}_K(x)$, if necessary), and thus so is the target. \square

We close this subsection by establishing a variant of Lemma 3.2.19. We first define a certain subcategory of $\text{Rep}_{\text{top.c}}(Z^+)$. By assumption, Z^+ contains an open subgroup Z_0 of Z . We let $\text{Rep}_{\text{la.c}}^z(Z^+)$ denote the full subcategory of $\text{Rep}_{\text{top.c}}(Z^+)$ consisting of objects which may be written as a union of an increasing sequence of Z^+ -invariant BH -subspaces, and on which Z_0 acts through a locally analytic representation (for one, or equivalently any, choice of such open subgroup Z_0).

Let us now fix a closed subgroup Y of Z , as well as a submonoid Y^+ of $Y \cap Z^+$, such that Y^+ contains an open subgroup of Y and generates Y as a group, and such that $YZ^+ = Z$. If V is an object of $\text{Rep}_{\text{top.c}}(Z^+)$, then by restricting the Z^+ -action to Y^+ , we may also regard V as an object of $\text{Rep}_{\text{top.c}}(Y^+)$. We write $V_{Z\text{-fs}}$ to denote the finite slope part of V as a Z^+ -representation, and write $V_{Y\text{-fs}}$ to denote the finite slope part of V as a Y^+ -representation. There is a natural Y -equivariant map

$$(3.2.26) \quad V_{Z\text{-fs}} \rightarrow V_{Y\text{-fs}}$$

(constructed either by a consideration of the universal property of $V_{Y\text{-fs}}$, or by considering the definitions directly, and restricting maps from $\mathcal{C}^{\text{an}}(\widehat{Z}, K)$ to $\mathcal{C}^{\text{an}}(\widehat{Y}, K)$). By functoriality, the Z^+ -action on V induces a Z^+ -action on $V_{Y\text{-fs}}$, and so $V_{Y\text{-fs}}$ is in fact naturally a $YZ^+ = Z$ -representation. The map (3.2.26) is clearly Z -equivariant.

PROPOSITION 3.2.27. – *If V is an object of $\text{Rep}_{\text{la.c}}^z(Z^+)$, then the map (3.2.26) is a Z -equivariant topological isomorphism.*

Proof. – Let W be an object of $\text{Rep}_{\text{la.c}}^z(Z)$. The universal property of $V_{Y\text{-fs}}$ yields an isomorphism $\mathcal{L}_Y(W, V_{Y\text{-fs}}) \xrightarrow{\sim} \mathcal{L}_{Y^+}(W, V)$, and hence (recalling that $Z = YZ^+$) an isomorphism $\mathcal{L}_Z(W, V_{Y\text{-fs}}) \xrightarrow{\sim} \mathcal{L}_{Z^+}(W, V)$. The proposition will follow provided that we prove that $V_{Y\text{-fs}}$ lies in $\text{Rep}_{\text{la.c}}^z(Z)$.

Proposition 3.2.7(i) shows that Z_0 acts locally analytically on $V_{Y\text{-fs}}$. By assumption (and [8, prop. 1.1.2]), we may write $V \cong \varinjlim_n V_n$, where each V_n is a K -Banach space equipped with a topological Z^+ -action. Also, we may write $\widehat{Y} = \bigcup_n \widehat{Y}_n$, where $\{\widehat{Y}_n\}$ is an increasing sequence of affinoid subdomains of \widehat{Y} , and hence write $\mathcal{C}^{\text{an}}(\widehat{Y}, K) \xrightarrow{\sim} \varinjlim_n \mathcal{C}^{\text{an}}(\widehat{Y}_n, K)$; each

$\mathcal{C}^{\text{an}}(\widehat{Y}_n, K)$ is a K -Banach space. From [3, prop. 1, p. I.20] we conclude that the natural map $\varinjlim_n \mathcal{L}_b(\mathcal{C}^{\text{an}}(\widehat{Y}_n, K), V_n) \rightarrow \mathcal{L}_b(\mathcal{C}^{\text{an}}(\widehat{Z}, K), V)$ is a Y -equivariant continuous bijection, and so we obtain a continuous bijection $\varinjlim_n \mathcal{L}_{b, Y^+}(\mathcal{C}^{\text{an}}(\widehat{Y}_n, K), V_n) \rightarrow V_{Y\text{-fs}}$. Since each of $\mathcal{C}^{\text{an}}(\widehat{Y}_n, K)$ and V_n is a Banach space, the same is true of the spaces $\mathcal{L}_{b, Y^+}(\mathcal{C}^{\text{an}}(\widehat{Y}_n, K), V_n)$. The action of Z^+ on V_n and of Y on $\mathcal{C}^{\text{an}}(\widehat{Y}_n, K)$ induces a topological action of $Z = YZ^+$ on $\mathcal{L}_{b, Y^+}(\mathcal{C}^{\text{an}}(\widehat{Y}_n, K), V_n)$. Thus $V_{Y\text{-fs}}$ is indeed an object of $\text{Rep}_{\text{la.c}}^z(Z)$. \square

The following lemma is helpful for verifying the hypothesis of the preceding proposition, in the context of Proposition 3.2.23.

LEMMA 3.2.28. – *If $V \in \text{Rep}_{\text{top.c}}(Z^+)$ may be written as the union of an increasing sequence of Z^+ -invariant FH-subspaces, then it may be written as the union of an increasing sequence of Z^+ -invariant BH-subspaces.*

Proof. – Appealing to [8, prop. 1.1.2], we may write $V \xrightarrow{\sim} \varinjlim_n W_n$ as the limit of an inductive sequence of Fréchet spaces, each equipped with a topological Z^+ -action, such that the transition maps $W_n \rightarrow W_{n+1}$ and the maps $W_n \rightarrow V$ are continuous, injective and Z^+ -equivariant. Since V is of compact type, it is in particular an LB -space, and so we may choose the sequence $\{W_n\}$ so that each transition map factors as $W_n \rightarrow X_n \rightarrow W_{n+1}$, where X_n is a Banach space. (See [8, props. 1.1.2 and 1.1.10].) Let Z_0 be a compact open subgroup of Z contained in Z^+ . The quotient Z^+/Z_0 is a submonoid of the finitely generated Abelian group Z/Z_0 , and so is itself a finitely generated monoid. Fix a subset S of Z^+ that is the union of Z_0 and a finite set whose image in Z^+/Z_0 is a generating set. Since Z_0 is compact, the set S acts equicontinuously on each of the spaces W_n .

Let A_{n+1} be an open neighbourhood of the origin of W_{n+1} , and choose an open neighbourhood B_{n+1} of the origin of W_{n+1} such that $SB_{n+1} \subset A_{n+1}$. If we let A_n (respectively B_n) denote the preimage in W_n of A_{n+1} (respectively B_{n+1}), then since the map $W_n \rightarrow W_{n+1}$ factors through the Banach space X_n , and is Z^+ -equivariant, the open subsets A_n and B_n of W_n are commensurable with one another (i.e. there exists $x \in K^\times$ such that $xA_n \subset B_n \subset x^{-1}A_n$), and $SB_n \subset A_n$. Thus if Y_n denotes the Banach space obtained by completing W_n with respect to the gauge of A_n (or equivalently, of B_n), then the S -action on W_n extends to an equicontinuous S -action on Y_n , and hence to a topological Z^+ -action. By construction, the map $W_n \rightarrow X_n$ factors through the Z^+ -equivariant map $W_n \rightarrow Y_n$. Thus we obtain an isomorphism $V \xrightarrow{\sim} \varinjlim_n Y_n$, where each Y_n is a Banach space equipped with a topological Z^+ -action. This proves the lemma. \square

In the context of Proposition 3.2.23, dualizing condition (iii) yields an isomorphism $V \xrightarrow{\sim} \varinjlim_n (U_n)'_b$. Since each U_n is of compact type, each of the strong duals $(U_n)'_b$ is a Fréchet space. Thus the hypothesis of Lemma 3.2.28 is satisfied.

3.3. We now return to the situation considered in the introduction. Fix a connected reductive linear algebraic group \mathbb{G} defined over L , as well as a parabolic subgroup \mathbb{P} of \mathbb{G} and a Levi factor $\mathbb{M} \subset \mathbb{P}$. Let \mathbb{N} denote the unipotent radical of \mathbb{P} , so that $\mathbb{P} = \mathbb{M}\mathbb{N}$. Write $G := \mathbb{G}(L)$, $P := \mathbb{P}(L)$, $M := \mathbb{M}(L)$, and $N := \mathbb{N}(L)$. Let $\mathbb{Z}_{\mathbb{G}}$ (respectively $\mathbb{Z}_{\mathbb{M}}$) denote the centre of \mathbb{G} (respectively \mathbb{M}), and write $Z_G := \mathbb{Z}_{\mathbb{G}}(L)$ (respectively $Z_M := \mathbb{Z}_{\mathbb{M}}(L)$), so that Z_G (respectively Z_M) is the centre of G (respectively M). Taking T to be Z_M in the discussion of Section 1.4, let Z_M^0 denote the maximal compact subgroup of Z_M , and choose a sublattice Λ of Z_M which splits the bottom row of (1.4.1).

LEMMA 3.3.1. – *If N_0 and N'_0 are two compact open subgroups of N , and if we write $\Lambda' := \{\lambda \in \Lambda \mid \lambda N'_0 \lambda^{-1} \subset N_0\}$, then Λ' generates Λ as a group.*

Proof. – This is a standard consequence of the theory of roots. Indeed, there exists a positive constant C such that if $\text{ord}_L(\alpha(\lambda)) > C$ for every positive restricted root α of Z_M , then $\lambda \in \Lambda'$. (See for example [5, prop. 1.4.3].) The subsemigroup of elements satisfying this inequality generates Λ as a group. \square

PROPOSITION 3.3.2. – *Let $\{(N_i, N'_i)\}_{1 \leq i \leq n}$ be any finite sequence of pairs of compact open subgroups of N .*

- (i) *If $Y = \{z \in Z_M \mid zN_iz^{-1} \subset N'_i \text{ for all } 1 \leq i \leq r\}$, then Y generates Z_M as a group.*
- (ii) *If $Y' = \{m \in M \mid mN_im^{-1} \subset N'_i \text{ for all } 1 \leq i \leq r\}$, then Y' and Z_M together generate M as a semigroup. In particular, Y' generates M as a group.*

Proof. – If N_0 denotes the compact open subgroup of N generated by all the N_i , and N'_0 denotes the intersection of all the N'_i , then it suffices to prove the proposition with the sequence of pairs $\{(N_i, N'_i)\}$ replaced by the single pair (N_0, N'_0) . Let N''_0 denote the compact open subgroup of N generated by the elements $z_0n_0z_0^{-1}$, as z_0 and n_0 range over all elements of Z_M^0 and N_0 respectively. Lemma 3.3.1 gives a subsemigroup Λ' of Λ that generates Λ as a group, such that $\lambda N''_0 \lambda^{-1} \subset N_0$ for $\lambda \in \Lambda'$. Then $\Lambda' Z_M^0 \subset Y$, and $\Lambda' Z_M^0$ generates Z_M as a group. This proves (i).

If $m \in M$, then Lemma 3.3.1 applied to the pair mN_0m^{-1} and N'_0 of compact open subgroups of N gives an element $z \in Z_M$ such that $z(mN_0m^{-1})z^{-1} \subset N'_0$. Thus $zm \in Y'$, and so $m \in z^{-1}Y' \subset Z_M Y'$. This proves (ii).

Since, by definition, $Y = Y' \cap Z_M \subset Y'$, it follows from (i) and (ii) together that Y' generates M as a group. \square

Fix a compact open subgroup P_0 of P , and write $M_0 := M \cap P_0$, $N_0 := N \cap P_0$, $M^+ := \{m \in M \mid mN_0m^{-1} \subset N_0\}$, and $Z_M^+ := M^+ \cap Z_M$. There are inclusions $M_0 \subset M^+$ and $Z_G \subset Z_M^+$.

COROLLARY 3.3.3. – *The Abelian group Z_M is generated by its submonoid Z_M^+ .*

Proof. – This is a special case of Proposition 3.3.2. \square

DEFINITION 3.3.4. – Let $M^+ \times_{Z_M^+} Z_M$ denote the quotient of the monoid $M^+ \times Z_M$ by the equivalence relation generated by the relations $(mz^+, z) \sim (m, z^+z)$, for any $m \in M^+$, $z^+ \in Z_M^+$ and $z \in Z_M$.

The morphism of monoids $M^+ \times Z_M \rightarrow M$ defined by $(m, z) \mapsto mz$ clearly factors through $M^+ \times_{Z_M^+} Z_M$, yielding a morphism

$$(3.3.5) \quad M^+ \times_{Z_M^+} Z_M \rightarrow M.$$

PROPOSITION 3.3.6. – *The morphism (3.3.5) is an isomorphism.*

Proof. – Suppose that (m_1, z_1) and (m_2, z_2) are two elements of the product $M^+ \times Z_M$, such that m_1z_1 and m_2z_2 are equal as elements of M . Corollary 3.3.3 implies that we may find an element $z \in Z_M^+$ such that $z_1z_2^{-1}z$ also lies in Z_M^+ , yielding the relations $(m_1, z_1) \sim (m_1z_1z_2^{-1}z, z^{-1}z_2) = (m_2z, z^{-1}z_2) \sim (m_2, z_2)$. Thus (m_1, z_1) and (m_2, z_2) are related by the kernel of the canonical map $M^+ \times Z_M \rightarrow M^+ \times_{Z_M^+} Z_M$, and (3.3.5) is injective. Proposition 3.3.2(ii) shows that M^+ and Z_M together generate M , and thus that (3.3.5) is surjective. \square

3.4. We are now ready to define our Jacquet module functors. We retain the notation of the previous subsection.

Let \mathfrak{n} denote the Lie algebra of N . If V is a locally analytic representation of P , then let $V^{\mathfrak{n}}$ denote the closed subspace of \mathfrak{n} -invariant elements of V . The subspace $V^{\mathfrak{n}}$ is P -invariant (since \mathfrak{n} is invariant under the adjoint action of P). In particular, the subgroup N of P acts smoothly on $V^{\mathfrak{n}}$ [8, cor. 4.1.7].

DEFINITION 3.4.1. – Let V be a locally analytic representation of P . If N_0 is any compact open subgroup of N , then we define the projection operator $\pi_{N_0} : V^{\mathfrak{n}} \rightarrow V^{N_0}$ as follows: $\pi_{N_0} v = \int_{N_0} nv \, dn$, for all $v \in V^{\mathfrak{n}}$. (The measure dn is Haar measure on N , normalized so that N_0 has measure one.)

More concretely, if $v \in V^{\mathfrak{n}}$, then v is fixed by some subgroup N'_0 of N_0 , and so $\pi_{N_0} v = [N_0 : N'_0]^{-1} \sum_{n \in N_0/N'_0} nv$.

We now fix a compact open subgroup P_0 of P , and define M_0, N_0, M^+ and Z_M^+ as they were defined preceding Corollary 3.3.3. That corollary shows that Z_M^+ generates Z_M as a group. Also, Z_M^+ contains the compact open subgroup $Z_M \cap P_0$ of Z_M .

DEFINITION 3.4.2. – If $m \in M^+$, define $\pi_{N_0, m} : V^{\mathfrak{n}} \rightarrow V^{N_0}$ as follows: $\pi_{N_0, m}(v) = \pi_{N_0}(mv)$, for all $v \in V^{\mathfrak{n}}$.

The alternative formula $\pi_{N_0, m}(v) = m\pi_{m^{-1}N_0m}(v)$ is also useful. Note that the restriction of $\pi_{N_0, m}$ to V^{N_0} induces an endomorphism of V^{N_0} .

LEMMA 3.4.3. – If $m \in M^+$, then the endomorphism $\pi_{N_0, m}$ of V^{N_0} is continuous.

Proof. – Since $m \in M^+$, we have an inclusion $mN_0m^{-1} \subset N_0$. The operator $\pi_{N_0, m}$ is defined by the formula $\pi_{N_0, m}(v) = \frac{1}{[N_0 : mN_0m^{-1}]} \sum_{n \in N_0/mN_0m^{-1}} nmv$, for any $v \in V^{N_0}$. Since P acts via continuous automorphisms of V , the lemma follows. \square

LEMMA 3.4.4. – If $m, m' \in M^+$, then $\pi_{N_0, m}\pi_{N_0, m'} = \pi_{N_0, mm'}$.

Proof. – Since $m \in M^+$ there is an inclusion $N_0 \subset m^{-1}N_0m$. Thus $\pi_{m^{-1}N_0m}\pi_{N_0} = \pi_{m^{-1}N_0m}$, and so we compute, for any $v \in V$,

$$\begin{aligned} \pi_{N_0, m}\pi_{N_0, m'}(v) &= m\pi_{m^{-1}N_0m}\pi_{N_0}(m'v) \\ &= m\pi_{m^{-1}N_0m}(m'v) = \pi_{N_0}(mm'v) = \pi_{N_0, mm'}(v), \end{aligned}$$

as claimed. \square

The preceding two lemmas show that the operators $\pi_{N_0, m}$ define a topological action of the monoid M^+ on V^{N_0} . In particular, we obtain a topological action of Z_M^+ on V^{N_0} . Suppose now that V is of compact type (so that V lies in the category $\text{Rep}_{\text{la.c}}(P)$ defined in Section 3.1). The closed subspace V^{N_0} of V is then also of compact type, and is equipped with a topological Z_M^+ -action. Thus it is an object of the category $\text{Rep}_{\text{top.c}}(Z_M^+)$, and following Definition 3.2.1 (with $Z = Z_M$ and $Z^+ = Z_M^+$), we may take its finite slope part.

DEFINITION 3.4.5. – For any object V of $\text{Rep}_{\text{la.c}}(P)$, we define $J_P(V) := (V^{N_0})_{\text{fs}}$.

Proposition 3.2.4 shows that V^{N_0} is an object of $\text{Rep}_{\text{la.c}}^z(Z_M)$, and so in particular is a convex K -vector space of compact type, equipped with a locally analytic representation of Z_M .

PROPOSITION 3.4.6. – If V is an object of $\text{Rep}_{\text{la.c}}(P)$, then the locally analytic Z_M -representation on $J_P(V)$ extends in a natural way to a locally analytic M -representation.

Proof. – Since Z_M^+ lies in the centre of M^+ , the M^+ -action on V^{N_0} induces an M^+ -action on $(V^{N_0})_{\text{fs}}$. Proposition 3.3.6 then implies that the M^+ -action and Z_M action on $(V^{N_0})_{\text{fs}}$ -action together determine an M -action on $(V^{N_0})_{\text{fs}}$. It remains to show that this action is locally analytic. Since M_0 is compact open in M , it suffices to show that the M_0 -action on $(V^{N_0})_{\text{fs}}$ is locally analytic.

The group M_0 normalizes N_0 . This implies that $M_0 \subset M^+$, and that for $m \in M_0$, the operator $\pi_{N_0, m}$ on V^{N_0} coincides with multiplication by m . Since V^{N_0} is a closed M_0 -invariant subspace of V , which is locally analytic as a P -representation, and so in particular as an M_0 -representation, we see that the M_0 -action on V^{N_0} is locally analytic. The operators $\pi_{N_0, m}$ commute with the action of Z_M^+ on V^{N_0} , and so Lemma 3.2.7(i) shows that the M_0 -action on $(V^{N_0})_{\text{fs}}$ is locally analytic. \square

Thus J_P defines a functor $\text{Rep}_{\text{la.c}}(P) \rightarrow \text{Rep}_{\text{la.c}}^z(M)$, which in Section 3.5 will be characterized as an adjoint functor. Since any object of $\text{Rep}_{\text{la.c}}(G)$ gives rise to an object of $\text{Rep}_{\text{la.c}}(P)$ by restricting the group action from G to P , we also obtain a functor $J_P : \text{Rep}_{\text{la.c}}(G) \rightarrow \text{Rep}_{\text{la.c}}^z(M)$, which in Section 4.2 will be shown to restrict to a functor $\text{Rep}_{\text{es}}(G) \rightarrow \text{Rep}_{\text{es}}(M)$. The remainder of this subsection is devoted to proving some more elementary properties of J_P .

LEMMA 3.4.7. –

- (i) *The functor J_P is additive.*
- (ii) *If $U \rightarrow V$ is a continuous injection in $\text{Rep}_{\text{la.c}}(P)$, then the morphism induced by functoriality, $J_P(U) \rightarrow J_P(V)$, is again injective.*
- (iii) *If $U \rightarrow V$ is a closed embedding in $\text{Rep}_{\text{la.c}}(P)$, then the morphism induced by functoriality, $J_P(U) \rightarrow J_P(V)$, is again a closed embedding. Furthermore, in this situation, the morphism $J_P(V)/J_P(U) \rightarrow J_P(V/U)$ induced by functoriality is a continuous injection.*
- (iv) *If $\{V_n\}_{n \geq 0}$ is an inductive sequence in the category $\text{Rep}_{\text{la.c}}(P)$, having injective transition maps, and if V denotes the locally convex inductive limit $V = \varinjlim_n V_n$ (an object of $\text{Rep}_{\text{la.c}}(P)$, by Lemma 3.1.2), then the natural map $\varinjlim_n J_P(V_n) \rightarrow J_P(V)$ is a topological isomorphism.*

Proof. – The additivity of the formation of N_0 -invariants is clear. If $U \rightarrow V$ is injective (respectively a closed embedding), then the same is true of the induced map $U^{N_0} \rightarrow V^{N_0}$. Also, if $V = \varinjlim_n V_n$, then $V^{N_0} = \varinjlim_n V_n^{N_0}$ (taking into account the fact that a continuous bijection between spaces of compact type is an isomorphism). The lemma now follows from Proposition 3.2.6. \square

If we compose the map $J_P(V) = (V^{N_0})_{\text{fs}} \rightarrow V^{N_0}$, provided by (3.2.2), with the inclusion $V^{N_0} \rightarrow V$, we obtain a natural map

$$(3.4.8) \quad J_P(V) \rightarrow V.$$

It is immediate from the construction that this map is $M_0 N_0$ -equivariant (where the compact open subgroup $M_0 N_0$ of P is regarded as acting on $J_P(V)$ through its quotient M_0). As will become clear in Section 4.3, this is a generalization of the so-called “canonical lifting” of [5, p. 40]. It also provides the map of property 0.9 stated in the introduction.

If χ is a locally analytic K -valued character of Z_M , and V is an object of $\text{Rep}_{\text{la.c}}(P)$, then we write

$$V^{N_0, Z_M^+ = \chi} = \{v \in V^{N_0} \mid \pi_{N_0, z} v = \chi(z)v \text{ for all } z \in Z_M^+\}.$$

As in the introduction, we let $J_P^\chi(V)$ denote the closed subrepresentation of $J_P(V)$ on which Z_M acts through the character χ .

PROPOSITION 3.4.9. – *If V is an object of $\text{Rep}_{\text{la.c.}}(P)$, then (3.4.8) induces an isomorphism $J_P^\chi(V) \xrightarrow{\sim} V^{N_0, Z_M^+ = \chi}$.*

Proof. – This follows from Proposition 3.2.12. \square

We now determine the extent to which the functor J_P depends on the choice of the Levi factor M and the open subgroup P_0 of P .

If M' is another lift of M to a Levi subgroup of P , then we may write $M' = nMn^{-1}$ for some (uniquely determined) $n \in N$. If we set $N'_0 = nN_0n^{-1}$ and $(M')^+ = nM^+n^{-1}$, then $(M')^+ = \{m' \in M' \mid m'N'_0(m')^{-1} \subset N'_0\}$. Thus for any object V of $\text{Rep}_{\text{la.c.}}(P)$, we may define Hecke operators $\pi_{N'_0, m'} : V^{N'_0} \rightarrow V^{N'_0}$ for $m' \in (M')^+$, and so regard $V^{N'_0}$ as an $(M')^+$ -module.

Let us use the isomorphism given by conjugation by n to identify M and M' (and hence M^+ and $(M')^+$). (This is the same identification that arises by identifying each of M and M' with the Levi quotient of P .)

PROPOSITION 3.4.10. – *For any object V of $\text{Rep}_{\text{la.c.}}(P)$, multiplication by n induces an isomorphism of M^+ -modules $V^{N_0} \xrightarrow{\sim} V^{N'_0}$, and hence an isomorphism $(V^{N_0})_{\text{fs}} \xrightarrow{\sim} (V^{N'_0})_{\text{fs}}$ of objects in $\text{Rep}_{\text{la.c.}}(M)$.*

Proof. – This is immediate. \square

Proposition 3.4.10 shows that the functor J_P is independent (up to a natural isomorphism) of the choice of Levi factor of P used in its construction. We next show that it is similarly independent of the choice of compact open subgroup P_0 .

PROPOSITION 3.4.11. – *If P'_0 is an open subgroup of P_0 , and if $N'_0 = N \cap P'_0$, then the projection $\pi_{N_0} : V^{N'_0} \rightarrow V^{N_0}$ induces an M -equivariant topological isomorphism $(V^{N'_0})_{\text{fs}} \xrightarrow{\sim} (V^{N_0})_{\text{fs}}$.*

Proof. – Choose $z \in Z_M^+$ such that $zN_0z^{-1} \subset N'_0$, and write $\phi := \pi_{N_0} : V^{N'_0} \rightarrow V^{N_0}$, $\psi := \pi_{N'_0, z} : V^{N_0} \rightarrow V^{N'_0}$. Let $(M^+)'$ denote the submonoid of M that conjugates N'_0 into itself, and write $Y' = M^+ \cap (M^+)'$. One checks easily that ϕ and ψ intertwine the Y' -action on each of $V^{N'_0}$ and V^{N_0} . (Here $m \in Y'$ acts on these spaces through the operators $\pi_{N'_0, m}$ and $\pi_{N_0, m}$ respectively.) Also, $\psi \circ \phi = \pi_{N'_0, z}$, while $\phi \circ \psi = \pi_{N_0, z}$. Proposition 3.3.2(i) shows that $Y := Y' \cap Z_M$ generates Z_M as a group, and thus, by Proposition 3.2.20, we see that ϕ induces a Z_M -equivariant isomorphism $(V^{N'_0})_{\text{fs}} \xrightarrow{\sim} (V^{N_0})_{\text{fs}}$ that intertwines the Y' -action on its source and target. Since Proposition 3.3.2(ii) shows that Y' generates M as a group, this isomorphism is in fact M -equivariant. \square

Although the preceding results show that the functor J_P is determined, up to a natural isomorphism, independently of any particular choice of Levi factor and compact open subgroup of P , the “canonical lifting” (3.4.8) does in general depend on these choices. (In light of this, the term “canonical lifting” is a little misleading.)

Of course, there is one obvious example in which the construction of J_P , and hence also the canonical lifting, does not require any choices.

PROPOSITION 3.4.12. – *Suppose that V is an object of $\text{Rep}_{\text{la.c.}}^{\tilde{z}}(G)$. (For example, V could be an object of $\text{Rep}_{\text{es}}(G)$.) Taking P to be G , the morphism $J_G(V) \rightarrow V$ provided by (3.4.8) is a topological isomorphism.*

Proof. – In this case $N_0 = N = 1$, $Z_G^+ = Z_G$, and Lemma 3.2.8 gives the result. \square

We finish this subsection by establishing property 0.8 of the introduction.

PROPOSITION 3.4.13. – *The diagram of 0.8 is commutative.*

Proof. – Although M^+ is not a group, it contains the compact open subgroup $M_0 = M \cap P_0$. Thus we write $\text{Rep}_{\text{la.c.}}(M^+)$ to denote the category of convex K -vector spaces of compact type, equipped with a topological action of M^+ , which are locally analytic as representations of M_0 . Note that $Z_M^+ \times \mathfrak{z}(\mathfrak{m})$ acts as symmetries of the category $\text{Rep}_{\text{la.c.}}(M^+)$.

The functor J_P is the composite of two functors: the functor $V \mapsto V^{N_0}$, mapping $\text{Rep}_{\text{la.c.}}(G)$ to $\text{Rep}_{\text{la.c.}}(M^+)$, and the functor $W \mapsto W_{\text{fs}}$, mapping $\text{Rep}_{\text{la.c.}}(M^+)$ to $\text{Rep}_{\text{la.c.}}(M)$. Thus to verify that the diagram of 0.8 commutes, it suffices to verify the commutativity of the diagrams

$$(3.4.14) \quad \begin{array}{ccc} Z_G \times \mathfrak{z}(\mathfrak{g}) & \longrightarrow & Z_M^+ \times \mathfrak{z}(\mathfrak{m}) \\ \downarrow & & \downarrow \\ \text{Aut}(V) \times \text{End}(V) & \longrightarrow & \text{Aut}(V^{N_0}) \times \text{End}(V^{N_0}) \end{array}$$

(where V is any object of $\text{Rep}_{\text{la.c.}}(G)$, and the bottom horizontal arrow is induced by the functoriality of the formation of N_0 -invariants), and

$$(3.4.15) \quad \begin{array}{ccc} Z_M^+ \times \mathfrak{z}(\mathfrak{m}) & \longrightarrow & Z_M \times \mathfrak{z}(\mathfrak{m}) \\ \downarrow & & \downarrow \\ \text{Aut}(W) \times \text{End}(W) & \longrightarrow & \text{Aut}(W_{\text{fs}}) \times \text{End}(W_{\text{fs}}) \end{array}$$

(where W is any object of $\text{Rep}_{\text{la.c.}}(M^+)$, and the bottom arrow is induced by the functoriality of the formation of finite slope parts). Here we are taking into account the fact that the map $Z_G \rightarrow Z_M$ factors through the inclusion $Z_M^+ \rightarrow Z_M$.

Let us begin by verifying the commutativity of (3.4.14). This diagram is the direct product of two diagrams, the first involving Z_G and Z_M^+ , and the second involving $\mathfrak{z}(\mathfrak{g})$ and $\mathfrak{z}(\mathfrak{m})$. We verify the commutativity of each factor separately.

If V is an object of $\text{Rep}_{\text{la.c.}}(G)$, then the action of Z_G on V preserves V^{N_0} . Thus for any $z \in Z_G$, the operator $\pi_{N_0, z}$ coincides with the action of z on V^{N_0} . Recalling that the action of Z_M^+ on V^{N_0} is defined via the operators $\pi_{N_0, z}$, the commutativity of the first factor of (3.4.14) follows.

Let $\gamma' : \mathfrak{z}(\mathfrak{g}) \rightarrow \mathfrak{z}(\mathfrak{m})$ denote the map of Proposition 1.3.2. If $X \in \mathfrak{z}(\mathfrak{g})$, then we may write $X = \gamma'(X) + X'$, with $X' \in U(\mathfrak{g})\mathfrak{n}$. If $v \in V^{N_0} \subset V^n$, then we see that $Xv = \gamma'(X)v$. Recalling that the $U(\mathfrak{m})$ -action, and hence the $\mathfrak{z}(\mathfrak{m})$ -action, on V^{N_0} is defined by differentiating the M_0 -action on this space, the commutativity of the second factor of (3.4.14) follows.

We now turn to establishing the commutativity of (3.4.15). Again, we regard this diagram as the product of two factors, and treat each factor separately. We must first show that the Z_M^+ -action on W_{fs} , induced by functoriality from the Z_M^+ -action on W , coincides with the Z_M^+ -action on W_{fs} obtained from regarding Z_M^+ as a submonoid of Z_M . This follows from the definition of W_{fs} as $\mathcal{L}_{b, Z_M^+}(\mathcal{C}^{\text{an}}(\widehat{Z}_M, K), W)$, or alternatively, from the universal property of W_{fs} .

Secondly, we must show that the $\mathfrak{z}(\mathfrak{m})$ -action on W_{fs} , induced by functoriality from the $\mathfrak{z}(\mathfrak{m})$ -action on W , coincides with the $\mathfrak{z}(\mathfrak{m})$ -action on W_{fs} , obtained by differentiating the M_0 -action on W_{fs} . This follows from Lemma 3.2.7(ii), and so we are done. \square

The preceding result implies in particular that when V is an object of $\text{Rep}_{\text{la.c.}}(G)$, the canonical lifting (3.4.8) is both Z_G and $\mathfrak{z}(\mathfrak{g})$ -equivariant (if Z_G and $\mathfrak{z}(\mathfrak{g})$ act on $J_P(V)$ through the maps $Z_G \rightarrow Z_M$ and $\mathfrak{z}(\mathfrak{g}) \rightarrow \mathfrak{z}(\mathfrak{m})$ considered above).

3.5. In this subsection we characterize J_P as an adjoint functor. We begin by describing the functor to which it is adjoint. As in the preceding section we fix a parabolic subgroup P of G , and a Levi factor M of P , so that $P = MN$.

For U an object of $\text{Rep}_{\text{la.c}} M$, let $\mathcal{C}_c^{\text{sm}}(N, U)$ denote the space of locally constant, compactly supported functions on N , with values in U . Since N is the union of its compact open subgroups, we may regard $\mathcal{C}_c^{\text{sm}}(N, U)$ as the inductive limit

$$(3.5.1) \quad \mathcal{C}_c^{\text{sm}}(N, U) \xleftarrow{\sim} \varinjlim_{N_0, N_1} \mathcal{F}(N_1/N_0, U),$$

where N_0 is an open subgroup of the compact open subgroup N_1 of N , and the inductive limit is taken as N_0 shrinks to the identity while N_1 grows without bound. The quotient N_1/N_0 is finite, and so $\mathcal{F}(N_1/N_0, U)$ (the space of U -valued functions on N_1/N_0) is isomorphic to a finite direct sum of copies of U . In this way, $\mathcal{C}_c^{\text{sm}}(N, U)$ is written as an inductive limit of finite direct sums of copies of U . The transition maps are obviously continuous and injective, and thus $\mathcal{C}_c^{\text{sm}}(N, U)$ may be regarded as an inductive limit with injective transition maps of spaces of compact type. In particular, it is naturally a space of compact type.

The right regular action of N on $\mathcal{C}_c^{\text{sm}}(N, U)$ equips this space with a smooth (and so locally analytic) action of N . We extend this to an action of $P = MN$ by defining the following action of an element mn (with $m \in M$ and $n \in N$) on an element f of $\mathcal{C}_c^{\text{sm}}(N, U)$:

$$(mnf)(n') = mf(m^{-1}n'mn)$$

(the right-hand side denoting the action of m on the element $f(m^{-1}n'mn)$ of U). One easily checks that this does yield an action of P on $\mathcal{C}_c^{\text{sm}}(N, U)$.

If $N_0 \subset N_1$ are compact open subgroups of N , then a sufficiently small open subgroup of M will leave both N_0 and N_1 invariant under conjugation. The description (3.5.1) of $\mathcal{C}_c^{\text{sm}}(N, U)$ and the fact that M acts locally analytically on U then implies that $\mathcal{C}_c^{\text{sm}}(N, U)$ is an object of $\text{Rep}_{\text{la.c}} P$. Its construction is clearly functorial in U , and so defines a functor $\text{Rep}_{\text{la.c}} M \rightarrow \text{Rep}_{\text{la.c}} P$. We will show that the functor J_P is right adjoint (up to a twist) to (the restriction to $\text{Rep}_{\text{la.c}}^z M$ of) the functor $\mathcal{C}_c^{\text{sm}}(N, -)$.

As usual, we denote by δ the modulus character of P , which describes the effect of right multiplication by elements of P on a left-invariant Haar measure of P . This is a smooth character that factors through the natural surjection $P \rightarrow M$; concretely, if $m \in M$, then $\delta(m)$ is equal to $[N_0 : mN_0m^{-1}]^{-1}$, for any compact open subgroup N_0 of N . For an object U of $\text{Rep}_{\text{la.c}} M$, we let $U(\delta)$ denote the object of $\text{Rep}_{\text{la.c}} M$ obtained by twisting the M -action on U by the character δ .

LEMMA 3.5.2. – *If U is an object of $\text{Rep}_{\text{la.c}}^z M$, then there is a natural isomorphism $J_P(\mathcal{C}_c^{\text{sm}}(N, U)) \xrightarrow{\sim} U(\delta)$.*

Proof. – Fix a compact open subgroup P_0 of P , with respect to which we will compute J_P , and let M_0, M^+, N_0, Z_M^+ and so on have the same meanings as in Section 3.4. Composing the map (3.4.8) (with V taken to be $\mathcal{C}_c^{\text{sm}}(N, U)$) with the map induced by evaluating at the identity of N yields a continuous map

$$(3.5.3) \quad J_P(\mathcal{C}_c^{\text{sm}}(N, U)) \rightarrow U.$$

We will show that this map is a topological isomorphism of vector spaces, that intertwines the action of M up to a twist by δ .

If u is an object of U , let f_u denote the element of $\mathcal{C}_c^{\text{sm}}(N, U)$ that is constant on N_0 , with value equal to u , and that vanishes elsewhere. Clearly f_u is invariant under N_0 , and one easily

computes that $\pi_{N_0, m}(f_u) = \delta(m)f_{mu} = f_{\delta(m)mu}$ for each $m \in M^+$. Thus the association of f_u to u induces an M^+ -equivariant closed embedding

$$(3.5.4) \quad U(\delta) \rightarrow (\mathcal{C}_c^{\text{sm}}(N, U))^{N_0},$$

and hence (taking into account Proposition 3.2.6(iii) and Lemma 3.2.8) an M -equivariant closed embedding

$$(3.5.5) \quad U(\delta) \rightarrow J_P(\mathcal{C}_c^{\text{sm}}(N, U)).$$

It is easily checked that (3.5.5) provides an inverse to (3.5.3), and hence we have obtained our required isomorphism. (It is easily seen that this isomorphism is independent of the choice of the compact open subgroup P_0 of P , up to the natural isomorphism provided by Proposition 3.4.11.) \square

We can now prove the main result of this subsection.

THEOREM 3.5.6. – *If V is an object of $\text{Rep}_{\text{la.c}} P$ and U is an object of $\text{Rep}_{\text{la.c}}^z M$, then passing to Jacquet modules yields a natural isomorphism*

$$(3.5.7) \quad \mathcal{L}_P(\mathcal{C}_c^{\text{sm}}(N, U), V) \xrightarrow{\sim} \mathcal{L}_M(U(\delta), J_P(V)).$$

Proof. – Lemma 3.5.2, and functoriality of the formation of Jacquet modules, yield the map (3.5.7). We first prove that (3.5.7) is surjective. As in the proof of Lemma 3.5.2, we fix P_0, N_0 and so on, and consider the corresponding canonical lifting $J_P(V) \rightarrow V^{N_0}$. If we are given $\phi \in \mathcal{L}_M(U(\delta), J_P(V))$, we may compose it with the canonical lifting to obtain a map $\tilde{\phi}: U(\delta) \rightarrow V^{N_0}$, which intertwines the action of M^+ on the source with the action of the Hecke operators $\pi_{N_0, m}$ (for $m \in M^+$) on the target. For any $z \in Z_M^+$ and $f \in \mathcal{C}^{\text{sm}}(N_0, U)$ define $I_z(f) \in V$ via

$$I_z(f) = \delta(z)^{-1} \int_N n^{-1} z \tilde{\phi}(z^{-1} f(n)) dn.$$

(Here z^{-1} acts on the values $f(n)$ via the action of $Z_M \subset M$ on U , and dn is Haar measure on N , normalized to give N_0 measure one.) One checks (using the equivariance property of $\tilde{\phi}$, and the fact that it takes values in V^{N_0}) that if $z \in Z_M^+$ is chosen such that f is locally constant on the left cosets of zN_0z^{-1} in N , then $I_z(f) = I_{zz'}(f)$ for any $z' \in Z_M^+$. Thus the net $I_z(f)$ (indexed by the elements of Z_M^+ , directed by the relation of divisibility) is eventually constant, and in particular the limit $I(f) := \varinjlim_{z \in Z_M^+} I_z(f)$ exists. The map $f \mapsto I(f)$ is easily seen to yield an element of $\mathcal{L}_P(\mathcal{C}_c^{\text{sm}}(N, U), V)$, which yields the given map ϕ upon passage to Jacquet modules. Thus (3.5.7) is surjective.

We turn to proving that (3.5.7) is injective. Let P^+ denote the product $N_0M^+ \subset P$, and let $\mathcal{C}^{\text{sm}}(N_0, U)$ denote the subspace of $\mathcal{C}_c^{\text{sm}}(N, U)$ consisting of locally constant U -valued functions on N_0 . One sees that P^+ is a subsemigroup of P , that the P -action on $\mathcal{C}_c^{\text{sm}}(N, U)$ restricts to a P^+ -action on $\mathcal{C}^{\text{sm}}(N_0, U)$, and that the embedding $\mathcal{C}^{\text{sm}}(N_0, U) \rightarrow \mathcal{C}_c^{\text{sm}}(N, U)$ induces an isomorphism

$$K[N] \otimes_{K[N_0]} \mathcal{C}^{\text{sm}}(N_0, U) \xrightarrow{\sim} \mathcal{C}_c^{\text{sm}}(N, U),$$

and so also (since M^+ generates M as a group) an isomorphism

$$(3.5.8) \quad \mathcal{L}_P(\mathcal{C}_c^{\text{sm}}(N, U), V) \xrightarrow{\sim} \mathcal{L}_{P^+}(\mathcal{C}^{\text{sm}}(N_0, U), V).$$

The map (3.5.4) induces a map

$$(3.5.9) \quad \mathcal{L}_{P^+}(\mathcal{C}^{\text{sm}}(N_0, U), V) \rightarrow \mathcal{L}_{M^+}(U(\delta), V^{N_0}).$$

Composing this on the left with (3.5.8), and on the right with the isomorphism

$$\mathcal{L}_{M^+}(U(\delta), V^{N_0}) \xrightarrow{\sim} \mathcal{L}_M(U(\delta), J_P(V))$$

induced by the adjointness property of $(V^{N_0})_{\text{fs}}$, we obtain (3.5.7). Thus to prove the injectivity of (3.5.7), it suffices to show that (3.5.9) is injective. This follows from the fact that the constant functions on N_0 obtained as the image of (3.5.4) clearly generate $\mathcal{C}^{\text{sm}}(N_0, U)$ as a P^+ -representation. \square

The adjointness isomorphism of the preceding theorem does not depend on the choice of the compact open subgroup P_0 of P that is used to compute the functor J_P , since this is true of the isomorphism of Lemma 3.5.2. However, it does depend on the choice of the Levi factor M of P (or equivalently, on the particular choice of \overline{P}), since the very definition of the functor $\mathcal{C}_c^{\text{sm}}(N, -)$ depends on this choice.

If we take $U = J_P(V)$ in Theorem 3.5.6 then the identity map of $J_P(V)$ to itself gives rise to a P -equivariant morphism

$$(3.5.10) \quad \mathcal{C}_c^{\text{sm}}(N, J_P(V)) \rightarrow V.$$

If we fix a compact open subgroup P_0 of P , then the constant $J_P(V)$ -valued functions on N_0 form an M_0N_0 -invariant subspace of the source of (3.5.10), which is isomorphic to $J_P(V)$ itself (regarded as an M_0N_0 -representation by having this product act through its quotient M_0). Thus (3.5.10) induces an M_0N_0 -equivariant map $J_P(V) \rightarrow V$, which is immediately seen to be the canonical lifting (3.4.8).

4. Some properties of the Jacquet module functor

4.1. This subsection is preliminary to the proof of Theorem 0.5. We first recall some constructions (and the attendant terminology and notation) introduced in [8, §§3.5, 5.2], and then apply them in the context of our reductive group G .

We begin with a discussion that applies to an arbitrary locally L -analytic group, and so for the moment we let G denote any such group. Suppose that (ϕ, H, \mathbb{H}) is a locally analytic chart of G —thus H is a compact open subset of G , \mathbb{H} is an affinoid rigid analytic space over L isomorphic to a closed ball, and ϕ is a locally analytic isomorphism $\phi: H \xrightarrow{\sim} \mathbb{H}(L)$ —with the additional property that H is a subgroup of G . Since H is Zariski dense in \mathbb{H} , there is at most one rigid analytic group structure on \mathbb{H} giving rise to the group structure on H . If such a rigid analytic group structure exists on \mathbb{H} , we will refer to the chart (ϕ, H, \mathbb{H}) as an analytic open subgroup of G . Usually, we will suppress the isomorphism ϕ , and simply refer to an analytic open subgroup H of G , and write \mathbb{H} to denote the corresponding rigid analytic group determined by H .

The analytic open subgroups of G form a directed set in an obvious fashion: if $H' \subset H$ is an inclusion of open subgroups of G each of which is equipped with the structure of an analytic open subgroup of G , then we say that it is an inclusion of analytic open subgroups if it lifts to a rigid analytic map $\mathbb{H}' \rightarrow \mathbb{H}$ on the associated rigid analytic groups. (Since H' and H are Zariski dense in \mathbb{H}' and \mathbb{H} respectively, such a lift is uniquely determined, if it exists, and is automatically a homomorphism of rigid analytic groups.) Forgetting the chart structure yields

an order-preserving map from the directed set of analytic open subgroups of G to the set of all open subgroups of G . Since the group structure on G is locally analytic, the image of this map is cofinal in the directed set of all open subgroups of G .

We now introduce a particular class of analytic open subgroups of G (the “good analytic open subgroups”), obtained by applying the Campbell–Baker–Hausdorff formula to certain \mathcal{O}_L -Lie subalgebras of \mathfrak{g} . We begin with some terminology. By an \mathcal{O}_L -lattice in \mathfrak{g} we mean an \mathcal{O}_L -submodule \mathfrak{h} of \mathfrak{g} which is free of finite rank as an \mathcal{O}_L -module, and which spans \mathfrak{g} over L . The gauge of \mathfrak{h} defines a norm on \mathfrak{g} , and we denote by \mathbb{H} the rigid analytic closed unit ball in the affine space underlying \mathfrak{g} defined by this norm (so by definition we have $\mathfrak{h} = \mathbb{H}(L)$). By a Lie sublattice of \mathfrak{g} , we mean an \mathcal{O}_L -lattice \mathfrak{h} in \mathfrak{g} that is also an \mathcal{O}_L -Lie subalgebra of \mathfrak{g} . Note that if \mathfrak{h} is any \mathcal{O}_L -lattice in \mathfrak{g} , then $a\mathfrak{h}$ is a Lie sublattice of \mathfrak{g} for any $a \in K^\times$ of sufficiently large valuation. It follows from [22, LG Ch. V, §4] (or more precisely, from the proof of that result) that even more is true: if $a \in K^\times$ has sufficiently large valuation, then not only is $a\mathfrak{h}$ a Lie subalgebra of \mathfrak{g} , but the Baker–Campbell–Hausdorff formula converges on \mathfrak{h} and so defines a rigid analytic group structure on \mathbb{H} . In particular, \mathfrak{g} admits a basis of neighbourhoods of the origin consisting of Lie sublattices \mathfrak{h} for which the Baker–Campbell–Hausdorff formula defines a rigid analytic group structure on \mathbb{H} .

If \mathfrak{h} is such a Lie sublattice, giving rise to the rigid analytic group \mathbb{H} , we write H to denote the locally L -analytic group underlying \mathbb{H} (so $H = \mathbb{H}(L) = \mathfrak{h}$ as sets). Assuming that \mathfrak{h} is sufficiently small, it is proved in [22, LG 5.35, cor. 2] that we may construct an embedding of locally L -analytic groups $\exp: H \rightarrow G$, and thus realize H as an analytic open subgroup of G . We refer to analytic open subgroups of G constructed in this manner as good analytic open subgroups. Our discussion shows that the set of good analytic open subgroups is cofinal in the directed set of all analytic open subgroups of G . Furthermore, if H' and H are two good analytic open subgroups of G , corresponding to Lie sublattices \mathfrak{h}' and \mathfrak{h} of \mathfrak{g} respectively, then $H' \subset H$ if and only if $\mathfrak{h}' \subset \mathfrak{h}$, and if these equivalent conditions hold, then the inclusion $H' \subset H$ is necessarily an inclusion of analytic open subgroups.

We continue to let \mathfrak{h} denote a Lie sublattice of \mathfrak{g} corresponding to a good analytic open subgroup H of G . For any real number $0 < r < 1$, we let \mathbb{H}_r denote the rigid analytic closed ball of radius r in \mathfrak{g} with respect to the gauge of \mathfrak{h} . The open subdomain \mathbb{H}_r of \mathfrak{h} is in fact a rigid analytic subgroup of \mathbb{H} , and if r equals the absolute value of an element of the algebraic closure \overline{L} of L , then \mathbb{H}_r is furthermore affinoid [2, thm. 6.1.5/4]. (Note that such elements r are dense in the interval $(0, 1)$.) We let H_r denote the locally analytic group of L -valued points of \mathbb{H}_r . If we write $\mathbb{H}^\circ = \bigcup_{0 < r < 1} \mathbb{H}_r$, then \mathbb{H}° is a strictly σ -affinoid rigid analytic open subgroup of \mathbb{H} (i.e. it is the union of an increasing sequence of open affinoid subdomains, each relatively compact in the next).

For any element $g \in G$, the conjugate gHg^{-1} is naturally a good analytic open subgroup of G , corresponding to the Lie sublattice $\text{Ad}_g(\mathfrak{h})$. We denote the rigid analytic group that underlies gHg^{-1} by $g\mathbb{H}g^{-1}$. Conjugation by g induces an isomorphism of rigid analytic groups $\mathbb{H} \xrightarrow{\sim} g\mathbb{H}g^{-1}$. In particular, we see that $gHg^{-1} \subset H$ if and only if $\text{Ad}_g(\mathfrak{h}) \subset \mathfrak{h}$, and that if these equivalent conditions hold, then the action of g on H by conjugation extends to a rigid analytic endomorphism of \mathbb{H} , and also of the rigid analytic groups \mathbb{H}_r , for each $r \in (0, 1)$, and of \mathbb{H}° .

We now introduce some notation similar to that introduced in Sections 5.3 and 6.1 of [8]. Let us fix for the moment a compact open subgroup G_0 of G . The space $\mathcal{C}^{1a}(G_0, K)$ is a compact type space equipped with a locally analytic action of G_0 . (In fact, $\mathcal{C}^{1a}(G_0, K)$ is the most fundamental example of a strongly admissible locally analytic G_0 -representation.) Let H be a good analytic open subgroup of G , contained in G_0 , such that G_0 normalizes H . The discussion of the preceding paragraph implies that the conjugation action of G_0 on H extends to an action by rigid analytic automorphisms on \mathbb{H}_r , for each $r \in (0, 1)$, and of \mathbb{H}° . (We say that

G_0 “normalizes” these rigid analytic groups.) Following [8, def. 3.4.1], we define

$$(4.1.1) \quad \mathcal{C}^{\text{la}}(G_0, K)_{\mathbb{H}^\circ - \text{an}} = \varinjlim_r \mathcal{C}^{\text{la}}(G_0, K)_{\mathbb{H}_r - \text{an}} \xrightarrow{\sim} \varinjlim_r \bigoplus_{g \in G_0/H_r} \mathcal{C}^{\text{an}}(g\mathbb{H}_r, K).$$

(Here r ranges through those elements of the interval $(0, 1)$ that are absolute values of elements of \overline{L} . If $g \in G_0$, then we let $g\mathbb{H}_r$ denote the coset gH_r , endowed with the structure of a rigid analytic space via the locally L -analytic isomorphism $H_r \xrightarrow{\sim} gH_r$ obtained via multiplication by g , together with the equality $H_r = \mathbb{H}_r(L)$. The second isomorphism then follows from the natural isomorphism $\mathcal{C}^{\text{la}}(G_0, K) \xrightarrow{\sim} \bigoplus_{g \in G_0/H_r} \mathcal{C}^{\text{la}}(gH_r, K)$ together with [8, cor. 3.3.26].) Since the inclusion $\mathbb{H}_r \subset \mathbb{H}_{r'}$ is relatively compact if $r < r'$, we see that $\mathcal{C}^{\text{la}}(G_0, K)_{\mathbb{H}^\circ - \text{an}}$ is the projective limit of K -Banach spaces under compact transition maps, and so is a nuclear Fréchet space. (See [8, prop. 2.1.16], for example.) Since G_0 normalizes each \mathbb{H}_r , we see that $\mathcal{C}^{\text{la}}(G_0, K)_{\mathbb{H}^\circ - \text{an}}$ is naturally equipped with a continuous G_0 -action.

If $D(\mathbb{H}^\circ, G_0)$ denotes the strong dual to $\mathcal{C}^{\text{la}}(G_0, K)_{\mathbb{H}^\circ - \text{an}}$, then $D(\mathbb{H}^\circ, G_0)$ is a locally convex topological K -vector space of compact type. In fact the definition of $\mathcal{C}^{\text{la}}(G_0, K)_{\mathbb{H}^\circ - \text{an}}$ induces an isomorphism

$$(4.1.2) \quad D(\mathbb{H}^\circ, G_0) := (\mathcal{C}^{\text{la}}(H, K)_{\mathbb{H}^\circ - \text{an}})'_b \xrightarrow{\sim} \varinjlim_r \left(\bigoplus_{g \in G_0/H_r} \mathcal{C}^{\text{an}}(g\mathbb{H}_r, K) \right)'_b \\ \xrightarrow{\sim} \varinjlim_r \bigoplus_{g \in G_0/H_r} \delta_g * \mathcal{D}^{\text{an}}(\mathbb{H}_r, K),$$

where $\mathcal{D}^{\text{an}}(\mathbb{H}_r, K)$ denotes the strong dual to $\mathcal{C}^{\text{an}}(\mathbb{H}_r, K)$, and δ_g denotes the δ -function supported at g . Each of the spaces $\mathcal{D}^{\text{an}}(\mathbb{H}_r, K)$ is naturally a K -Banach algebra [8, cor. 5.1.8]. Since G_0 normalizes each \mathbb{H}_r , each of the direct sums $\bigoplus_{g \in G_0/H_r} \delta_g * \mathcal{D}^{\text{an}}(\mathbb{H}_r, K)$ also has a natural structure of K -Banach algebra. Finally, the space $D(\mathbb{H}^\circ, G_0)$ is realized as the inductive limit of these topological K -algebras under continuous K -algebra maps, and so is again a topological K -algebra. (The multiplication map on $D(\mathbb{H}^\circ, G_0)$, while *a priori* separately continuous, is in fact jointly continuous, since $D(\mathbb{H}^\circ, G_0)$ is of compact type [8, prop. 1.1.31].)

We now return to the setting of the introduction, in which G is supposed to be the group of L -valued points of a connected reductive linear algebraic group \mathbb{G} . We will exploit the reductive group structure of G to show that G admits a neighbourhood basis of the identity consisting of a descending sequence $\{H_n\}_{n \geq 0}$ of good analytic open subgroups with certain additional nice properties. Among these is the existence of certain rigid analytic Iwahori decompositions for each of the groups H_n . We begin by explaining what we mean by this.

Let H_n be a good analytic open subgroup belonging to our sort-for neighbourhood basis, let \mathbb{P} be a parabolic subgroup of \mathbb{G} , let $\overline{\mathbb{P}}$ be a parabolic subgroup opposite to \mathbb{P} , and let $\mathbb{M} = \mathbb{P} \cap \overline{\mathbb{P}}$ be the common Levi factor of each of \mathbb{P} and $\overline{\mathbb{P}}$. Let \mathbb{N} (respectively $\overline{\mathbb{N}}$) denote the unipotent radical of \mathbb{P} (respectively $\overline{\mathbb{P}}$). Write P, \overline{P}, M, N , and \overline{N} to denote the corresponding groups of L -valued points. Set $M_n := H_n \cap M$, $N_n := H_n \cap N$, and $\overline{N}_n := H_n \cap \overline{N}$, and let $\mathbb{M}_n, \mathbb{N}_n$, and $\overline{\mathbb{N}}_n$ denote the rigid analytic Zariski closure of M_n, N_n , and \overline{N}_n in \mathbb{H}_n (the rigid analytic group underlying the good analytic subgroup H_n). Let \mathfrak{h}_n denote the Lie sublattice of \mathfrak{g} corresponding to H_n , and write $\mathfrak{m}_n := \mathfrak{h}_n \cap \mathfrak{m}$, $\mathfrak{n}_n := \mathfrak{h}_n \cap \mathfrak{n}$, and $\overline{\mathfrak{n}}_n := \mathfrak{h}_n \cap \overline{\mathfrak{n}}$.

DEFINITION 4.1.3. – We will say that the good analytic open subgroup H_n admits a rigid analytic Iwahori decomposition with respect to \mathbb{P} and $\overline{\mathbb{P}}$ if:

- (i) Under the identification $H_n = \mathfrak{h}_n$, the groups M_n, N_n , and \overline{N}_n are identified with $\mathfrak{m}_n, \mathfrak{n}_n$, and $\overline{\mathfrak{n}}_n$ respectively. Thus M_n, N_n , and \overline{N}_n coincide with the good analytic open

subgroups of M , N , and \overline{N} obtained by exponentiating \mathfrak{m}_n , \mathfrak{n}_n , and $\overline{\mathfrak{n}}_n$, respectively, and \mathbb{M}_n , \mathbb{N}_n , and $\overline{\mathbb{N}}_n$ coincide with the rigid analytic groups underlying M_n , N_n , and \overline{N}_n , respectively.

(ii) The rigid analytic map

$$(4.1.4) \quad \overline{\mathbb{N}}_n \times \mathbb{M}_n \times \mathbb{N}_n \rightarrow \mathbb{H}_n$$

induced by multiplication in \mathbb{H}_n is in fact a rigid analytic isomorphism.

Note that if (4.1.4) is an isomorphism, then the corresponding map

$$(4.1.5) \quad \overline{N}_n \times M_n \times N_n \rightarrow H_n$$

obtained by passing to L -valued points is also an isomorphism.

We fix a minimal parabolic subgroup \mathbb{P}_θ of \mathbb{G} , as well as a maximal split torus \mathbb{A}_θ in \mathbb{P}_θ . The centralizer \mathbb{M}_θ of \mathbb{A}_θ is a Levi factor of \mathbb{P}_θ . Denote by $\overline{\mathbb{P}}_\theta$ the opposite parabolic to \mathbb{P}_θ , chosen so that $\mathbb{M}_\theta := \mathbb{P}_\theta \cap \overline{\mathbb{P}}_\theta$.

The following result refines [5, prop. 1.4.4].

PROPOSITION 4.1.6. – *We may find a decreasing sequence $\{H_n\}_{n \geq 0}$ of good analytic open subgroups of G , cofinal in the directed set of all analytic open subgroups of G , and satisfying the following conditions:*

- (i) *For each $n \geq 0$, the inclusion $H_{n+1} \subset H_n$ extends to a relatively compact rigid analytic map $\mathbb{H}_{n+1} \subset \mathbb{H}_n$.*
- (ii) *For each $n \geq 0$, the subgroup H_n of H_0 is normal. The remaining properties refer to any pair \mathbb{P} and $\overline{\mathbb{P}}$ of opposite parabolic subgroups of \mathbb{G} , chosen so that \mathbb{P} contains \mathbb{P}_θ and $\overline{\mathbb{P}}$ contains $\overline{\mathbb{P}}_\theta$. We employ the notation introduced in the discussion preceding Definition 4.1.3.*
- (iii) *Each H_n admits a rigid analytic Iwahori decomposition with respect to \mathbb{P} and $\overline{\mathbb{P}}$.*
- (iv) *If $z \in Z_M$ is such that $z^{-1}\overline{N}_0z \subset \overline{N}_0$, then $z^{-1}\overline{N}_nz \subset \overline{N}_n$ for each $n \geq 0$.*
- (v) *If $z \in Z_M$ is such that $zN_0z^{-1} \subset N_0$, then $zN_nz^{-1} \subset N_n$ for each $n \geq 0$.*
- (vi) *We may find $z \in Z_M$ such that $z^{-1}\overline{N}_0z \subset \overline{N}_0$ and $zN_0z^{-1} \subset N_0$, and such that, for each $n \geq 0$, the embedding of part (iv) factors through the inclusion $\overline{N}_{n+1} \subset \overline{N}_n$.*

Proof. – Let \mathfrak{a}_θ denote the Lie algebra of \mathbb{A}_θ , let \mathfrak{m}_θ denote Lie algebra of \mathbb{M}_θ , and write $\mathfrak{g} = \mathfrak{m}_\theta \oplus \bigoplus_{\alpha \in \pm \Delta(\mathbb{G}, \mathbb{A}_\theta)} \mathfrak{g}_\alpha$, where $\Delta(\mathbb{G}, \mathbb{A}_\theta)$ denotes the set of positive restricted roots of \mathbb{A}_θ acting on \mathfrak{g} , and for any positive or negative restricted root α , we denote by \mathfrak{g}_α the corresponding restricted root space. Fix an \mathcal{O}_L -lattice \mathfrak{h}' in \mathfrak{g} , fix $a \in K^\times$ of sufficiently large valuation, and set

$$\mathfrak{h} := (a\mathfrak{h}' \cap \mathfrak{m}_\theta) \oplus \bigoplus_{\alpha \in \pm \Delta(\mathbb{G}, \mathbb{A}_\theta)} (a\mathfrak{h}' \cap \mathfrak{g}_\alpha).$$

More generally, if π is a uniformizer of L , set $\mathfrak{h}_n := \pi^n \mathfrak{h}$. Then, provided that the valuation of a was chosen large enough, we see that each \mathfrak{h}_n is a Lie sublattice of \mathfrak{g} which exponentiates to a good analytic open subgroup H_n of G . Since the sequence $\{\mathfrak{h}_n\}$ forms a neighbourhood basis of zero in the Lie algebra \mathfrak{g} , we see that $\{H_n\}$ forms a neighbourhood basis of the identity in G . As above, for each $n \geq 0$, let \mathbb{H}_n denote the rigid analytic group underlying H_n . By construction, the inclusion $H_{n+1} \rightarrow H_n$ extends to a rigid analytic embedding $\mathbb{H}_{n+1} \rightarrow \mathbb{H}_n$, for each value of n , and (again by construction) this embedding is relatively compact. Thus we have established (i). Property (ii) is clear from our construction.

By construction, the natural map

$$(4.1.7) \quad (\mathfrak{h}_n \cap \mathfrak{m}_\emptyset) \oplus \bigoplus_{\alpha \in \pm \Delta(\mathbb{G}, \mathbb{A}_\emptyset)} (\mathfrak{h}_n \cap \mathfrak{g}_\alpha) \xrightarrow{\sim} \mathfrak{h}_n$$

is an isomorphism, for each n . Suppose now that \mathbb{P} is a parabolic subgroup of \mathbb{G} that contains \mathbb{P}_\emptyset , that $\overline{\mathbb{P}}$ is the corresponding opposite parabolic, chosen to contain $\overline{\mathbb{P}}_\emptyset$, and that \mathbb{M} is the common Levi factor of these two parabolics. As above, we let \mathbb{N} denote the unipotent radical of \mathbb{P} , and let $\overline{\mathbb{N}}$ denote the unipotent radical of $\overline{\mathbb{P}}$. We write \mathfrak{m} , \mathfrak{n} , and $\overline{\mathfrak{n}}$ for the Lie algebras of \mathbb{M} , \mathbb{N} , and $\overline{\mathbb{N}}$. The structure theory of parabolic subgroups of reductive groups shows that there is a subset Φ of $\Delta(\mathbb{G}, \mathbb{A}_\emptyset)$ such that $\mathfrak{m} = \mathfrak{m}_\emptyset \oplus \bigoplus_{\alpha \in \pm \Phi} \mathfrak{g}_\alpha$, $\mathfrak{n} = \bigoplus_{\alpha \in \Delta(\mathbb{G}, \mathbb{A}_\emptyset) \setminus \Phi} \mathfrak{g}_\alpha$, and $\overline{\mathfrak{n}} = \bigoplus_{\alpha \in \Delta(\mathbb{G}, \mathbb{A}_\emptyset) \setminus \Phi} \mathfrak{g}_{-\alpha}$. It thus follows from the fact that (4.1.7) is an isomorphism that the natural map

$$(4.1.8) \quad \overline{\mathfrak{n}}_n \oplus \mathfrak{m}_n \oplus \mathfrak{n}_n \rightarrow \mathfrak{h}_n$$

is an isomorphism, for each $n \geq 0$. If we fix a value of n , then each of $\overline{\mathfrak{n}}_n$, \mathfrak{m}_n , and \mathfrak{n}_n exponentiates to a good analytic open subgroup \overline{N}_n of \overline{N} , M_n of M , and N_n of N , respectively. Following the convention introduced above, we write \overline{N}_n , M_n , and N_n to denote the rigid analytic groups underlying the good analytic subgroups \overline{N}_n , M_n , and N_n . These groups are also equal to the Zariski closures in \mathbb{H}_n of \overline{N}_n , M_n , and N_n , respectively. Increasing the valuation of a if necessary, we furthermore find that $\overline{N}_n = H_n \cap \overline{N}$, that $M_n = H_n \cap M$, that $N_n = H_n \cap N$, and (taking into account the fact that (4.1.8) is an isomorphism) that the natural rigid analytic map (4.1.4) is an isomorphism. Altogether, we have established (iii).

Note that \overline{N}_0 is closed under conjugation by z^{-1} , for some $z \in Z_M$, if and only if $\mathfrak{h} \cap \mathfrak{n}$ is closed under the action of $\text{Ad}_{z^{-1}}$. This in turn is true if and only if $\pi^n \mathfrak{h} \cap \mathfrak{n}$ is closed under the action of $\text{Ad}_{z^{-1}}$, which holds if and only if \overline{N}_n is closed under conjugation by z^{-1} . Thus (iv) is proved, and (v) is proved similarly. The theory of roots shows that we may find $z \in Z_M$ such that $\text{Ad}_z(\mathfrak{h} \cap \mathfrak{n}) \subset \mathfrak{h} \cap \mathfrak{n}$ and $\text{Ad}_{z^{-1}}(\mathfrak{h} \cap \overline{\mathfrak{n}}) \subset \pi(\mathfrak{h} \cap \overline{\mathfrak{n}})$. This establishes (vi). \square

In the context of the preceding proposition, we may apply the construction of (4.1.2). Namely, taking the pair (G_0, H) to be the pair (H_0, H_n) for each $n \geq 0$ in turn, we obtain for each $n \geq 0$ a compact type topological K -algebra $D(\mathbb{H}_n^\circ, H_0)$ as the strong dual to the nuclear Fréchet space $\mathcal{C}^{\text{la}}(H_0, K)_{\mathbb{H}_n^\circ - \text{an}}$. The natural isomorphism $\mathcal{C}^{\text{la}}(H_0, K) \xrightarrow{\sim} \varprojlim_n \mathcal{C}^{\text{la}}(H_0, K)_{\mathbb{H}_n^\circ - \text{an}}$ induces a natural isomorphism

$$(4.1.9) \quad \mathcal{D}^{\text{la}}(H_0, K) \xrightarrow{\sim} \varprojlim_n D(\mathbb{H}_n^\circ, H_0).$$

(Compare the proof of [8, 5.3.1].) We may similarly apply the construction of (4.1.2) to the pair (M_0, M_n) , for each $n \geq 0$. Thus we obtain for each n a compact type topological K -algebra $D(\mathbb{M}_n^\circ, M_0)$, and a natural isomorphism

$$(4.1.10) \quad \mathcal{D}^{\text{la}}(M_0, K) \xrightarrow{\sim} \varprojlim_n D(\mathbb{M}_n^\circ, M_0).$$

The closed embedding $M_0 \rightarrow H_0$ gives rise to a closed embedding of K -Fréchet algebras

$$(4.1.11) \quad \mathcal{D}^{\text{la}}(M_0, K) \rightarrow \mathcal{D}^{\text{la}}(H_0, K).$$

This same closed embedding, together with the compatible rigid analytic closed immersion $\mathbb{M}_n^\circ \rightarrow \mathbb{H}_n^\circ$, gives rise to a closed embedding of topological K -algebras

$$(4.1.12) \quad D(\mathbb{M}_n^\circ, M_0) \rightarrow D(\mathbb{H}_n^\circ, H_0),$$

for each $n \geq 0$. The closed embeddings (4.1.11) and (4.1.12) are compatible with the isomorphisms (4.1.9) and (4.1.10), and passage to the projective limit in n .

We continue to suppose that we are in the context of Proposition 4.1.6, and define certain additional good analytic open subgroups of G . Let Z^+ denote the submonoid of Z_M consisting of those z such that $zN_0z \subset N_0$ and $z^{-1}\overline{N}_0z \subset \overline{N}_0$. An evident variant of Proposition 3.3.2(i) shows that Z^+ generates Z_M as a group.

If $n \geq 0$, and if $z \in Z^+$, then write $H(z)_n := (z^{-1}H_nz) \cap H_n$. Since H_n admits the Iwahori decomposition (4.1.5), we find that $z^{-1}H_nz$ admits the corresponding Iwahori decomposition $z^{-1}\overline{N}_nz \times M_n \times z^{-1}N_nz \xrightarrow{\sim} z^{-1}H_nz$, and thus (taking into account the inclusions of Proposition 4.1.6(iv) and (v), which hold, since $z \in Z^+$), that $H(z)_n$ admits the Iwahori decomposition

$$(4.1.13) \quad z^{-1}\overline{N}_nz \times M_n \times N_n \xrightarrow{\sim} H(z)_n.$$

If \mathfrak{h}_n is the \mathcal{O}_L -Lie subalgebra of \mathfrak{g} corresponding to the good analytic open subgroup H_n , then we see that $\mathfrak{h}(z)_n := \text{Ad}_{z^{-1}}\mathfrak{h}_n \cap \mathfrak{h}_n$ is an \mathcal{O}_L -Lie subalgebra of \mathfrak{h}_n , and that the identification of \mathfrak{h}_n with H_n yields an identification of $\mathfrak{h}(z)_n$ with $H(z)_n$. This latter identification gives $H(z)_n$ the structure of a good analytic open subgroup of H_n ; we let $\mathbb{H}(z)_n$ denote the corresponding rigid analytic group, of which $H(z)_n$ is the group of L -valued points. One sees that the Iwahori decomposition (4.1.13) extends to a rigid analytic Iwahori decomposition

$$(4.1.14) \quad z^{-1}\overline{\mathbb{N}}_nz \times \mathbb{M}_n \times \mathbb{N}_n \xrightarrow{\sim} \mathbb{H}(z)_n.$$

(Recall that $z^{-1}\overline{\mathbb{N}}_nz$ denotes the image of $\overline{\mathbb{N}}_n$ in itself under the map induced by conjugation by z^{-1} .)

The group $H(z)_n$ is a normal open subgroup of $H(z)_0$, and hence we may form the compact type algebra $D(\mathbb{H}(z)_n^\circ, H(z)_0)$. There is an isomorphism of topological K -algebras

$$\mathcal{D}^{\text{la}}(H(z)_0, K) \xrightarrow{\sim} \varprojlim_n D(\mathbb{H}(z)_n^\circ, H(z)_0).$$

The inclusion $H(z)_0 \subset H_0$ induces a continuous homomorphism of K -Fréchet algebras

$$\mathcal{D}^{\text{la}}(H(z)_0, K) \rightarrow \mathcal{D}^{\text{la}}(H_0, K),$$

and a consequent isomorphism

$$\mathcal{D}^{\text{la}}(H_0, K) \cong \bigoplus_{h \in H(z)_0 \setminus H_0} \mathcal{D}^{\text{la}}(H(z)_0, K) * \delta_h.$$

The same inclusion $H(z)_0 \subset H_0$, when coupled with the rigid analytic open immersion $\mathbb{H}(z)_n^\circ \rightarrow \mathbb{H}_n^\circ$, also induces a continuous homomorphism

$$(4.1.15) \quad D(\mathbb{H}(z)_n^\circ, H(z)_0) \rightarrow D(\mathbb{H}_n^\circ, H_0).$$

We define

$$\begin{aligned} D(\mathbb{H}(z)_n^\circ, H_0) &:= D(\mathbb{H}(z)_n^\circ, H(z)_0) \otimes_{\mathcal{D}^{\text{la}}(H(z)_0, K)} \mathcal{D}^{\text{la}}(H_0, K) \\ &\xrightarrow{\sim} \bigoplus_{h \in H(z)_0 \setminus H_0} D(\mathbb{H}(z)_n^\circ, H_0) * \delta_h. \end{aligned}$$

The compact type convex K -space $D(\mathbb{H}(z)_n^\circ, H_0)$ may be identified with the strong dual of $\mathcal{C}^{\text{la}}(H_0, K)_{\mathbb{H}(z)_n^\circ\text{-an}}$ (where the space of $\mathbb{H}(z)_n^\circ$ -analytic vectors in $\mathcal{C}^{\text{la}}(H_0, K)$ is defined as in [8, def. 3.4.1], with respect to the right-multiplication action of $H(z)_n$ on H_0). It is naturally a topological $(D(\mathbb{H}(z)_n^\circ, H(z)_0), \mathcal{D}^{\text{la}}(H_0, K))$ -bimodule. The homomorphism (4.1.15) extends to a continuous homomorphism of topological $(D(\mathbb{H}(z)_n^\circ, H(z)_0), \mathcal{D}^{\text{la}}(H_0, K))$ -bimodules

$$(4.1.16) \quad D(\mathbb{H}(z)_n^\circ, H_0) \rightarrow D(\mathbb{H}_n^\circ, H_0).$$

(The left $D(\mathbb{H}(z)_n^\circ, H(z)_0)$ -module structure on the target is defined via (4.1.15), and the right $\mathcal{D}^{\text{la}}(H_0, K)$ -module structure via the homomorphism $\mathcal{D}^{\text{la}}(H_0, K) \rightarrow D(\mathbb{H}_n^\circ, H_0)$ arising from the isomorphism (4.1.9).) Also, analogously to (4.1.12), there is a closed embedding of topological K -algebras

$$(4.1.17) \quad D(\mathbb{M}_n^\circ, M_0) \rightarrow D(\mathbb{H}(z)_n^\circ, H(z)_0),$$

whose composite with (4.1.15) is equal to the map (4.1.12). In particular, we obtain a left action of $D(\mathbb{M}_n^\circ, M_0)$ on $D(\mathbb{H}(z)_n^\circ, H(z)_0)$, and also on $D(\mathbb{H}(z)_n^\circ, H_0)$. Since (4.1.16) is compatible with the left $D(\mathbb{H}(z)_n^\circ, H(z)_0)$ -module structure on its source and target, it is in particular a map of left $D(\mathbb{M}_n^\circ, M_0)$ -modules.

Finally, suppose that $z \in Z^+$ is chosen so that $z^{-1}\overline{N}_n z \subset \overline{N}_{n+1}$. We then define $H(z)_{n,n+1} := (z^{-1}H_n z) \cap H_{n+1}$. Just as with the open subgroup $H(z)_n$, we see that $H(z)_{n,n+1}$ admits the structure of a good analytic open subgroup associated to the \mathcal{O}_L -Lie subalgebra $\text{Ad}_{z^{-1}}\mathfrak{h}_n \cap \mathfrak{h}_{n+1}$ of \mathfrak{g} . Our assumption on z shows that $H(z)_{n,n+1}$ furthermore admits the rigid analytic Iwahori decomposition

$$(4.1.18) \quad z^{-1}\overline{N}_n z \times \mathbb{M}_{n+1} \times \mathbb{N}_{n+1} \xrightarrow{\sim} \mathbb{H}(z)_{n,n+1}.$$

Since $H(z)_{n,n+1}$ is normal in $H(z)_0$, we may define the compact type algebra $D(\mathbb{H}(z)_{n,n+1}^\circ, H(z)_0)$, as well as the $(D(\mathbb{H}(z)_{n,n+1}^\circ, H(z)_0), \mathcal{D}^{\text{la}}(H_0, K))$ -bimodule

$$\begin{aligned} D(\mathbb{H}(z)_{n,n+1}^\circ, H_0) &:= D(\mathbb{H}(z)_{n,n+1}^\circ, H(z)_0) \otimes_{\mathcal{D}^{\text{la}}(H(z)_0, K)} \mathcal{D}^{\text{la}}(H_0, K) \\ &\xrightarrow{\sim} \bigoplus_{h \in H(z)_0 \setminus H_0} D(\mathbb{H}(z)_{n,n+1}^\circ, H_0) * \delta_h. \end{aligned}$$

Analogously to (4.1.17), there is a closed embedding of topological K -algebras

$$D(\mathbb{M}_{n+1}^\circ, M_0) \rightarrow D(\mathbb{H}(z)_n^\circ, H(z)_0),$$

and thus $D(\mathbb{H}(z)_{n,n+1}^\circ, H_0)$ may also be regarded as a $(D(\mathbb{M}_{n+1}^\circ, M_0), \mathcal{D}^{\text{la}}(H_0, K))$ -bimodule. The diagram of inclusions of good analytic open subgroups

$$\begin{array}{ccc} H(z)_{n,n+1} & \longrightarrow & H(z)_n \\ \downarrow & & \downarrow \\ H_{n+1} & \longrightarrow & H_n \end{array}$$

induces a diagram of continuous maps of topological $(D(\mathbb{M}_n^\circ, M_0), \mathcal{D}^{\text{la}}(H_0, K))$ -bimodules

$$(4.1.19) \quad \begin{array}{ccc} D(\mathbb{M}_n^\circ, M_0) \widehat{\otimes}_{D(\mathbb{M}_{n+1}^\circ, M_0)} D(\mathbb{H}(z)_{n,n+1}^\circ, H_0) & \longrightarrow & D(\mathbb{H}(z)_n^\circ, H_0) \\ \downarrow & & \downarrow \\ D(\mathbb{M}_n^\circ, M_0) \widehat{\otimes}_{D(\mathbb{M}_{n+1}^\circ, M_0)} D(\mathbb{H}_{n+1}^\circ, H_0) & \longrightarrow & D(\mathbb{H}_n^\circ, H_0) \end{array}$$

4.2. In this subsection we prove Theorem 0.5, that is, that J_P restricts to a functor $\text{Rep}_{\text{es}}(G) \rightarrow \text{Rep}_{\text{es}}(M)$. Equivalently, we will show that if V is an object of $\text{Rep}_{\text{es}}(G)$, then $J_P(V)_b'$ is a coadmissible $\mathcal{C}^{\text{an}}(\widehat{Z}_M, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(M_0, K)$ -module. Our strategy will be to apply Proposition 3.2.23. In order to explain how that result will be applied, we must first establish some notation.

Choose a Levi factor \mathbb{M} of \mathbb{P} , let $\overline{\mathbb{P}}$ denote the opposite parabolic to \mathbb{P} with respect to \mathbb{M} (so that $\mathbb{M} = \mathbb{P} \cap \overline{\mathbb{P}}$), and let $\overline{\mathbb{N}}$ denote the nilpotent radical of $\overline{\mathbb{P}}$. Also let $\overline{P} := \overline{\mathbb{P}}(L)$ and $\overline{N} := \overline{\mathbb{N}}(L)$. We apply Proposition 4.1.6 to an opposite pair of minimal parabolic subgroups of \mathbb{G} that are compatible with \mathbb{P} and $\overline{\mathbb{P}}$, so as to find a decreasing sequence $\{H_n\}_{n \geq 0}$ of good analytic open subgroups of \mathbb{G} satisfying conditions (i) to (vi) of that proposition. (We will apply conditions (iii), (iv), (v), and (vi) with \mathbb{P} and $\overline{\mathbb{P}}$ taken to be the particular parabolic subgroups just introduced.) We will use the compact type algebras $D(\mathbb{H}_n^\circ, H_0)$ and $D(\mathbb{M}_n^\circ, M_0)$ introduced in Section 4.1, as well as the $(D(\mathbb{M}_n^\circ, M_0), \mathcal{D}^{\text{la}}(H_0, K))$ -bimodules $D(\mathbb{H}(z)_n^\circ, H_0)$ (for $n \geq 0$, and $z \in Z^+$) and the $(D(\mathbb{M}_{n+1}^\circ, M_0), \mathcal{D}^{\text{la}}(H_0, K))$ -bimodules $D(\mathbb{H}(z)_{n,n+1}^\circ, H_0)$ (for $n \geq 0$ and $z \in Z^+$ such that $z^{-1}\overline{N}_n z \subset \overline{N}_{n+1}$).

Choose an exhaustive increasing sequence $\{\widehat{Z}_{G,n}\}_{n \geq 0}$ of admissible open affinoid subdomains of \widehat{Z}_G , such that each inclusion $\widehat{Z}_{G,n} \subset \widehat{Z}_{G,n+1}$ is relatively compact. If $\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger$ denotes the space of overconvergent (in \widehat{Z}_G) rigid analytic functions on $\widehat{Z}_{G,n}$, then $\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger$ is a locally convex topological K -algebra of compact type, and there is a natural isomorphism

$$(4.2.1) \quad \mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \xrightarrow{\sim} \varinjlim_n \mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger.$$

Combining this with (4.1.10) yields (as a special case of (3.2.21)) an isomorphism

$$\mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(M_0, K) \xrightarrow{\sim} \varinjlim_n \mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0).$$

This isomorphism describes a weak Fréchet–Stein structure (in the sense of [8, def. 1.2.6]) on the nuclear Fréchet algebra $\mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(M_0, K)$.

Recall that Z_M^+ denotes the submonoid of Z_M consisting of elements z for which $zN_0z^{-1} \subset N_0$, and that the operators $\pi_{N_0,z}$ (for $z \in Z_M^+$) define an action of Z_M^+ on V^{N_0} . We let Z^+ denote the submonoid of Z_M^+ consisting of elements z such that $z^{-1}\overline{N}_0z \subset \overline{N}_0$. As mentioned in the preceding section, Z^+ generates Z_M as a group, and so Lemma 3.2.19 shows that we may compute $J_P(V)$ as the finite slope part of V^{N_0} , regarded as a Z^+ -module. We will apply Proposition 3.2.23 to V^{N_0} , taking A to be the nuclear Fréchet algebra $\mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(M_0, K)$, and A_n to be $\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0)$, for each $n \geq 0$.

For Proposition 3.2.23 to apply, we must introduce the following additional data:

4.2.2. – For each $n \geq 0$, a $\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0)$ -module U_n as well as a $\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n+1}, K)^\dagger \widehat{\otimes}_K D(\mathbb{M}_{n+1}^\circ, M_0)$ -linear transition map $U_{n+1} \rightarrow U_n$, and a $\mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(M, K)$ -linear topological isomorphism $(V^{N_0})_b' \xrightarrow{\sim} \varinjlim_n U_n$.

4.2.3. – A $\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0)$ -linear action of Z^+ on each U_n , such that the transition maps $U_{n+1} \rightarrow U_n$ and the maps $(V^{N_0})'_b \rightarrow U_n$ are Z^+ -equivariant.

We must then establish the following hypotheses:

4.2.4. – For each $n \geq 0$, the induced map

$$\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0) \xrightarrow{\widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n+1}, K)^\dagger \widehat{\otimes}_K D(\mathbb{M}_{n+1}^\circ, M_0)}} U_{n+1} \rightarrow U_n$$

is $\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0)$ -compact (in the sense of Definition 2.3.3).

4.2.5. – There exists $z \in Z^+$ such that the endomorphism of each U_n induced by z factors through the map of 4.2.4, so as to render commutative the diagram

$$\begin{array}{ccc} \mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0) & \xrightarrow{\widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n+1}, K)^\dagger \widehat{\otimes}_K D(\mathbb{M}_{n+1}^\circ, M_0)}} & U_{n+1} \longrightarrow U_n \\ \text{id} \widehat{\otimes} z \downarrow & \swarrow & \downarrow z \\ \mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0) & \xrightarrow{\widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n+1}, K)^\dagger \widehat{\otimes}_K D(\mathbb{M}_{n+1}^\circ, M_0)}} & U_{n+1} \longrightarrow U_n \end{array}$$

Let us begin by specifying the modules U_n . Combining (4.2.1) with (4.1.9), we obtain as a special case of (3.2.21) an isomorphism

$$\mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(H_0, K) \xrightarrow{\sim} \varinjlim_n \mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_K D(\mathbb{H}_n^\circ, H_0).$$

This isomorphism describes a weak Fréchet–Stein structure (in the sense of [8, def. 1.2.6]) on the nuclear Fréchet algebra $\mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(H_0, K)$.

The strong dual V'_b is a coadmissible $\mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(H_0, K)$ -module (since V is an object of $\text{Rep}_{\text{es}}(G)$) and so [8, thm. 1.2.11(i)] yields a natural isomorphism

$$V'_b \xrightarrow{\sim} \varinjlim_n \mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_K D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(H_0, K)} V'_b,$$

where each of the tensor products

$$\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_K D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(H_0, K)} V'_b$$

is a finitely generated topological $\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_K D(\mathbb{H}_n^\circ, H_0)$ -module. Since the strong dual to V^{N_0} is naturally identified with the Hausdorff N_0 -coinvariants $(V'_b)_{N_0}$ of the strong dual to V , we obtain an isomorphism

$$(4.2.6) \quad (V^{N_0})'_b \xrightarrow{\sim} \varinjlim_n \left(\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{Z}_G, K)} (D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} (V'_b)_{N_0}) \right).$$

(Here $(D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} (V'_b)_{N_0})$ denotes the Hausdorff N_0 -coinvariants of the completed tensor product $D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} (V'_b)_{N_0}$.)

For each $n \geq 0$, define

$$U_n := (\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{Z}_{G,K})} (D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H,K)} V'_b)_{N_0}).$$

The completed tensor product $D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0,K)} V'_b$ is naturally a topological $D(\mathbb{H}_n^\circ, H_0)$ -module, and hence is also (taking into account (4.1.12)) a topological $D(\mathbb{M}_n^\circ, M_0)$ -module. As M_0 normalizes N_0 , the space $(D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0,K)} V'_b)_{N_0}$ inherits a natural quotient $D(\mathbb{M}_n^\circ, M_0)$ -module structure (this is “dual” to the fact that V^{N_0} is an M_0 -invariant subspace of V), and hence U_n is naturally a compact type topological $\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0)$ -module for each value of n . Furthermore, the isomorphism (4.2.6) may be rewritten as an isomorphism

$$(V^{N_0})'_b \xrightarrow{\sim} \varinjlim_n U_n.$$

We next turn to explaining the action of the monoid Z^+ on the modules U_n . The discussion is somewhat involved, since it will incorporate some constructions necessary for the verification of 4.2.4 and 4.2.5.

For each $z \in Z^+$, the map (4.1.16) induces a map

$$(4.2.7) \quad D(\mathbb{H}(z)_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0,K)} V'_b \rightarrow D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0,K)} V'_b$$

of left $D(\mathbb{H}(z)_n^\circ, H(z)_0)$ -modules. The preceding discussion generalizes to show that the $D(\mathbb{M}_n^\circ, M_0)$ -action on $D(\mathbb{H}(z)_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0,K)} V'_b$ induced by (4.1.17) descends to an action of $D(\mathbb{M}_n^\circ, M_0)$ on $(D(\mathbb{H}(z)_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0,K)} V'_b)_{N_0}$, and that (4.2.7) induces a map of $D(\mathbb{M}_n^\circ, M_0)$ -modules

$$(4.2.8) \quad (D(\mathbb{H}(z)_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0,K)} V'_b)_{N_0} \rightarrow (D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0,K)} V'_b)_{N_0}.$$

For each $z \in Z^+$, we are going to define a continuous $D(\mathbb{M}_n^\circ, M_0)$ -linear map

$$(4.2.9) \quad (D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0,K)} V'_b)_{N_0} \rightarrow (D(\mathbb{H}(z)_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0,K)} V'_b)_{N_0},$$

in such a way that the diagram

$$(4.2.10) \quad \begin{array}{ccccc} (V^{N_0})'_b & \xrightarrow{\sim} & (V'_b)_{N_0} & \xrightarrow{v' \mapsto 1 \otimes v'} & (D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0,K)} V'_b)_{N_0} \\ \downarrow (\pi_{N_0,z})' & & & & \downarrow (4.2.9) \\ (V^{N_0})'_b & \xrightarrow{\sim} & (V'_b)_{N_0} & \xrightarrow{v' \mapsto 1 \otimes v'} & (D(\mathbb{H}(z)_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0,K)} V'_b)_{N_0} \\ & & & \searrow v' \mapsto 1 \otimes v' & \downarrow (4.2.8) \\ & & & & (D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0,K)} V'_b)_{N_0} \end{array}$$

commutes. (Here, as indicated on the diagram, the left-hand vertical arrow is given by the action of $z \in Z^+$ on $(V'_b)_{N_0}$, obtained as the transpose of the operator $\pi_{N_0,z}$ on V^{N_0} .)

Since the horizontal arrows have dense image (because the maps $\mathcal{D}^{\text{la}}(H_0, K) \rightarrow D(\mathbb{H}_n^\circ, H_0)$ and $\mathcal{D}^{\text{la}}(H_0, K) \rightarrow D(\mathbb{H}(z)_n^\circ, H_0)$ have dense image, as one sees by noting that their strong duals are the injections $\mathcal{C}(H_0, K)_{\mathbb{H}_n^\circ\text{-an}} \rightarrow \mathcal{C}^{\text{la}}(H_0, K)$ and $\mathcal{C}(H_0, K)_{\mathbb{H}(z)_n^\circ\text{-an}} \rightarrow \mathcal{C}^{\text{la}}(H_0, K)$ respectively), the map (4.2.9) is uniquely determined, if it exists.

LEMMA 4.2.11. – *The morphism (4.2.9) can be constructed so as to make (4.2.10) commute.*

Proof. – Let $\lambda: H_0 \rightarrow \overline{N}_0 \times M_0 =: \overline{P}_0$ denote the inverse of the isomorphism (4.1.5) (for $n = 0$), followed by projection onto the first two factors. If x is an element of N_0 (we use the symbol x rather than n so as to avoid confusion with the index n of the analytic open subgroup H_n), then

$$(4.2.12) \quad h \mapsto z^{-1}\lambda(x^{-1}\lambda(h^{-1}))^{-1}z$$

describes a locally analytic map $H_0 \rightarrow z^{-1}\overline{P}_0z \subset H_0$. This map induces a continuous linear endomorphism $s_{x,z}$ of $\mathcal{C}^{\text{la}}(H_0, K)$, defined for $f \in \mathcal{C}^{\text{la}}(H_0, K)$ via

$$(s_{x,z}f)(h) = f(z^{-1}\lambda(x^{-1}\lambda(h^{-1}))^{-1}z).$$

We regard H_0 as the locally analytic group underlying the rigid analytic group

$$(4.2.13) \quad \coprod_{h \in H_0/\mathbb{H}_n^\circ(L)} h\mathbb{H}_n^\circ.$$

(Note that Proposition 4.1.6(ii) implies that we obtain the same decomposition of H_0 if we consider instead left cosets, and so this disjoint union does indeed describe a rigid analytic group.) The space of rigid analytic K -valued functions on this rigid analytic group is precisely $\mathcal{C}^{\text{la}}(H_0, K)_{\mathbb{H}_n^\circ\text{-an}}$. We simultaneously regard H_0 as the locally analytic space underlying the rigid analytic space

$$(4.2.14) \quad \coprod_{h \in H_0/\mathbb{H}(z)_n^\circ(L)} h\mathbb{H}(z)_n^\circ.$$

The space of rigid analytic K -valued functions on this rigid analytic space is precisely $\mathcal{C}^{\text{la}}(H_0, K)_{\mathbb{H}(z)_n^\circ\text{-an}}$.

Since H_n admits a rigid analytic Iwahori decomposition, the restriction to H_n of λ induces a rigid analytic map $\mathbb{H}_n \rightarrow \overline{N}_n \times \mathbb{M}_n$, and hence also a rigid analytic map $\mathbb{H}_n^\circ \rightarrow \overline{N}_n^\circ \times \mathbb{M}_n^\circ$. Thus (taking into account Proposition 4.1.6(iv) as well) the map (4.2.12) is in fact a rigid analytic map from the rigid analytic group (4.2.13) to the rigid analytic space (4.2.14), and so $s_{x,z}$ induces a continuous linear map $\mathcal{C}^{\text{la}}(H_0, K)_{\mathbb{H}(z)_n^\circ\text{-an}} \rightarrow \mathcal{C}^{\text{la}}(H_0, K)_{\mathbb{H}_n^\circ\text{-an}}$. Passing to the transpose, we obtain a continuous linear map $s'_{x,z}: D(\mathbb{H}_n^\circ, H_0) \rightarrow D(\mathbb{H}(z)_n^\circ, H_0)$. Since (4.2.12) is left M_0 -equivariant, this map is in fact $D(\mathbb{M}_n^\circ, M_0)$ -linear.

As in Section 3.5, let δ denote the modulus character of P . Let x run over a set of coset representatives for zN_0z^{-1} in N_0 , and define a map

$$(4.2.15) \quad D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_K V'_b \rightarrow D(\mathbb{H}(z)_n^\circ, H_0) \widehat{\otimes}_K V'_b$$

via

$$(4.2.16) \quad \mu \otimes v' \mapsto \delta(z) \sum_x s'_{x,k}(\mu) \otimes z^{-1}x^{-1}v'.$$

We claim that the map (4.2.15) descends to the required morphism (4.2.9). To see this, we note that formula (4.2.16) also defines a map

$$(4.2.17) \quad K[H_0] \otimes_K V'_b \rightarrow K[H_0] \otimes_K V'_b.$$

In Lemma 4.2.19 below we will establish the commutativity of the diagram

$$(4.2.18) \quad \begin{array}{ccccc} K[H_0] \otimes_K V'_b & \longrightarrow & (V'_b)_{N_0} & \xrightarrow{\sim} & (V^{N_0})'_b \\ \downarrow (4.2.17) & & & & \downarrow (\pi_{N_0, z})' \\ K[H_0] \otimes_K V'_b & \longrightarrow & (V'_b)_{N_0} & \xrightarrow{\sim} & (V^{N_0})'_b \end{array}$$

(in which the left horizontal arrows are induced by the action of H_0 on V'_b , together with the quotient map $V'_b \rightarrow (V'_b)_{N_0}$). In particular, the map (4.2.17) descends to a map $(V'_b)_{N_0} \rightarrow (V'_b)_{N_0}$. Since $K[H_0] \otimes_K V'_b$ is dense in each of $D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_K V'_b$ and $D(\mathbb{H}(z)_n^\circ, H_0) \widehat{\otimes}_K V'_b$, we infer that (4.2.15) similarly descends to a map $(D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} V'_b)_{N_0} \rightarrow (D(\mathbb{H}(z)_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} V'_b)_{N_0}$. If we take this map to be our sought-after morphism (4.2.9), then the commutativity of (4.2.10) follows from the commutativity of (4.2.18). \square

LEMMA 4.2.19. – *The diagram (4.2.18) commutes.*

Proof. – It suffices to show that for any $v \in V^{N_0}$, $v' \in V'_b$, and $h \in H_0$, there is an equality

$$(4.2.20) \quad \delta(z) \sum_x \langle s'_{x,z}(\delta_h) \otimes z^{-1}x^{-1}v', v \rangle = \langle hv', \pi_{N_0, z}(v) \rangle.$$

(Here we have denoted by δ_h the element $h \in H_0$ regarded as an element of the group ring $K[H_0]$, since this element is identified with the delta function supported at h under the natural embedding $K[H_0] \rightarrow \mathcal{D}^{\text{la}}(H_0, K)$; it has nothing to do with the modulus δ .) It is an immediate consequence of the definition of $s_{x,z}$ that $s'_{x,z}(\delta_h) = \delta_{z^{-1}\lambda(x^{-1}\lambda(h^{-1}))^{-1}z}$. Thus we compute

$$\begin{aligned} & \delta(z) \sum_x \langle s'_{x,z}(\delta_h) \otimes z^{-1}x^{-1}v', v \rangle \\ &= \delta(z) \sum_x \langle \delta_{z^{-1}\lambda(x^{-1}\lambda(h^{-1}))^{-1}z} \otimes z^{-1}x^{-1}v', v \rangle \\ &= \delta(z) \sum_x \langle v', xzz^{-1}\lambda(x^{-1}\lambda(h^{-1}))zv \rangle \\ &= \delta(z) \sum_x \langle v', x\lambda(x^{-1}\lambda(h^{-1}))zv \rangle \\ &= \delta(z) \sum_x \langle v', h^{-1}hx\lambda(x^{-1}\lambda(h^{-1}))zv \rangle. \end{aligned}$$

Lemma 4.2.21 below shows that as x ranges over a set of right coset representatives of zN_0z^{-1} in N_0 , the same is true of the elements $hx\lambda(x^{-1}\lambda(h^{-1}))$. Thus if we take into account the definition of $\pi_{N_0, z}$, we find that

$$\delta(z) \sum_x \langle s'_{x,z}(\delta_h) \otimes x^{-1}z^{-1}v', v \rangle = \langle v', h^{-1}\pi_{N_0, z}v \rangle.$$

This establishes (4.2.20). \square

LEMMA 4.2.21. – *If x ranges over a set of right coset representatives of zN_0z^{-1} in N_0 , then for any $h \in H_0$, the elements $hx\lambda(x^{-1}\lambda(h^{-1}))$ also range through a set of right coset representatives of zN_0z^{-1} in N_0 .*

Proof. – Define $\rho: H_0 \rightarrow N_0$ via the equation $h = \lambda(h)\rho(h)$ for any $h \in H_0$. (Thus the product $\lambda \times \rho$ yields the isomorphism (4.1.5) for $n = 0$.) Note that $h\lambda(h^{-1}) = \rho(h^{-1})^{-1} \in N_0$, and thus that the lemma is true if and only if the elements

$$\lambda(h^{-1})^{-1}x\lambda(x^{-1}\lambda(h^{-1})) = \rho(x^{-1}\lambda(h^{-1}))^{-1}$$

run over a set of right coset representatives of zN_0z^{-1} in N_0 . Write $\bar{p} = \lambda(h^{-1}) \in \bar{P}_0$. Since the inverses of a system of right coset representatives form a system of left coset representatives, it is equivalent to show that, as x ranges over a set of *left* coset representatives of zN_0z^{-1} in N_0 , the elements $\rho(x\bar{p})$ range over such a set of left coset representatives.

Let x and x' be two elements of N_0 that lie in the given set of left coset representatives of zN_0z^{-1} in N_0 . Then $x\bar{p}$ and $x'\bar{p}$ are both elements of H_0 , and so we may write $x\bar{p} = \lambda(x\bar{p})\rho(x\bar{p})$ and $x'\bar{p} = \lambda(x'\bar{p})\rho(x'\bar{p})$. Thus

$$x(x')^{-1} = \lambda(x\bar{p})\rho(x\bar{p})\rho(x'\bar{p})^{-1}\lambda(x'\bar{p})^{-1},$$

and so $(zN_0z^{-1})x = (zN_0z^{-1})x'$ if and only if $x(x')^{-1} \in zN_0z^{-1}$, which holds if and only if $\lambda(x\bar{p})\rho(x\bar{p})\rho(x'\bar{p})^{-1}\lambda(x'\bar{p})^{-1} \in zN_0z^{-1}$, which holds if and only if $\rho(x\bar{p})\rho(x'\bar{p})^{-1}$ lies in $\lambda(x\bar{p})^{-1}zN_0z^{-1}\lambda(x'\bar{p}) \cap N_0$.

Now $H_0 \cap zH_0z^{-1} = \bar{P}_0zN_0z^{-1}$. Thus $\bar{p}'zN_0z^{-1}\bar{p}'' \cap N_0 \subset H_0 \cap zH_0z^{-1} \cap N_0 = zN_0z^{-1}$ for any pair of elements \bar{p}' and \bar{p}'' of \bar{P}_0 . Hence $(zN_0z^{-1})x = (zN_0z^{-1})x'$ implies that $\rho(x\bar{p})\rho(x'\bar{p})^{-1} \in zN_0z^{-1}$. Conversely, if $\rho(x\bar{p})\rho(x'\bar{p})^{-1} \in zN_0z^{-1}$, then $x(x')^{-1} \in \lambda(x\bar{p})zN_0z^{-1}\lambda(x'\bar{p}) \cap N_0 \subset zN_0z^{-1}$. This proves the lemma. \square

Let ϕ_z denote the composite of (4.2.8) with (4.2.9). (Thus ϕ_z is the composite of the two right-hand vertical arrows in the diagram (4.2.10).) As observed, ϕ_z is uniquely determined by the fact that it makes the diagram (4.2.10) commute. Thus, since the operators $(\pi_{N_0,z})'$ define a topological action of Z^+ on $(V'_b)_{N_0}$, we see that the operators ϕ_z define a topological action of Z^+ on $(D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} V'_b)_{N_0}$. Taking the completed tensor product with $\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger$ over $\mathcal{C}^{\text{an}}(\widehat{Z}_G, K)$, we obtain continuous $\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0)$ -linear endomorphisms $\text{id} \widehat{\otimes} \phi_z$ of $U_n = \mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{Z}_G, K)} (D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} V'_b)_{N_0}$, defining the required topological action of Z^+ on U_n .

This completes our specification of the data 4.2.2 and 4.2.3. Since V is a locally analytic G -representation, the closed subspace V^{N_0} is certainly a locally analytic M_0 -representation, and so also a locally analytic representation of $Z_0 := M_0 \cap Z^+$. Together with the remarks following the proof of Lemma 3.2.28, this shows that V^{N_0} lies in the category $\text{Rep}_{\text{la.c}}^z(Z^+)$ (as defined in the discussion preceding the statement of Proposition 3.2.27).

We now turn to verifying the hypotheses 4.2.4 and 4.2.5. Since M_0 normalizes N_0 , we see that the space of Hausdorff N_0 -coinvariants $D(\mathbb{H}_n^\circ, H_0)_{N_0}$ is naturally a compact type $D(\mathbb{M}_n^\circ, M_0)$ -module, for each $n \geq 0$.

PROPOSITION 4.2.22. – *For any $n \geq 0$, the continuous $D(\mathbb{M}_n^\circ, M_0)$ -linear map*

$$(4.2.23) \quad D(\mathbb{M}_n^\circ, M_0) \widehat{\otimes}_{D(\mathbb{M}_{n+1}^\circ, M_0)} D(\mathbb{H}_{n+1}^\circ, H_0)_{N_0} \rightarrow D(\mathbb{H}_n^\circ, H_0)_{N_0}$$

induced by the natural map $D(\mathbb{H}_{n+1}^\circ, H_0)_{N_0} \rightarrow D(\mathbb{H}_n^\circ, H_0)_{N_0}$ is a $D(\mathbb{M}_n^\circ, M_0)$ -compact map (in the sense of Definition 2.3.3).

Proof. – For each $n \geq 0$ the Iwahori decomposition of H_0 , and the rigid analytic Iwahori decomposition of \mathbb{H}_n° , give rise to a topological isomorphism

$$D(\mathbb{H}_n^\circ, H_0) \xrightarrow{\sim} D(\overline{\mathbb{N}}_n^\circ, \overline{N}_0) \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0) \widehat{\otimes}_K D(\mathbb{N}_n^\circ, N_0)$$

(where each of the factors in the target is defined via (4.1.2)), and hence an isomorphism of $D(\mathbb{M}_n^\circ, M_0)$ -modules

$$(4.2.24) \quad D(\mathbb{H}_n^\circ, H_0)_{N_0} \xrightarrow{\sim} D(\overline{\mathbb{N}}_n^\circ, \overline{N}_0) \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0),$$

provided that the $D(\mathbb{M}_n^\circ, M_0)$ -module structure on the target of this map is defined by having M_0 act on the first factor via the action of M_0 on N_0 given by conjugation, and on the second factor by regarding $D(\mathbb{M}_n^\circ, M_0)$ as a left module over itself.

The untwisting lemma [8, 3.6.4] yields (after passing to the appropriate limits, and then taking strong duals) a natural isomorphism of the completed tensor product $D(\overline{\mathbb{N}}_n^\circ, \overline{N}_0) \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0)$, equipped with the $D(\mathbb{M}_n^\circ, M_0)$ -module structure that appears in (4.2.24), and the same completed tensor product, equipped with the evident $D(\mathbb{M}_n^\circ, M_0)$ -structure defined by the $D(\mathbb{M}_n^\circ, M_0)$ -structure on the second factor. Composing this isomorphism with (4.2.24) yields an isomorphism of $D(\mathbb{M}_n^\circ, M_0)$ -modules

$$(4.2.25) \quad D(\mathbb{H}_n^\circ, H_0)_{N_0} \xrightarrow{\sim} D(\overline{\mathbb{N}}_n^\circ, \overline{N}_0) \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0),$$

in which the $D(\mathbb{M}_n^\circ, M_0)$ -module structure on the target is defined simply by the $D(\mathbb{M}_n^\circ, M_0)$ -module structure on the second factor of the completed tensor product. The naturality of this isomorphism (for the index $n + 1$ as well as for the index n) shows that we may embed (4.2.23) into the commutative diagram

$$\begin{array}{ccc} D(\mathbb{M}_n^\circ, M_0) \widehat{\otimes}_{D(\mathbb{M}_{n+1}^\circ, M_0)} D(\mathbb{H}_{n+1}^\circ, H_0)_{N_0} & \xrightarrow{\text{id} \widehat{\otimes} (4.2.25)} & D(\overline{\mathbb{N}}_{n+1}^\circ, \overline{N}_0) \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0) \\ \downarrow (4.2.23) & & \downarrow \\ D(\mathbb{H}_n^\circ, H_0)_{N_0} & \xrightarrow{(4.2.25)} & D(\overline{\mathbb{N}}_n^\circ, \overline{N}_0) \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0) \end{array}$$

To see that (4.2.23) is $D(\mathbb{M}_n^\circ, M_0)$ -compact, it now suffices (by Lemma 2.3.4(ii) and the remark following Definition 2.3.3) to note that the map $\overline{\mathbb{N}}_{n+1}^\circ \rightarrow \overline{\mathbb{N}}_n^\circ$ is relatively compact (by Proposition 4.1.6(i)), and thus that the map $D(\overline{\mathbb{N}}_{n+1}^\circ, \overline{N}_0) \rightarrow D(\overline{\mathbb{N}}_n^\circ, \overline{N}_0)$ is a compact map of convex K -vector spaces. \square

COROLLARY 4.2.26. – For each $n \geq 0$, the map

$$\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0) \xrightarrow{\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n+1}, K)^\dagger \widehat{\otimes}_K D(\mathbb{M}_{n+1}^\circ, M_0)} U_{n+1} \rightarrow U_n$$

of 4.2.4 is a $\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0)$ -compact map (in the sense of Definition 2.3.3).

Proof. – By assumption V'_b is a coadmissible $\mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(H_0, K)$ -module, and so we may find a surjection

$$(4.2.27) \quad \mathcal{C}^{\text{an}}(\widehat{Z}_{G,n+1}, K)^\dagger \widehat{\otimes}_K D(\mathbb{H}_{n+1}^\circ, H_0)^r \rightarrow \mathcal{C}^{\text{an}}(\widehat{Z}_{G,n+1}, K)^\dagger \widehat{\otimes}_K D(\mathbb{H}_{n+1}^\circ, H_0) \widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(H_0, K)} V'_b,$$

for some natural number r , and hence also a surjection

$$\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n+1}, K)^\dagger \widehat{\otimes}_K D(\mathbb{H}_{n+1}^\circ, H_0)_{N_0}^r \rightarrow U_{n+1}.$$

Taking the completed tensor product of (4.2.27) with $\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_K D(\mathbb{H}_n^\circ, H_0)$ over $\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n+1}, K)^\dagger \widehat{\otimes}_K D(\mathbb{H}_{n+1}^\circ, H_0)$ yields a surjection

$$\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_K D(\mathbb{H}_n^\circ, H_0)^r \rightarrow \mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_K D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(H_0, K)} V'_b,$$

and hence also a surjection

$$\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_K D(\mathbb{H}_n^\circ, H_0)_{N_0}^r \rightarrow U_n.$$

The morphism of 4.2.4 may thus be covered by a morphism

$$\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K) \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0) \widehat{\otimes}_{D(\mathbb{M}_{n+1}^\circ, M_0)} D(\mathbb{H}_{n+1}^\circ, H_0)_{N_0}^r \rightarrow \mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K) \widehat{\otimes}_K D(\mathbb{H}_n^\circ, H_0)_{N_0}^r$$

of $\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_K D(\mathbb{H}(z)_n^\circ, H_0)$ -modules, and so the corollary follows from (ii) and (iii) of Lemma 2.3.4 together with Proposition 4.2.22. \square

PROPOSITION 4.2.28. – *If $n \geq 0$, and if $z \in Z^+$ is chosen so that $z^{-1}\overline{N}_n z \subset \overline{N}_{n+1}$, then the endomorphism of U_n induced by z factors through the map of 4.2.4, in such a way that the diagram of 4.2.5 commutes.*

Proof. – The action of z on U_n is defined as $\text{id} \widehat{\otimes} \phi_z$, where ϕ_z is the endomorphism of $(D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} V'_b)_{N_0}$ defined as the composite of the $D(\mathbb{M}_n^\circ, M_0)$ -linear maps (4.2.8) and (4.2.9). We will prove the proposition by showing that $\text{id} \widehat{\otimes} (4.2.8)$ factors through the map of 4.2.4. To this end we consider the diagram

$$(4.2.29) \quad \begin{array}{ccc} D(\mathbb{M}_n^\circ, M_0) \widehat{\otimes}_{D(\mathbb{M}_{n+1}^\circ, M_0)} U(z)_{n,n+1} & \xrightarrow{\hspace{2cm}} & U(z)_n \\ \downarrow & & \downarrow \text{id} \widehat{\otimes} (4.2.8) \\ \mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0) \widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n+1}, K)^\dagger \widehat{\otimes}_K D(\mathbb{M}_{n+1}^\circ, M_0)} U_{n+1} & \xrightarrow{4.2.4} & U_n \end{array}$$

induced by the diagram (4.1.19), where to ease typesetting we have written

$$U(z)_{n,n+1} := \mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{Z}_G, K)} (D(\mathbb{H}(z)_{n,n+1}^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} V'_b)_{N_0}$$

and similarly

$$U(z)_n := \mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{Z}_{G,K})} (D(\mathbb{H}(z)_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} V'_b)_{N_0}.$$

We will show below that the natural map

(4.2.30)

$$\begin{aligned} & D(\mathbb{M}_n^\circ, M_0) \widehat{\otimes}_{D(\mathbb{M}_{n+1}^\circ, M_0)} \mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{Z}_{G,K})} (D(\mathbb{H}(z)_{n,n+1}^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} S)_{N_0} \\ & \rightarrow \mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K)^\dagger \widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{Z}_{G,K})} (D(\mathbb{H}(z)_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} S)_{N_0} \end{aligned}$$

is an isomorphism for any coadmissible $\mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(H_0, K)$ -module S . In particular, the upper horizontal arrow of (4.2.29) is an isomorphism, and inverting this isomorphism then yields the required factorization of $\text{id} \widehat{\otimes} (4.2.8)$.

We now prove that (4.2.30) is an isomorphism. To ease notation, we denote by $M(z)_{n,n+1}$ the topological $(D(\mathbb{M}_{n+1}^\circ, M_0), \mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(H_0, K))$ -bimodule $\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K) \widehat{\otimes}_K D(\mathbb{H}(z)_{n,n+1}^\circ, H_0)$, and denote by $M(z)_n$ the topological $(D(\mathbb{M}_n^\circ, M_0), \mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(H_0, K))$ -bimodule $\mathcal{C}^{\text{an}}(\widehat{Z}_{G,n}, K) \widehat{\otimes}_K D(\mathbb{H}(z)_n^\circ, H_0)$. Since $\mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(H_0, K)$ is a Fréchet–Stein algebra, we may find a continuous K -algebra homomorphism $\mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(H_0, K) \rightarrow B$ for some Noetherian Banach algebra B such that the action of $\mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(H_0, K)$ on each of the topological modules $M(z)_{n,n+1}$ and $M(z)_n$ factors through a corresponding continuous action of B , and such that $S_1 := B \widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(H_0, K)} S$ is a finitely presented B -Banach module. It thus suffices to show that the natural map

$$(4.2.31) \quad D(\mathbb{M}_n^\circ, M_0) \widehat{\otimes}_{D(\mathbb{M}_{n+1}^\circ, M_0)} (M(z)_{n,n+1} \widehat{\otimes}_B S_1)_{N_0} \rightarrow (M(z)_n \widehat{\otimes}_B S_1)_{N_0}$$

is an isomorphism for any finitely presented B -Banach module S_1 .

The construction of (4.2.31) is evidently functorial in S_1 . If we apply this construction to the members of a finite presentation $B^r \rightarrow B^s \rightarrow S_1 \rightarrow 0$ of S_1 , we obtain the diagram

$$\begin{array}{ccc} D(\mathbb{M}_n^\circ, M_0) \widehat{\otimes}_{D(\mathbb{M}_{n+1}^\circ, M_0)} (M(z)_{n,n+1}^r)_{N_0} & \longrightarrow & (M(z)_n^r)_{N_0} \\ \downarrow & & \downarrow \\ D(\mathbb{M}_n^\circ, M_0) \widehat{\otimes}_{D(\mathbb{M}_{n+1}^\circ, M_0)} (M(z)_{n,n+1}^s)_{N_0} & \longrightarrow & (M(z)_n^s)_{N_0} \\ \downarrow & & \downarrow \\ D(\mathbb{M}_n^\circ, M_0) \widehat{\otimes}_{D(\mathbb{M}_{n+1}^\circ, M_0)} (M(z)_{n,n+1} \widehat{\otimes}_B S_1)_{N_0} & \longrightarrow & (M(z)_n \widehat{\otimes}_B S_1)_{N_0} \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

(the horizontal maps being induced by (4.2.31)). The columns of this diagram may not be exact as sequences of maps of abstract K -vector spaces (since we have formed *completed* tensor products,

and then passed to Hausdorff N_0 -coinvariants), but they are exact in the category of Hausdorff locally convex K -spaces (i.e. the third member of each column is isomorphic to the quotient of the second member of that column by the closure of the image of the first member). Thus to show that (4.2.31) is an isomorphism, it suffices to show that the natural map

$$D(\mathbb{M}_n^\circ, M_0) \widehat{\otimes}_{D(\mathbb{M}_{n+1}^\circ, M_0)} (M(z)_{n,n+1})_{N_0} \rightarrow (M(z)_n)_{N_0}$$

is an isomorphism. For this, it suffices in turn to note that the natural map

$$D(\mathbb{M}_n^\circ, M_0) \widehat{\otimes}_{D(\mathbb{M}_{n+1}^\circ, M_0)} D(\mathbb{H}(z)_{n,n+1}^\circ, H_0)_{N_0} \rightarrow D(\mathbb{H}(z)_n^\circ, H_0)_{N_0}$$

is an isomorphism, as follows from a consideration of the rigid analytic Iwahori decompositions (4.1.14) and (4.1.18) of each of the good analytic open subgroups $H(z)_{n,n+1}$ and $H(z)_n$. \square

We may now prove the main theorem of this subsection.

THEOREM 4.2.32. – *If V is an object of $\text{Rep}_{\text{es}}(G)$, then the Jacquet module $J_P(V)$ is an object of $\text{Rep}_{\text{es}}(M)$. Thus J_P induces an additive, left exact functor $\text{Rep}_{\text{es}}(G) \rightarrow \text{Rep}_{\text{es}}(M)$.*

Proof. – Proposition 4.1.6(vi) yields $z \in Z^+$ such that $z^{-1}\overline{N}_n z \subset \overline{N}_{n+1}$ for each $n \geq 0$. Corollary 4.2.26 and Proposition 4.2.28 establish 4.2.4 and 4.2.5, and so Proposition 3.2.23 implies that $J_P(V)'_b$ is a coadmissible module over the nuclear Fréchet algebra $\mathcal{C}^{\text{an}}(\widehat{Z}_M, K) \widehat{\otimes}_K \mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(M_0, K)$. Since the two actions of $\mathcal{C}^{\text{an}}(\widehat{Z}_G, K)$ on $J_P(V)'_b$ (one through the embedding $\mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \rightarrow \mathcal{C}^{\text{an}}(\widehat{Z}_M, K)$ and the other through the first factor of $\mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(M_0, K)$) coincide, the action of the algebra $\mathcal{C}^{\text{an}}(\widehat{Z}_M, K) \widehat{\otimes}_K \mathcal{C}^{\text{an}}(\widehat{Z}_G, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(M_0, K)$ on $J_P(V)'_b$ factors through its quotient $\mathcal{C}^{\text{an}}(\widehat{Z}_M, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(M_0, K)$ (in which the two copies of $\mathcal{C}^{\text{an}}(\widehat{Z}_G, K)$ described above are identified). Thus $J_P(V)'_b$ is a coadmissible $\mathcal{C}^{\text{an}}(\widehat{Z}_M, K) \widehat{\otimes}_K \mathcal{D}^{\text{la}}(M_0, K)$ -module, and so J_P restricts to a functor $\text{Rep}_{\text{es}}(G) \rightarrow \text{Rep}_{\text{es}}(M)$, as required. That this functor is additive and left exact follows from Lemma 3.4.7. \square

We now suppose that \mathbb{G} is quasi-split, and that \mathbb{P} is a Borel subgroup of \mathbb{G} , and turn to proving 0.11. The Levi factor \mathbb{M} is then a torus, and so $M = Z_M$.

PROPOSITION 4.2.33. – *If V is an object of $\text{Rep}_{\text{ad}}(G)$, then the map $\text{Exp}(J_P(V)) \rightarrow \mathfrak{m}$ has discrete fibres.*

Proof. – Write $\widehat{M} = \bigcup \widehat{M}_n$ as a union of affinoid subdomains. We have to show that the intersection of $\text{Exp}(J_P(V))$ with any of the affinoid subdomains \widehat{M}_n has a finite fibre over any point of \mathfrak{m} . For this, it suffices to show that for any character $\chi \in \mathfrak{m}$, the tensor product $(\mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger \widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{M}, K)} J_P(V)'_{m=\chi})$ is finite dimensional for each n , or equivalently, for each sufficiently large value of n . (Here the subscript indicates the maximal Hausdorff quotient where m acts through χ .) The proof will be a variant of that given for the preceding theorem.

We begin by normalizing our various choices of indices. For each $n \geq 0$, the composite $\mathcal{D}^{\text{la}}(M_0, K) \rightarrow \mathcal{C}^{\text{an}}(\widehat{M}, K) \rightarrow \mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger$ factors through the natural map $\mathcal{D}^{\text{la}}(M_0, K) \rightarrow D(\mathbb{M}_m^\circ, M_0)$, for some sufficiently large value of m . Replacing the sequence \mathbb{H}_n of analytic open subgroups of G by a cofinal subsequence, and relabelling if necessary, we may assume that in fact the map $\mathcal{D}^{\text{la}}(M_0, K) \rightarrow \mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger$ factors through $D(\mathbb{M}_n^\circ, M_0)$. Thus for each $n \geq 0$, the continuous homomorphism $\mathcal{D}^{\text{la}}(M_0, K) \rightarrow \mathcal{C}^{\text{an}}(\widehat{M}, K)$ determines a continuous

homomorphism

$$(4.2.34) \quad D(\mathbb{M}_n^\circ, M_0) \rightarrow \mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger.$$

Since V is admissible as a locally analytic G -representation (rather than just essentially admissible) we may strengthen the isomorphism (4.2.6) as follows:

$$(V^{N_0})'_b \xrightarrow{\sim} \varinjlim_n (D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} V'_b)_{N_0}.$$

Take A_n to be $D(\mathbb{M}_n^\circ, M_0)$ and U_n to be $(D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} V'_b)_{N_0}$, for $n \geq 0$; variants of Corollary 4.2.26 and Proposition 4.2.28 then show that the hypotheses of Proposition 3.2.23 are satisfied. The proof of that proposition thus yields an isomorphism

$$\begin{aligned} J_P(V)'_b &= ((V^{N_0})'_{\text{fs}})_b \xrightarrow{\sim} \mathcal{C}^{\text{an}}(\widehat{M}, K) \widehat{\otimes}_{K[Z^+]} (V^{N_0})'_b \\ &\xrightarrow{\sim} \mathcal{C}^{\text{an}}(\widehat{M}, K) \widehat{\otimes}_{K[Z^+]} \varinjlim_n (D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} V'_b)_{N_0} \\ &\xrightarrow{\sim} \varinjlim_n \mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger \widehat{\otimes}_{K[Z^+]} (D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} V'_b)_{N_0}, \end{aligned}$$

where the space $\mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger \widehat{\otimes}_{K[Z^+]} (D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} V'_b)_{N_0}$ is finitely generated as a $\mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0)$ -module, for each $n \geq 0$. The continuous homomorphism (4.2.34) induces a continuous surjection

$$\mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0) \rightarrow \mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger,$$

and by construction the $\mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0)$ -action on

$$\mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger \widehat{\otimes}_{K[Z^+]} (D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} V'_b)_{N_0}$$

factors through this surjection. Thus we see that for each $n \geq 0$, the space

$$\mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger \widehat{\otimes}_{K[Z^+]} (D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} V'_b)_{N_0}$$

is finitely generated as a $\mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger$ -module. (In short, we have just repeated the argument of Theorem 4.2.32, proving that $J_P(V)$ lies in $\text{Rep}_{\text{es}}(M)$, in the particular situation of the current proposition.)

We deduce from [8, thm. 1.2.11(i)] that there is a topological isomorphism

$$\mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger \widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{M}, K)} J_P(V)'_b \xrightarrow{\sim} \mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger \widehat{\otimes}_{K[Z^+]} (D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} V'_b)_{N_0},$$

and hence a topological isomorphism

$$\begin{aligned} &(\mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger \widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{M}, K)} J_P(V)'_b)_{\mathfrak{m}=\chi} \\ &\xrightarrow{\sim} (\mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger \widehat{\otimes}_{K[Z^+]} (D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} V'_b)_{N_0})_{\mathfrak{m}=\chi}, \end{aligned}$$

for each $n \geq 0$.

If n is chosen large enough, then χ integrates to a rigid analytic character $\tilde{\chi}$ of \mathbb{M}_n° , and the natural map $(\mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger)_{\mathfrak{m}=\chi} \rightarrow (\mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger)_{M_n^\circ=\tilde{\chi}}$ is an isomorphism (as follows from the fact that the natural map $\mathcal{D}^{\text{la}}(M_0, K) \rightarrow \mathcal{C}^{\text{an}}(\widehat{M}, K)$ factors through the map (4.2.34)). Thus there is an isomorphism

$$(\mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger \widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{M}, K)} J_P(V)'_b)_{\mathfrak{m}=\chi} \xrightarrow{\sim} (\mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger \widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{M}, K)} J_P(V)'_b)_{M_n^\circ=\tilde{\chi}},$$

and hence an isomorphism

(4.2.35)

$$\begin{aligned} & (\mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger \widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{M}, K)} J_P(V)'_b)_{\mathfrak{m}=\chi} \\ & \xrightarrow{\sim} (\mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger \widehat{\otimes}_{K[Z^+]} (D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} V'_b)_{N_0})_{M_n^\circ=\tilde{\chi}} \\ & \xrightarrow{\sim} \mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger \widehat{\otimes}_{K[Z^+]} (D(\mathbb{M}_n^\circ, M)_{M_n^\circ=\tilde{\chi}} \widehat{\otimes}_{D(\mathbb{M}_n^\circ, M)} (D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} V'_b)_{N_0}). \end{aligned}$$

If $z \in Z^+$ is such that ϕ_z induces a $D(\mathbb{M}_n^\circ, M_0)$ -compact endomorphism of $(D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} V'_b)_{N_0}$, then by Lemma 2.3.4(ii), we see that ϕ_z induces a $D(\mathbb{M}_n^\circ, M)_{M_n^\circ=\tilde{\chi}}$ -compact endomorphism of

$$D(\mathbb{M}_n^\circ, M)_{M_n^\circ=\tilde{\chi}} \widehat{\otimes}_{D(\mathbb{M}_n^\circ, M)} (D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} V'_b)_{N_0}.$$

Since M_n° has finite index in M_0 , we see that $D(\mathbb{M}_n^\circ, M)_{M_n^\circ=\tilde{\chi}}$ is in fact a finite dimensional K -algebra, and thus that ϕ_z induces simply a compact endomorphism of

$$D(\mathbb{M}_n^\circ, M)_{M_n^\circ=\tilde{\chi}} \widehat{\otimes}_{D(\mathbb{M}_n^\circ, M)} (D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} V'_b)_{N_0}.$$

Proposition 2.3.6 implies that

$$\mathcal{C}^{\text{an}}(\widehat{M}_n, K)^\dagger \widehat{\otimes}_{K[Z^+]} (D(\mathbb{M}_n^\circ, M)_{M_n^\circ=\tilde{\chi}} \widehat{\otimes}_{D(\mathbb{M}_n^\circ, M)} (D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{\mathcal{D}^{\text{la}}(H_0, K)} V'_b)_{N_0})$$

is finite dimensional over K . Taking into account (4.2.35), this completes the proof of the proposition. \square

We close this subsection by strengthening the preceding proposition, under suitable hypotheses. We continue to assume that \mathbb{P} is a Borel subgroup of \mathbb{G} .

PROPOSITION 4.2.36. – *Let V be an object of $\text{Rep}_{\text{ad}}(G)$. If there is a compact open subgroup H of G such that $V \xrightarrow{\sim} \mathcal{C}^{\text{la}}(H, K)^r$ as an H representation, for some natural number r (so that in particular V is a strongly admissible locally analytic representation of G), then $\text{Exp}(J_P(V))$ is equidimensional of dimension d , where d is the dimension of the maximal torus \mathbb{M} of \mathbb{G} .*

Proof. – The key point is that our assumption on V will allow us to reduce to a situation in which we may apply the relative theory of compact operators developed in [6], for compact operators on orthonormalizable Banach modules. A careful explanation of how we make this reduction is somewhat involved, and accounts for the length of the following argument.

If H' is an open subgroup of H , then there is an H' -equivariant isomorphism $\mathcal{C}^{\text{la}}(H, K)^r \xrightarrow{\sim} \mathcal{C}^{\text{la}}(H', K)^{r[H:H']}$, and so in the statement of the proposition, we may replace H by any such open subgroup H' . In particular, after relabelling the members of the sequence $\{H_n\}$ if necessary, we may assume that the open subgroup H appearing in the statement of the proposition is equal to H_0 .

We choose $z \in Z^+$ such that $z^{-1}\overline{N}_n z \subset \overline{N}_{n+1}$ for each $n \geq 0$, and such that z^{-1} and Z^+ together generate the group Z_M . (A consideration of the proof of Proposition 4.1.6(vi) shows that we may do this.) Let Y^+ (respectively Y) denote the submonoid (respectively subgroup) of Z_M generated by z . The discussion preceding the statement of Proposition 4.2.22 shows that V^{N_0} lies in $\text{Rep}_{\text{la.c}}^z(Z^+)$, and so by Proposition 3.2.27 there is a natural isomorphism

$$J_P(V) \xrightarrow{\sim} \mathcal{L}_{b, Y^+}(\mathcal{C}^{\text{an}}(\widehat{Y}, K), V^{N_0}).$$

Dualizing, we obtain an isomorphism

$$(4.2.37) \quad J_P(V)'_b \xrightarrow{\sim} K\{\{z, z^{-1}\}\} \widehat{\otimes}_{K[z]} (V^{N_0})'_b,$$

where $K\{\{z, z^{-1}\}\}$ denotes the ring of entire functions on \widehat{Y} (which is isomorphic to \mathbb{G}_m , thought of as a rigid analytic space).

As in the proof of Proposition 4.2.33, we set $U_n := (D(\mathbb{H}_n^\circ, H_0) \widehat{\otimes}_{D^{\text{la}}(H_0, K)} V'_b)_{N_0}$, for each $n \geq 0$. Also, set $U(z)_n = (D(\mathbb{H}(z)_n^\circ, H_0) \widehat{\otimes}_{D^{\text{la}}(H_0, K)} V'_b)_{N_0}$ for each value of n . There is an isomorphism

$$(4.2.38) \quad (V^{N_0})'_b \xrightarrow{\sim} \varprojlim_n U_n.$$

By assumption, $V \xrightarrow{\sim} \mathcal{C}^{\text{la}}(H_0, K)^r$. The untwisting isomorphism (4.2.25) thus yields isomorphisms of $D(\mathbb{M}_n^\circ, M_0)$ -modules

$$(4.2.39) \quad U_n \xrightarrow{\sim} D(\overline{N}_n^\circ, \overline{N}_0)^r \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0)$$

and

$$(4.2.40) \quad U(z)_n \xrightarrow{\sim} D(z^{-1}\overline{N}_n^\circ z, \overline{N}_0)^r \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0),$$

where $D(\mathbb{M}_n^\circ, M_0)$ acts on the targets of these isomorphisms via its action on the right-hand factor in the completed tensor product. We may fit these isomorphisms into the following commutative diagram:

$$\begin{array}{ccc}
 U_n & \xrightarrow[(4.2.39)]{\sim} & D(\overline{N}_n^\circ, \overline{N}_0)^r \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0) \\
 \downarrow (4.2.9) & & \downarrow \alpha_n \\
 U(z)_n & \xrightarrow[(4.2.40)]{\sim} & D(z^{-1}\overline{N}_n^\circ z, \overline{N}_0)^r \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0) \\
 \downarrow (4.2.8) & & \downarrow \beta_n \\
 U_n & \xrightarrow[(4.2.39)]{\sim} & D(\overline{N}_n^\circ, \overline{N}_0)^r \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0)
 \end{array}$$

ϕ_z (curved arrow from top-left to bottom-left)

where the vertical arrow labelled β_n is obtained by taking the completed tensor product of the natural map

$$(4.2.41) \quad D(z^{-1}\mathbb{N}_n^\circ z, \overline{N}_0)^r \rightarrow D(\mathbb{N}_n^\circ, \overline{N}_0)^r$$

(induced by the embedding $z^{-1}\mathbb{N}_n^\circ z \rightarrow \mathbb{N}_n^\circ$) with $D(\mathbb{M}_n^\circ, M_0)$ over K . By assumption, the embedding $z^{-1}\mathbb{N}_n^\circ z \rightarrow \mathbb{N}_n^\circ$ factors through the embedding $\mathbb{N}_{n+1}^\circ \rightarrow \mathbb{N}_n^\circ$, and so is relatively compact (by Proposition 4.1.6(i)). Hence (4.2.41) is a compact map, and so can be factored as

$$D(z^{-1}\mathbb{N}_n^\circ z, \overline{N}_0)^r \xrightarrow{\iota_{n,1}} W_n \xrightarrow{\iota_{n,2}} D(\mathbb{N}_n^\circ, \overline{N}_0)^r,$$

where W_n is a Banach space. Consequently, the map β_n factors as

$$\begin{aligned} D(z^{-1}\mathbb{N}_n^\circ z, \overline{N}_0)^r \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0) &\xrightarrow{\beta_{n,1}} W_n \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0) \\ &\xrightarrow{\beta_{n,2}} D(\mathbb{N}_n^\circ, \overline{N}_0)^r \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0), \end{aligned}$$

where $\beta_{i,n} := \iota_{i,n} \widehat{\otimes} \text{id}$. The composite $\beta_n \circ \alpha_n$ defines a $D(\mathbb{M}_n^\circ, M_0)$ -linear action of z (and thus Y^+) on $D(\overline{N}_n^\circ, \overline{N}_0)^r \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0)$, compatible with the isomorphism (4.2.39) and the action of z (and thus Y^+) on U_n defined by ϕ_z . The composite $\beta_{n,1} \circ \alpha_n \circ \beta_{n,2}$ defines a $D(\mathbb{M}_n^\circ, M_0)$ -linear action of z (and thus Y^+) on $W_n \widehat{\otimes}_K D(\mathbb{M}_n^\circ, M_0)$. By construction, the maps $\beta_{n,1} \circ \alpha_n$ and $\beta_{n,2}$ are Y^+ -equivariant and $D(\mathbb{M}_n^\circ, M_0)$ -linear.

The isomorphisms (4.2.37), (4.2.38), and (4.2.39), [8, prop. 1.1.29], and Proposition 2.1.9, together yield isomorphisms

$$(4.2.42) \quad \begin{aligned} J_P(V)'_b &\xrightarrow{\sim} \varprojlim_n K \{ \{z, z^{-1}\} \} \widehat{\otimes}_{K[z]} U_n \\ &\xrightarrow{\sim} \varprojlim_n K \{ \{z, z^{-1}\} \} \widehat{\otimes}_{K[z]} D(\overline{N}_n^\circ, \overline{N}_0)^r \widehat{\otimes}_K D(M_n^\circ, M_0) \\ &\xrightarrow{\sim} \varprojlim_n K \{ \{z, z^{-1}\} \} \widehat{\otimes}_{K[z]} W_n \widehat{\otimes}_K D(M_n^\circ, M_0) \end{aligned}$$

(the third isomorphism being induced by the maps $\beta_{n,1} \circ \alpha_n$).

The isomorphism $\mathcal{D}^{\text{la}}(M_0, K) \xrightarrow{\sim} \mathcal{C}^{\text{an}}(\widehat{M}_0, K)$ of [8, prop. 6.4.6] shows that $\mathcal{D}^{\text{la}}(M_0, K)$ is equal to the Fréchet algebra of rigid analytic functions on the rigid analytic space \widehat{M}_0 . Writing \widehat{M}_0 as the union of an increasing sequence of admissible open affinoid subdomains, we may write $\mathcal{D}^{\text{la}}(M_0, K) \xrightarrow{\sim} \varprojlim_n A_n$, where $\{A_n\}$ is a projective sequence of Tate algebras over K . We also have the isomorphism $\mathcal{D}^{\text{la}}(M_0, K) \xrightarrow{\sim} \varprojlim_n D(\mathbb{M}_n^\circ, M_0)$. The two projective sequences $\{A_n\}$ and $\{D(\mathbb{M}_n^\circ, M_0)\}$ are cofinal [8, prop. 1.2.7]. Passing to subsequences, and relabelling if necessary, we may assume that the map $\mathcal{D}^{\text{la}}(M_0, K) \rightarrow A_n$ factors through the map $\mathcal{D}^{\text{la}}(M_0, K) \rightarrow D(\mathbb{M}_n^\circ, M_0)$, for each n . Tensoring the n -th term of the projective sequence with A_n over $D(\mathbb{M}_n^\circ, M_0)$, the isomorphism (4.2.42) yields an isomorphism

$$(4.2.43) \quad J_P(V)'_b \xrightarrow{\sim} \varprojlim_n K \{ \{z, z^{-1}\} \} \widehat{\otimes}_{K[z]} W_n \widehat{\otimes}_K A_n.$$

Since $\iota_{n,2}$ is compact (being a map from a Banach space to the compact type space $D(\mathbb{M}_n^\circ, M_0)$), we find that z acts on $W_n \widehat{\otimes}_K A_n$ through an A_n -compact map.

In order to show that the support of $J_P(V)'_b$ is equidimensional of dimension d , it follows from (4.2.43) that it suffices to show that this is true for each of the A_n -modules

$$K \{ \{z, z^{-1}\} \} \widehat{\otimes}_{K[z]} W_n \widehat{\otimes}_K A_n.$$

Since $W_n \widehat{\otimes}_K A_n$ is an orthonormalizable A_n -Banach module on which z acts via an A_n -compact map, this follows from the theory of [6, §A] and [7, §1]. \square

4.3. In this subsection we establish property 0.13. We maintain the notation of the preceding subsections.

Let V be an admissible smooth representation of G defined over K . If we equip V with its finest convex topology, then it becomes an object of $\text{Rep}_{\text{ad}}(G)$ [8, prop. 6.3.2]. Our first goal is to show that $J_P(V)$ is isomorphic as an M -representation to the N -coinvariants V_N of V . In order to compute $J_P(V)$, we fix a lift of the Levi quotient M to a subgroup of P , as well as a compact open subgroup P_0 of P . Let \overline{P} denote the opposite parabolic to P (chosen so that $P \cap \overline{P} = M$) and let \overline{N} denote the unipotent radical of \overline{P} .

LEMMA 4.3.1. – *The Z_M^+ -representation V^{N_0} is the locally convex inductive limit of an increasing sequence of finite dimensional Z_M^+ -invariant subspaces.*

Proof. – Let $v \in V^{N_0}$, and choose an open subgroup H_1 of G that leaves v invariant. Shrinking H_1 if necessary, we may assume that H_1 admits an Iwahori decomposition $H_1 = \overline{N}_1 M_1 N_1$ with $N_1 \subset N_0$. Set $Y := \{y \in Z_M^+ \mid yN_1y^{-1} \subset N_1, y^{-1}\overline{N}_1y \subset \overline{N}_1\}$. Since V is an admissible smooth G -representation, the space V^{H_1} is finite dimensional. Furthermore, the Hecke operators $\pi_{N_1,y}$ for $y \in Y$ leave V^{H_1} invariant. (If we let π_{H_1} denote averaging over H_1 , then a consideration of the Iwahori decomposition of H_1 shows that $\pi_{H_1}(yv_1) = \pi_{N_1}(yv_1) =: \pi_{N_1,y}(v_1)$ for any $y \in Y$ and $v_1 \in V^{H_1}$.) Hence the span of the set $\{\pi_{N_1,y}(v) \mid y \in Y\}$ is finite dimensional. A variant of Proposition 3.3.2(i) shows that Y generates Z_M as a group, and so we may find a finite subset $\{z_1, \dots, z_n\}$ of Z_M^+ so that $Z_M^+ = \bigcup_{i=1}^n z_i Y$. Since $\pi_{N_0,y} = \pi_{N_0} \pi_{N_1,y}$ for $y \in Y$, we conclude that the span of the set $\{\pi_{N_0,z}(v) \mid z \in Z_M^+\}$ is also finite dimensional, and hence that V^{N_0} is the union of finite dimensional Z_M^+ -invariant subspaces. Since V , and hence V^{N_0} , is equipped with its finest convex topology, we see that V^{N_0} is even isomorphic to the locally convex inductive limit of these invariant subspaces. \square

PROPOSITION 4.3.2. –

- (i) *The natural map $(V^{N_0})_{\text{null}} \oplus J_P(V) \rightarrow V^{N_0}$ is an isomorphism.*
- (ii) *Each of $(V^{N_0})_{\text{null}}$ and $J_P(V)$ is an M^+ -invariant subspace of V^{N_0} .*

Proof. – Part (i) follows from Lemmas 4.3.1 and 3.2.18 and Propositions 3.2.6(iv) and 3.2.15(ii) (once we recall that, by definition, $J_P(V) = (V^{N_0})_{\text{fs}}$). Part (ii) follows by functoriality of the formation of $(V^{N_0})_{\text{null}}$ and $J_P(V)$. \square

We now recall some ideas from [5, §3]. Since V is a smooth representation of G , it is in particular smooth as a representation of N , and consequently, for any compact open subgroup N'_0 of N the map $\pi_{N'_0}$ is defined on all of V . As in [5, §3] we let $V(N)$ denote the subspace of V consisting of those vectors v for which $\pi_{N'_0}v = 0$ for some compact open subgroup N'_0 of N ; it is easily checked that $V(N)$ is equal to the kernel of the natural map

$$(4.3.3) \quad V \rightarrow V_N$$

[5, prop. 3.2.1]. (This description of the kernel shows that the functor $V \mapsto V_N$, which is right exact by construction, is in fact an exact functor on smooth N -representations [5, prop. 3.2.3].)

By Proposition 3.3.2, any compact open subgroup of N is contained in $z^{-1}N_0z$ for some $z \in Z_M^+$. The equation $\pi_{N_0,z} = z\pi_{z^{-1}N_0z}$ thus shows that $V(N)$ may also be described as the subspace of V consisting of those vectors that are annihilated by $\pi_{N_0,z}$ for some element $z \in Z_M^+$. In particular, $V(N) \cap V^{N_0} = (V^{N_0})_{\text{null}}$.

PROPOSITION 4.3.4. –

- (i) *The restriction of the morphism (4.3.3) to V^{N_0} induces an M^+ -equivariant surjective map $V^{N_0} \rightarrow V_N$ (the M^+ -action being defined on the source via the operators $\pi_{N_0, m}$, and on the target via restricting the natural M -action).*
- (ii) *The map of (i) restricts to an M -equivariant isomorphism $J_P(V) \xrightarrow{\sim} V_N$.*

Proof. – For any $v \in V$, the elements v and $\pi_{N_0} v$ have the same image in V_N . This proves (i). The discussion preceding the lemma shows that $(V^{N_0})_{\text{null}}$ is the kernel of the map of (i). Thus we obtain the isomorphism of (ii). Since it is an M^+ -equivariant map between M -modules, it is automatically M -equivariant. \square

PROPOSITION 4.3.5. – *For any admissible smooth representation V of G , the isomorphism $J_P(V) \xrightarrow{\sim} V_N$ of Proposition 4.3.4(ii) is independent of the choices of Levi factor M and compact open subgroup P_0 of P that are used to compute $J_P(V)$.*

Proof. – If we compute $J_P(V)$ with respect to a different choice of M (respectively of P_0), then the natural isomorphism of Proposition 3.4.10 (respectively Proposition 3.4.11) involves conjugation by an element of N (respectively the application of the projection operator π_{N_0}). This operation reduces to the identity of V_N , and so the present proposition follows. \square

We now generalize Proposition 4.3.4 so as to give a description of the Jacquet module of certain locally algebraic G -representations.

PROPOSITION 4.3.6. – *If W is a finite dimensional algebraic representation of \mathbb{G} , with $B = \text{End}_G(W)$, and if X is an admissible smooth representation of G over B , then there is a natural isomorphism $J_P(X \otimes_B W) \xrightarrow{\sim} X_N \otimes_B W^N$.*

Proof. – Since X is a smooth representation of G , and hence of N , the inclusion $X^n \subset X$ is in fact an equality. Since W is a locally algebraic representation of \mathbb{G} , and hence of \mathbb{N} , the inclusions $W^N \subset W^{N_0} \subset W^n$ are both equalities. Combining these two observations, we obtain the equality $X^{N_0} \otimes_B W^N = (X \otimes_B W)^{N_0}$.

If $m \in M^+$, then the operator $\pi_{N_0, m}$ on $(X \otimes_B W)^{N_0}$ can be described in terms of this tensor product decomposition as $\pi_{N_0, m} : u \otimes w \mapsto \pi_{N_0, m}^X(u) \otimes mw$ (where $\pi_{N_0, m}^X$ denotes the corresponding operator on X^{N_0}). Thus as an M^+ -representation, the space $(X \otimes_B W)^{N_0}$ is isomorphic to the tensor product of the M^+ -representation on X^{N_0} and the space W^N , equipped with the M^+ -action obtained by restricting the natural M -action on this space.

Proposition 3.2.9 thus yields an M -equivariant isomorphism $((X \otimes_B W)^{N_0})_{\text{fs}} \xrightarrow{\sim} (X^{N_0})_{\text{fs}} \otimes_B W^N$. The proposition now follows from the definition of the functor J_P , together with Proposition 4.3.4(ii). \square

Recall that a locally algebraic representation V of G is said to be an admissible locally algebraic G -representation if it becomes an admissible locally analytic G -representation when endowed with its finest convex topology [8, def. 6.3.9]. We will always regard an admissible locally algebraic representation as being an object of $\text{Rep}_{\text{ad}}(G)$, by endowing it with its finest convex topology. By [8, prop. 6.3.11], any admissible locally algebraic G -representation admits an isomorphism $V \xrightarrow{\sim} \bigoplus_n X_n \otimes_{B_n} W_n$, where W_n runs over a sequence of isomorphism class representatives for the irreducible algebraic representations of \mathbb{G} , $B_n = \text{End}_G(W_n)$, and X_n is an admissible smooth representation of G over B_n . Proposition 4.3.6 together with Lemma 3.4.7(i) and (iv) thus yields an isomorphism $J_P(V) \xrightarrow{\sim} \bigoplus_n (X_n)_N \otimes_{B_n} W_n^N$.

4.4. In this subsection we will establish property 0.14. As above we fix a compact open subgroup P_0 of P , as well as a Levi factor M of P , and we define N_0 , M_0 , M^+ and Z_M^+ as they

were defined preceding Corollary 3.3.3. If V is a locally analytic representation of G , we define the Hecke operators $\pi_{N_0, m}$ on V^{N_0} as in Definition 3.4.2. We also assume throughout this section that $L = \mathbb{Q}_p$. (This is not such a serious loss of generality, since restriction of scalars allows us to regard a reductive group over any finite extension of \mathbb{Q}_p as a reductive group over \mathbb{Q}_p .)

Let \mathbb{S} denote the maximal subtorus of $\mathbb{Z}_{\mathbb{M}}$ that splits over \mathbb{Q}_p , and as in Section 1.4, let Y^\bullet (respectively Y_\bullet) denote the character lattice (respectively the cocharacter lattice) of \mathbb{S} . If χ is a character of Z_M , then the discussion of that subsection allows us to define the element $\text{slope}(\chi)$ of $\mathbb{Q} \otimes_{\mathbb{Z}} Y^\bullet$.

Let $\Delta(\mathbb{G}, \mathbb{S})$ denote the set of positive restricted roots of \mathbb{S} (i.e. the characters of \mathbb{S} appearing in the adjoint action of \mathbb{S} on the Lie algebra of \mathbb{N}), and let $\Delta(\mathbb{G}, \mathbb{S})_s$ denote the subset of positive simple restricted roots. Let R^\bullet denote the sublattice of Y^\bullet spanned by $\Delta(\mathbb{G}, \mathbb{S})_s$, and let $(\mathbb{Q} \otimes_{\mathbb{Z}} R^\bullet)^{\geq 0}$ denote the $\mathbb{Q}^{\geq 0}$ -invariant cone in $\mathbb{Q} \otimes_{\mathbb{Z}} R^\bullet$ generated by the elements of $\Delta(\mathbb{G}, \mathbb{S})_s$.

LEMMA 4.4.1. – *If χ is an element of \widehat{Z}_M , then χ satisfies the inequality $|\chi(a)| \leq 1$ for every $a \in Z_M^+$ if and only if $\text{slope}(\chi) \in (\mathbb{Q} \otimes_{\mathbb{Z}} R^\bullet)^{\geq 0}$.*

Proof. – Taking $T = Z_M$ in the diagram (1.4.1) yields a surjection $\text{ord} : Z_M \rightarrow Y'_\bullet$, where Y'_\bullet is a sublattice of $\mathbb{Q} \otimes_{\mathbb{Z}} Y_\bullet$ containing Y_\bullet with finite index. Consider the submonoid

$$(Y'_\bullet)^+ = \{y \in Y'_\bullet \mid \langle y, r \rangle \geq 0 \text{ for all } r \in (\mathbb{Q} \otimes_{\mathbb{Z}} R^\bullet)^{\geq 0}\}$$

of Y'_\bullet . The image of Z_M^+ under the map ord is contained in $(Y'_\bullet)^+$, and is cofinal in this submonoid (when the latter is directed by the relation of divisibility; compare the proof of Lemma 3.3.1). This cofinality, together with the equation $\text{ord}(\chi(a)) = \langle \text{ord}(a), \text{slope}(\chi) \rangle$, shows that $\text{ord}(\chi(a)) \geq 0$ for all $a \in Z_M^+$ if and only if $\langle y, \text{slope}(\chi) \rangle \geq 0$ for all $y \in (Y'_\bullet)^+$, which holds if and only if $\text{slope}(\chi) \in (\mathbb{Q} \otimes_{\mathbb{Z}} R^\bullet)^{\geq 0}$, proving the lemma. \square

As in the preceding subsections, we let δ denote the modulus character of P , regarded as a smooth character of M . If we write $\rho := \rho(\mathbb{G}, \mathbb{S})$ to denote one-half the sum of the elements of $\Delta(\mathbb{G}, \mathbb{S})$, each counted with the multiplicity with respect to which it appears in the adjoint action of \mathbb{S} on \mathbb{N} , then $\text{slope}(\delta|_{\mathbb{S}}) = -\rho$. (This is where the assumption that $L = \mathbb{Q}_p$ is used.)

LEMMA 4.4.2. – *Let V be an object of $\text{Rep}_{\text{la.c.}}(P)$, and suppose that V admits a P -invariant norm. If $\chi \in \widehat{Z}_M$ is such that $V^{N_0, Z_M^+ = \chi}$ is non-zero, then for every $a \in Z_M^+$ we have $|\delta(a)^{-1}\chi(a)| \leq 1$. Equivalently, $\rho + \text{slope}(\chi)$ lies in $(\mathbb{Q} \otimes_{\mathbb{Z}} R^\bullet)^{\geq 0}$.*

Proof. – Let $\|-\|$ denote the P -invariant norm on V , whose existence we are assuming, and let v be a non-zero vector in $V^{N_0, Z_M^+ = \chi}$. If $a \in Z_M^+$ then $\chi(a)v = \pi_{N_0, a}v = [N_0 : aN_0a^{-1}]^{-1} \sum_{n \in N_0/aN_0a^{-1}} nav = \delta(a) \sum_{n \in N_0/aN_0a^{-1}} nav$, hence $|\delta(a)^{-1}\chi(a)| \|v\| = \|\sum_{n \in N_0/aN_0a^{-1}} nav\| \leq \|v\|$, and so $|\delta(a)^{-1}\chi(a)| \leq 1$. The equivalent formulation in terms of slopes follows from Lemma 4.4.1. \square

For the remainder of the subsection, assume that \mathbb{G} is split over K , and fix a finite dimensional irreducible algebraic representation W of \mathbb{G} over K . As was noted in Section 1.2, highest weight theory shows that the space $W^{\mathbb{N}}$ of \mathbb{N} -invariants of W is an irreducible algebraic representation of \mathbb{M} , and so $\mathbb{Z}_{\mathbb{M}}$ must act on $W^{\mathbb{N}}$ through a character $\psi \in X^\bullet(\mathbb{Z}_{\mathbb{M}})$, which we also regard as an element of $\widehat{Z}_M(K)$. We also fix a smooth (i.e. locally constant) character $\theta \in \widehat{Z}_M(K)$, and write $\chi = \psi\theta$.

Let us for the moment choose a Borel subgroup \mathbb{B} of \mathbb{G} , defined over K , and let \mathbb{T} be a maximal torus of this Borel subgroup (again, defined over K), contained in \mathbb{M} . The intersection $\mathbb{M} \cap \mathbb{B}$ is then a Borel subgroup of \mathbb{M} . Let \mathfrak{n}' denote the Lie algebra of the unipotent radical of

\mathbb{B} , and let $\bar{\mathfrak{n}}'$ denote the Lie algebra of unipotent radical of the opposite Borel to \mathbb{B} (with respect to the chosen torus \mathbb{T}).

Since \mathbb{G} is split over K , the highest weight theory recalled in Section 1.1 applies to W . Let $\tilde{\psi} \in X^\bullet(\mathbb{T})$ denote the highest weight of the representation W with respect to \mathbb{B} ; this is also the highest weight of the \mathbb{M} -representation $W^{\mathbb{N}}$ with respect to $\mathbb{M} \cap \mathbb{B}$. Note that $\tilde{\psi}|_{\mathbb{Z}_{\mathbb{M}}} = \psi$. If α is a simple restricted root of $\mathbb{Z}_{\mathbb{M}}$ acting on \mathfrak{n} , then denote by $\tilde{\alpha}$ the (unique, by Lemma 1.2.3) simple root of \mathbb{T} acting on \mathfrak{n}' that lifts α . If $\tilde{\rho} := \rho(\mathbb{G}, \mathbb{T})$ denotes one-half the sum of the positive roots of \mathbb{T} acting on \mathfrak{n}' , then $\tilde{\rho}|_{\mathbb{S}} = \rho$.

DEFINITION 4.4.3. – We say that $\chi = \psi\theta$ is of critical slope with respect to the representation $W^{\mathbb{N}}$ if for some positive simple root $\alpha \in \Delta(\mathbb{G}, \mathbb{Z}_{\mathbb{M}})$, the element $s_{\tilde{\alpha}}(\tilde{\psi} + \tilde{\rho})|_{\mathbb{S}} + \text{slope}(\theta) + \rho$ of $\mathbb{Q} \otimes_{\mathbb{Z}} Y^\bullet$ in fact lies in $(\mathbb{Q} \otimes_{\mathbb{Z}} R^\bullet)^{\geq 0}$. Otherwise, we say that χ is of non-critical slope with respect to $W^{\mathbb{N}}$. (By Lemma 1.2.6 this definition is independent of the choice of Borel subgroup \mathbb{B} .)

Let V be an object of $\text{Rep}_{\text{la.c.}}(P)$ admitting a P -invariant norm. If the eigenspace of $V^{N_0, Z_M^+ = \chi}$ is non-zero, then Lemma 4.4.2 shows that $\psi|_{\mathbb{S}} + \rho + \text{slope}(\theta) + \rho$ lies in $(\mathbb{Q} \otimes_{\mathbb{Z}} R^\bullet)^{\geq 0}$. The element $s_{\tilde{\alpha}}(\tilde{\psi} + \tilde{\rho})|_{\mathbb{S}} + \text{slope}(\theta) + \rho$ thus also lies in $\mathbb{Q} \otimes_{\mathbb{Z}} R^\bullet$. The key point of Definition 4.4.3 in this case is thus whether or not this element actually lies in the positive cone of $\mathbb{Q} \otimes_{\mathbb{Z}} R^\bullet$.

If \mathbb{G} is quasi-split over \mathbb{Q}_p , and if \mathbb{P} is chosen to be a Borel subgroup of \mathbb{G} defined over \mathbb{Q}_p , then $\mathbb{Z}_{\mathbb{M}} = \mathbb{M} = \mathbb{T}$ (using the above notation), and so $\tilde{\psi} = \psi$. In this case the character χ thus determines the representation $W^{\mathbb{N}}$ (indeed, the latter is just equal to the character ψ), and so we may say that the character χ is of critical or non-critical slope, without making explicit reference to the representation $W^{\mathbb{N}}$.

If V is an object of $\text{Rep}_{\text{la.c.}}(G)$, then $V^{N_0, Z_M^+ = \chi}$ is a locally analytic representation of M_0 . Thus we may consider its subspace $(V^{N_0, Z_M^+ = \chi})_{W^{\mathbb{N}}\text{-lalg}}$ of locally $W^{\mathbb{N}}$ -algebraic vectors (as defined in [8, prop.-def. 4.2.2]; note that if we had not chosen χ so that it coincided with ψ locally, then this space would necessarily vanish). We may also consider its subspace $(V^{N_0, Z_M^+ = \chi})^{\mathfrak{m} \cap \mathfrak{b} = d\tilde{\psi}}$, the subspace of vectors on which the Lie algebra $\mathfrak{m} \cap \mathfrak{b}$ of $\mathbb{B} \cap \mathbb{M}$ acts through the character $d\tilde{\psi}$ of its quotient \mathfrak{t} (the Lie algebra of \mathbb{T}).

The proof of the following proposition is somewhat analogous to a computation appearing in the proof of [17, prop. 6.2] (for the special case $\mathbb{G} = \text{GL}_2$).

PROPOSITION 4.4.4. – Let $\tilde{\alpha}$ be a positive simple root of \mathbb{T} acting on \mathfrak{n}' , and let $X_{-\tilde{\alpha}} \in \bar{\mathfrak{n}}'$ be an element of the root space $\bar{\mathfrak{n}}_{-\tilde{\alpha}}$ corresponding to the negative root $-\tilde{\alpha}$. Let $\tilde{\alpha}^\vee$ denote the corresponding coroot, and write $m = \langle \tilde{\psi}, \tilde{\alpha}^\vee \rangle$. If V is a locally analytic representation of G then the action of $X_{-\tilde{\alpha}}^{m+1}$ on V^{N_0} restricts to a map $(V^{N_0, Z_M^+ = \chi})^{\mathfrak{m} \cap \mathfrak{b} = d\tilde{\psi}} \rightarrow V^{N_0, Z_M^+ = \tilde{\alpha}^{-m-1}\chi}$.

Proof. – Let v be an element of $(V^{N_0, Z_M^+ = \chi})^{\mathfrak{m} \cap \mathfrak{b} = d\tilde{\psi}}$. The action of \mathfrak{g} on v determines a $U(\mathfrak{g})$ -module morphism $\text{Ver}(d\tilde{\psi}) \rightarrow V$, given by mapping the highest weight vector $v(d\tilde{\psi})$ of $\text{Ver}(d\tilde{\psi})$ to v . (Here $\text{Ver}(d\tilde{\psi})$ denotes the Verma module of highest weight $d\tilde{\psi}$, as in Definition 1.1.5.) Proposition 1.1.6 shows that the element $X_{-\tilde{\alpha}}^{m+1}(1 \otimes v(d\tilde{\psi}))$ of $\text{Ver}(d\tilde{\psi})$ is fixed by the adjoint action of N . Thus if $n \in N_0$ we see that

$$\begin{aligned} nX_{-\tilde{\alpha}}^{m+1}v &= nX_{-\tilde{\alpha}}^{m+1}n^{-1}v = \phi(\text{Ad}_n X_{-\tilde{\alpha}}^{m+1}(1 \otimes v(d\tilde{\psi}))) \\ &= \phi(X_{-\tilde{\alpha}}^{m+1}(1 \otimes v(d\tilde{\psi}))) = X_{-\tilde{\alpha}}^{m+1}v, \end{aligned}$$

and so $X_{-\tilde{\alpha}}^{m+1}v$ lies in V^{N_0} . Furthermore, for any $a \in Z_M^+$ we compute that

$$\begin{aligned}
 \pi_{N_0,a} X_{-\tilde{\alpha}}^{m+1} v &= [N_0 : aN_0 a^{-1}]^{-1} \sum_{n \in N_0/aN_0 a^{-1}} na X_{-\tilde{\alpha}}^{m+1} v \\
 &= [N_0 : aN_0 a^{-1}]^{-1} \sum \text{Ad}_n \text{Ad}_a X_{-\tilde{\alpha}}^{m+1} nav \\
 &= [N_0 : aN_0 a^{-1}]^{-1} \sum \tilde{\alpha}(a)^{-m-1} X_{-\tilde{\alpha}}^{m+1} nav \\
 &= \tilde{\alpha}(a)^{-m-1} X_{-\tilde{\alpha}}^{m+1} [N_0 : aN_0 a^{-1}]^{-1} \sum nav \\
 &= \tilde{\alpha}(a)^{-m-1} X_{-\tilde{\alpha}}^{m+1} \pi_{N_0,a} v \\
 &= \tilde{\alpha}(a)^{-m-1} \chi(a) X_{-\tilde{\alpha}}^{m+1} v.
 \end{aligned}$$

(The third equality follows from the fact that $X_{-\tilde{\alpha}}$ is an eigenvector for the action of Ad_a , with eigenvalue $\tilde{\alpha}^{-1}$, together with the fact, already used above, that $X_{-\tilde{\alpha}}^{m+1} v$ is fixed by the adjoint action of N , while the final equality follows from the assumption that v lies in $V^{N_0, Z_M^+ = \chi}$.) This proves the proposition. \square

We can now prove the main result of this subsection.

THEOREM 4.4.5. – *Let V be an object of $\text{Rep}_{\text{la.c}}(G)$, and suppose that V admits a G -invariant norm. If χ is of non-critical slope, then the space $(V^{N_0, Z_M^+ = \chi})_{W^{\mathbb{N}}\text{-lalg}}$ consists of locally W -algebraic vectors of V .*

Proof. – Let v be an element of $(V^{N_0, Z_M^+ = \chi})_{W^{\mathbb{N}}\text{-lalg}}$. If U denotes the \mathfrak{m} -submodule of V^{N_0} generated by v , then by assumption U is isomorphic to a finite direct sum of copies of $W^{\mathbb{N}}$. Thus if v is not locally algebraic, then we may find a highest weight vector in the \mathfrak{m} -representation U which is also not locally algebraic. Such a vector lies in $(V^{N_0, Z_M^+ = \chi})^{\mathfrak{m} \cap \mathfrak{b} = d\tilde{\psi}}$. Replacing v by such a vector, if necessary, we may assume that v is a non-locally algebraic element of $(V^{N_0, Z_M^+ = \chi})^{\mathfrak{m} \cap \mathfrak{b} = d\tilde{\psi}}$, which is nevertheless $W^{\mathbb{N}}$ -locally algebraic under the action of M .

As in the proof of Proposition 4.4.4, the vector v determines a $U(\mathfrak{g})$ -module morphism $\text{Ver}(d\tilde{\psi}) \rightarrow V$, given by mapping the highest weight vector $v(d\tilde{\psi})$ of $\text{Ver}(d\tilde{\psi})$ to v . Since v is not locally algebraic, Corollary 1.1.4 shows that this map does not factor through the finite dimensional \mathfrak{g} -representation W . It thus follows from Proposition 1.1.6(ii) that we may find a positive simple root $\tilde{\alpha}$ of \mathbb{T} acting on \mathfrak{n}' , and an element $X_{-\tilde{\alpha}} \in \bar{\mathfrak{n}}_{-\tilde{\alpha}}$, such that $X_{-\tilde{\alpha}}^{m+1} v \neq 0$. (Here the notation is that of Proposition 4.4.4. In particular, $m = \langle \tilde{\psi}, \tilde{\alpha}^\vee \rangle$, and so $m + 1 = \langle \tilde{\psi} + \tilde{\rho}, \tilde{\alpha}^\vee \rangle$.) Since v is $W^{\mathbb{N}}$ -locally algebraic, the simple root $\tilde{\alpha}$ must appear in \mathfrak{n} , and thus restricts to an element $\alpha \in \Delta(\mathbb{G}, \mathbb{Z}_M)$. By Proposition 4.4.4 we thus obtain a non-zero element of $V^{N_0, Z_M^+ = \tilde{\alpha}^{-m-1}\chi}$. Applying Lemma 4.4.2 to $\tilde{\alpha}^{-m-1}\chi$, we find that $\rho - (m + 1)\tilde{\alpha}|_{\mathbb{S}} + \text{slope}(\chi)$ lies in $(\mathbb{Q} \otimes_{\mathbb{Z}} R^\bullet)^+$.

We compute that

$$\begin{aligned}
 \rho - (m + 1)\tilde{\alpha}|_{\mathbb{S}} + \text{slope}(\chi) &= \rho - \langle \tilde{\psi} + \tilde{\rho}, \tilde{\alpha}^\vee \rangle \tilde{\alpha}|_{\mathbb{S}} + \psi|_{\mathbb{S}} + \text{slope}(\theta) \\
 &= \psi|_{\mathbb{S}} + \rho - \langle \tilde{\psi} + \tilde{\rho}, \tilde{\alpha}^\vee \rangle \tilde{\alpha}|_{\mathbb{S}} + \text{slope}(\theta) + \rho \\
 &= s_{\tilde{\alpha}}(\tilde{\psi} + \tilde{\rho})|_{\mathbb{S}} + \text{slope}(\theta) + \rho
 \end{aligned}$$

(the last equality following from the definition of $s_{\tilde{\alpha}}$). Thus we see that χ is of critical slope, contradicting the hypothesis of the theorem. Hence $(V^{N_0, Z_M^+ = \chi})_{W^{\mathbb{N}}\text{-lalg}}$ must consist entirely of locally algebraic elements. \square

Property 0.14 follows from preceding theorem, together with Proposition 3.4.9.

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