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ON THE ZERO SET OF SEMI-INVARIANTS FOR QUIVERS

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ABSTRACT. – Let $\mathbf{d}$ be a prehomogeneous dimension vector for a finite quiver Q. We show that the set of common zeros of all semi-invariants of positive degree for the variety of representations of Q with dimension vector $N \cdot \mathbf{d}$ under the product of the general linear groups at all vertices is irreducible and a complete intersection for large natural numbers N.
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RÉSUMÉ. – Soient Q un carquois fini et $\mathbf{d}$ un vecteur de dimension pour Q tels que la variété des représentations de Q de dimension $\mathbf{d}$ contienne une orbite dense sous l'action du groupe $\mathrm{Gl}(\mathbf{d})$ des changements de base en chaque sommet. Nous montrons que l'ensemble des zéros des semi-invariants de degré positif sous $\mathrm{Gl}(N \cdot \mathbf{d})$ sur la variété des représentations de dimension $N \cdot \mathbf{d}$ est irréductible et une intersection complète pourvu que N soit suffisamment grand.
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1. Introduction

Let k be an algebraically closed field, and let $Q = (Q_0, Q_1, t, h)$ be a finite quiver, i.e. a finite set $Q_0 = \{1, \ldots, n\}$ of vertices and a finite set Q_1 of arrows $\alpha : t\alpha \to h\alpha$, where $t\alpha$ and $h\alpha$ denote the tail and the head of α, respectively.

A representation of Q over k is a collection $(X(i);\ i \in Q_0)$ of finite dimensional k-vector spaces together with a collection $(X(\alpha) : X(t\alpha) \to X(h\alpha);\ \alpha \in Q_1)$ of k-linear maps. A morphism $f : X \to Y$ between two representations is a collection $(f(i) : X(i) \to Y(i))$ of k-linear maps such that

$$f(h\alpha) \circ X(\alpha) = Y(\alpha) \circ f(t\alpha) \quad \text{for all } \alpha \in Q_1.$$

The dimension vector of a representation X of Q is the vector

$$\mathbf{dim}\, X = \big(\dim X(1), \ldots, \dim X(n)\big) \in \mathbb{N}^{Q_0}.$$

We denote the category of representations of Q by $\mathrm{rep}(Q)$, and for any vector $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^{Q_0}$

$$\mathrm{rep}(Q, \mathbf{d}) = \prod_{\alpha \in Q_1} \mathrm{Mat}(d_{h\alpha} \times d_{t\alpha}, k)$$

is the vector space of representations X of Q with $X(i) = k^{d_i}$, $i \in Q_0$. The group

$$\mathrm{Gl}(\mathbf{d}) = \prod_{i=1}^{n} \mathrm{Gl}(d_i, k)$$

acts on $\operatorname{rep}(Q,\mathbf{d})$ by

$$\big((g_1,\ldots,g_n)\star X\big)(\alpha)=g_{h\alpha}\cdot X(\alpha)\cdot g_{t\alpha}^{-1}.$$

Note that the $\mathrm{Gl}(\mathbf{d})$-orbit of X consists of the representations Y in $\operatorname{rep}(Q,\mathbf{d})$ which are isomorphic to X.

We call $\mathbf{d}$ a prehomogeneous dimension vector if $\operatorname{rep}(Q,\mathbf{d})$ contains an open orbit $\mathrm{Gl}(\mathbf{d})\star T$. Such a representation T is characterized by $\mathrm{Ext}^1_Q(T,T)=0$ [6]. If Q admits only finitely many indecomposable representations, or equivalently if the underlying graph of $\overline{Q}$ is a disjoint union of Dynkin diagrams $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$ [2], every vector $\mathbf{d}$ is prehomogeneous. Indeed, any representation is a direct sum of indecomposables in an essentially unique way by the theorem of Krull–Schmidt, and therefore $\operatorname{rep}(Q,\mathbf{d})$ contains finitely many orbits, one of which must be open.

Let $\mathbf{d}$ be prehomogeneous, and let $f_1,\ldots,f_s\in k[\operatorname{rep}(Q,\mathbf{d})]$ be the irreducible monic polynomials whose zeros $Z(f_1),\ldots,Z(f_s)$ are the irreducible components of codimension 1 of $\operatorname{rep}(Q,\mathbf{d})\setminus\mathrm{Gl}(\mathbf{d})\star T$, where $\mathrm{Gl}(\mathbf{d})\star T$ is the open orbit. It is easy to see that

$$g\cdot f_i=\chi_i(g)\cdot f_i$$

for $g\in\mathrm{Gl}(\mathbf{d})$, where $\chi_i:\mathrm{Gl}(\mathbf{d})\to k^*$ is a character. A regular function with this property is called a semi-invariant. By [8], any semi-invariant is a scalar multiple of a monomial in $f_1,\ldots,f_s$, and $f_1,\ldots,f_s$ are algebraically independent. We denote by

$$\mathcal{Z}_{Q,\mathbf{d}}=\big\{X\in\operatorname{rep}(Q,\mathbf{d});\ f_i(X)=0,\ i=1,\ldots,s\big\}$$

the set of common zeros of all semi-invariants of positive degree. Obviously we have $\operatorname{codim}\mathcal{Z}_{Q,\mathbf{d}}\leqslant s$, and equality means that $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection.

Our first main result is as follows.

THEOREM 1.1. – *Let $T_1,\ldots,T_r$ be pairwise non-isomorphic indecomposable representations in $\operatorname{rep}(Q)$ such that $\mathrm{Ext}^1_Q(T_i,T_j)=0$ for any $i,j\leqslant r$. Then there is a positive integer N such that $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection and an irreducible variety for any dimension vector $\mathbf{d}=\sum_{i=1}^r\lambda_i\cdot\mathbf{dim}\,T_i$ with $\lambda_i\geqslant N$, $i\leqslant r$.*

As an immediate consequence we derive the following fact.

COROLLARY 1.2. – *Let $\mathbf{d}$ be a prehomogeneous dimension vector in $\mathbb{N}^{Q_0}$. Then there is a positive integer N such that $\mathcal{Z}_{c\cdot\mathbf{d}}$ is a complete intersection and an irreducible variety for any $c\geqslant N$.*

We will prove in a forthcoming paper that we may choose $N=2$ if $\overline{Q}$ is a disjoint union of Dynkin diagrams and $N=3$ if $\overline{Q}$ is a disjoint union of Dynkin diagrams and extended Dynkin diagrams.

In order to put our results into the context of invariant theory, we recall a few definitions. We assume that k is the field of complex numbers. Let G be a reductive algebraic group acting regularly on a finite dimensional vector space V. By Hilbert's theorem, the ring $k[V]^G$ of G-invariant polynomials on V is a finitely generated algebra and thus is the algebra of polynomial functions on a variety $V//G$. The inclusion of $k[V]^G$ into $k[V]$ gives rise to a regular surjective map $\pi:V\to V//G$ which is constant on G-orbits, the so-called categorical quotient of V by G [3]. As G is completely reducible, the G-module $k[V]$ can be decomposed uniquely as a direct sum

(1.1) $$k[V]=\bigoplus_\lambda M_\lambda\otimes_k V_\lambda$$

where M_λ ranges over a set of representatives of the irreducible G-modules and V_λ is just a vector space, possibly infinite dimensional, which records the multiplicity with which M_λ arises in $k[V]$. As the action of G on $k[V]$ commutes with multiplication by G-invariants, each V_λ can be viewed as a $k[V]^G$-module. In fact, (1.1) is a decomposition as G-$k[V]^G$-bimodules; the group G acts only on M_λ and $k[V]^G$ only on V_λ. A covariant of weight λ is a G-linear map

$$\varphi : k[V] \to M_\lambda,$$

or equivalently a linear form on V_λ. The pair (V, G) is called:

- coregular if $V//G$ has no singularities,
- equidimensional if the fiber $\pi^{-1}(\pi(0))$ has the same dimension as the quotient $V//G$,
- cofree if $k[V]$ is free as a $k[V]^G$-module, or equivalently the module V_λ^* of covariants is free over $k[V]^G$ for all λ.

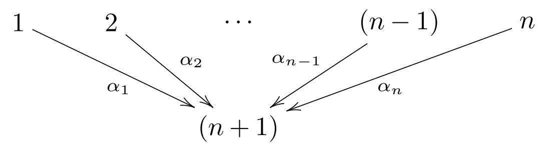
An equidimensional coregular pair (V, G) is automatically cofree ([5], [12] §17). G. Schwarz classified all coregular and cofree representations of connected simple algebraic groups [10,11]. In [4] P. Littelmann classified all cofree irreducible representations of semisimple groups.

Let us consider $V = \operatorname{rep}(Q, \mathbf{d})$ for a prehomogeneous $\mathbf{d}$ as a representation of the subgroup $\operatorname{Sl}(\mathbf{d}) = \prod_{i=1}^n \operatorname{Sl}(d_i)$ of $\operatorname{Gl}(\mathbf{d})$. For each arrow α, the set

$$V_\alpha = \{X \in V;\ X(\beta) = 0\ \forall \beta \neq \alpha\}$$

is an irreducible subrepresentation of V, and V is the direct sum $V = \bigoplus_{\alpha \in Q_1} V_\alpha$. The ring $k[V]^{\operatorname{Sl}(\mathbf{d})}$ of $\operatorname{Sl}(\mathbf{d})$-invariants is generated by the semi-invariants $f_1, \ldots, f_s$. So the categorical quotient $V// \operatorname{Sl}(\mathbf{d})$ is an s-dimensional affine space and $(V, \operatorname{Sl}(\mathbf{d}))$ is coregular. It is cofree in the cases for which our main results mentioned above hold. Surprisingly, the situation is better for big multiples of a given dimension vector. It can be bad otherwise, as the following example illustrates: For

$Q =$



the dimension vector $\mathbf{d} = (1, \ldots, 1, n-1)$ is prehomogeneous. Indeed, the open orbit in $\operatorname{rep}(Q, \mathbf{d})$ consists of those representations X for which none of the $(n-1) \times (n-1)$-minors $f_1, \ldots, f_n$ of the $(n-1) \times n$-matrix $[X(\alpha_1), \ldots, X(\alpha_n)]$ vanishes. On the other hand, the set $\mathcal{Z}_{Q,\mathbf{d}}$ of common zeros of $f_1, \ldots, f_n$ is the set of $(n-1) \times n$-matrices of rank less than $n-1$ and thus has codimension 2. In [1], Chang and Weyman examine quivers Q with underlying graph $\mathbb{A}_n$ and arbitrary dimension vectors. They find that $(\operatorname{rep}(Q, \mathbf{d}), \operatorname{Sl}(\mathbf{d}))$ is always equidimensional, but the set $\pi^{-1}(\pi(0)) = \mathcal{Z}_{Q,\mathbf{d}}$ is reducible in general.

2. Notations and preliminaries

We first recall Schofield's construction of semi-invariants from [9]. The Euler form is the $\mathbb{Z}$-bilinear form on $\mathbb{Z}^{Q_0}$ defined by

$$\langle \mathbf{d}, \mathbf{e} \rangle = \sum_{i \in Q_0} d_i e_i - \sum_{\alpha \in Q_1} d_{t\alpha} e_{h\alpha}.$$

For $\mathbf{d},\mathbf{e}\in\mathbb{N}^{Q_0}$, $X\in\operatorname{rep}(Q,\mathbf{d})$, $Y\in\operatorname{rep}(Q,\mathbf{e})$ we consider the linear map

$$\mathcal{F}_{X,Y}:\bigoplus_{i\in Q_0}\operatorname{Hom}_k\bigl(k^{d_i},k^{e_i}\bigr)\to\bigoplus_{\alpha\in Q_1}\operatorname{Hom}_k\bigl(k^{d_{t\alpha}},k^{e_{h\alpha}}\bigr),$$

which sends $(g_i;\ i\in Q_0)$ to $(h_\alpha;\ \alpha\in Q_1)$ with $h_\alpha=g_{h\alpha}\cdot X(\alpha)-Y(\alpha)\cdot g_{t\alpha}$. Note that

$$\ker\mathcal{F}_{X,Y}=\operatorname{Hom}_Q(X,Y)\quad\text{and}\quad\operatorname{coker}\mathcal{F}_{X,Y}\simeq\operatorname{Ext}^1_Q(X,Y).$$

This implies that

$$\langle\mathbf{dim}\,X,\mathbf{dim}\,Y\rangle=[X,Y]-{}^1[X,Y],$$

where we set

$$[X,Y]=\dim_k\operatorname{Hom}_Q(X,Y),\qquad {}^1[X,Y]=\dim_k\operatorname{Ext}^1_Q(X,Y).$$

If we assume that $\langle\mathbf{d},\mathbf{e}\rangle=0$, the linear map $\mathcal{F}_{X,Y}$ will be represented by a square matrix $M_{X,Y}$ (with respect to some bases), and the determinant $\det M_{X,Y}$ is a $\mathrm{Gl}(\mathbf{d})\times\mathrm{Gl}(\mathbf{e})$-semi-invariant on $\operatorname{rep}(Q,\mathbf{d})\times\operatorname{rep}(Q,\mathbf{e})$. It might vanish, however.

For a representation U of Q, the right perpendicular category $U^\perp$ and the left perpendicular category ${}^\perp U$ are the full subcategories of $\operatorname{rep}(Q)$ whose objects Y satisfy

$$[U,Y]={}^1[U,Y]=0\quad\text{and}\quad[Y,U]={}^1[Y,U]=0,$$

respectively. As ${}^1[X,Y]=[Y,\tau X]$ for any two representations X and Y of Q, where τ is the Auslander–Reiten translation (see [7] for relevant definitions), we have $U^\perp={}^\perp(\tau U)$.

Now assume that $T_1,\ldots,T_r$ are pairwise non-isomorphic with ${}^1[T_i,T_j]=0$, $i,j=1,\ldots,r$ and such that $T=\bigoplus_{i=1}^r T_i^{\lambda_i}$ is sincere, i.e., $T(e)\neq0$ for all $e\in Q_0$. Then the category $T^\perp$ is equivalent to the category of representations of a quiver $Q^\perp$ having $(n-r)$ vertices. Choose $Y\in T^\perp$, $Y\neq0$, and set $\mathbf{d}=\mathbf{dim}\,T$, $\mathbf{e}=\mathbf{dim}\,Y$. Observe that

$$\langle\mathbf{d},\mathbf{e}\rangle=[T,Y]-{}^1[T,Y]=0,$$

the dimension of $\bigoplus_{i\in Q_0}\operatorname{Hom}_k(k^{d_i},k^{e_i})$ is positive and $M_{T,Y}$ is invertible. Thus the $\mathrm{Gl}(\mathbf{d})$-semi-invariant $f_Y=\det M_{X,Y}$ is non-trivial on $\operatorname{rep}(Q,\mathbf{d})$. It is easy to see that $f_Y=f_{Y'}\cdot f_{Y''}$ for an exact sequence $0\to Y'\to Y\to Y''\to0$ in $T^\perp$. If the simple objects of $T^\perp$ are $S_1,\ldots,S_{n-r}$, the semi-invariants $f_{S_1},\ldots,f_{S_{n-r}}$ are algebraically independent and they generate the algebra of $\mathrm{Sl}(\mathbf{d})$-invariants. As a consequence we have

$$\begin{aligned}\mathcal{Z}_{Q,\mathbf{d}}&=\bigl\{X\in\operatorname{rep}(Q,\mathbf{d});\ [X,S_j]\neq0,\ j=1,\ldots,n-r\bigr\}\\&=\bigl\{X\in\operatorname{rep}(Q,\mathbf{d});\ [X,Y]\neq0\text{ for all }Y\in T^\perp,\ Y\neq0\bigr\}.\end{aligned}$$

We will keep the following assumptions and notations throughout the paper: T is a sincere representation of Q with ${}^1[T,T]=0$ and $\mathbf{dim}\,T=\mathbf{d}$. We can always make T sincere by considering the full subquiver which supports T instead of Q. Observe that Q does not contain oriented cycles. If the decomposition of T as a direct sum of pairwise non-isomorphic indecomposables is

$$T=\bigoplus_{i=1}^r T_i^{\lambda_i},\quad\lambda_i\geqslant1,$$

we set $\lambda = \min\{\lambda_i\colon\ i = 1,\ldots,r\}$. Note that $\mathcal{Z}_{Q,\mathbf{d}}$ is defined by $n-r$ polynomial equations. In order to prove it is a complete intersection it suffices to show

$$\operatorname{codim}\mathcal{Z}_{Q,\mathbf{d}} \geqslant n-r.$$

All varieties we consider will be quasi-projective over k, and we will look at codimensions for constructible subsets of affine space only.

3. Proof of Theorem 1.1

Our strategy is to first get rid of the set $\{X \in \mathcal{Z}_{Q,\mathbf{d}};\ {}^1[T,X] \neq 0\}$ by showing its codimension is big. In fact, it is for this we need our assumption on the multiplicities λ_i.

The following lemma will be used several times in our article.

LEMMA 3.1. – *Let* $\mathbf{d}'' \in \mathbb{N}^{Q_0} \setminus \{0\}$ *be such that* $\mathbf{d}'' \leqslant \mathbf{d}$, *i.e.,* $\mathbf{d}' = \mathbf{d} - \mathbf{d}'' \in \mathbb{N}^{Q_0}$, *and let* $V = V_1 \oplus \cdots \oplus V_b$ *belong to* $\operatorname{rep}(Q,\mathbf{d}'')$, *where* $V_1,\ldots,V_b$ *are indecomposable. Then the set*

$$\mathcal{A}_V = \bigl\{X \in \operatorname{rep}(Q,\mathbf{d});\ \exists \textit{ epimorphism } X \to V\bigr\}$$

is constructible, irreducible, and $\operatorname{codim}\mathcal{A}_V \geqslant b - \langle \mathbf{d},\mathbf{d}''\rangle$.

Proof. – Consider the subvariety

$$\mathcal{C} = \left\{\left(X, g = \begin{bmatrix} g' \\ g'' \end{bmatrix}\right);\ g'' \in \operatorname{Hom}_Q(X,V)\right\}$$

of $\operatorname{rep}(Q,\mathbf{d}) \times \operatorname{Gl}(\mathbf{d})$. Note that g'' is an epimorphism as g is invertible. This leads to the surjective regular map $\pi\colon\mathcal{C} \to \mathcal{A}_V$ given by the first projection. We see that the set $\mathcal{A}_V$ is constructible and that

$$\dim\pi^{-1}(X) = [X,V] + \sum_{i\in Q_0} d_i' d_i \geqslant b + \sum_{i\in Q_0} d_i' d_i$$

for $X \in \mathcal{A}_V$, since there exists an epimorphism $X \to V_1 \oplus \cdots \oplus V_b$.

On the other hand, sending (X,g) to $(g \star X, g)$ we obtain an isomorphism from $\mathcal{C}$ to the subvariety $\mathcal{D}$ of $\operatorname{rep}(Q,\mathbf{d}) \times \operatorname{Gl}(\mathbf{d})$ consisting of all pairs (Y,g) for which $Y(\alpha)$ is in the block form

$$Y(\alpha) = \begin{bmatrix} * & * \\ 0 & V(\alpha) \end{bmatrix}, \quad \alpha \in Q_1.$$

As $\mathcal{D}$ is just the product of an affine space of dimension $\sum_{\alpha\in Q_1} d'_{h\alpha} d_{t\alpha}$ with $\operatorname{Gl}(\mathbf{d})$, we conclude that $\mathcal{A}_V$ is irreducible and that

$$\sum_{\alpha\in Q_1} d'_{h\alpha} d_{t\alpha} + \dim\operatorname{Gl}(\mathbf{d}) - \dim\mathcal{A}_V = \dim\mathcal{C} - \dim\mathcal{A}_V \geqslant b + \sum_{i\in Q_0} d_i' d_i.$$

Our estimate follows from an easy computation. □

COROLLARY 3.2. – *Keeping the notations of the preceding lemma, we assume moreover that* V *is a subrepresentation of* τT. *Then we have*

$$\operatorname{codim}\mathcal{A}_V \geqslant 1 + \lambda.$$

Proof. – It suffices to show $-\langle \mathbf{d}, \mathbf{d}'' \rangle \geqslant \lambda_i$ for some i. Observe that

$$[T, V] \leqslant [T, \tau T] = {}^1[T, T] = 0 \quad \text{and} \quad {}^1[T, V] = [V, \tau T] > 0.$$

Consequently, we have ${}^1[T_i, V] \geqslant 1$ for some i and thus

$$-\langle \mathbf{d}, \mathbf{d}'' \rangle = -[T, V] + {}^1[T, V] = {}^1[T, V] \geqslant {}^1[T_i^{\lambda_i}, V] \geqslant \lambda_i. \qquad \square$$

For any $U \in \operatorname{rep}(Q)$, we denote by $\mathcal{X}_U$ the set

$$\mathcal{X}_U = \{X \in \operatorname{rep}(Q, \mathbf{d});\ [X, U] \neq 0\}.$$

LEMMA 3.3. – *Let U be a non-zero subrepresentation of τT. Then we have*

$$\operatorname{codim} \mathcal{X}_U \geqslant \lambda + 1 - \eta(\mathbf{dim}\, U),$$

where $\eta(\mathbf{e}) = \sum_{i \in Q_0} [e_i^2/4]$, for any $\mathbf{e} \in \mathbb{N}^{Q_0}$, and $[q]$ denotes the largest integer not exceeding q for $q \in \mathbb{Q}$.

Proof. – We want to exploit that for any non-zero homomorphism $\varphi : X \to U$ from $X \in \operatorname{rep}(Q, \mathbf{d})$ to U, X belongs to $\mathcal{A}_V$ for the representation $V = \varphi(X)$, which is a quotient of X as well as a subrepresentation of τT. Set $\mathbf{e} = \mathbf{dim}\, U$, and choose $\mathbf{f} \in \mathbb{N}^{Q_0}$ with $\mathbf{f} \leqslant \mathbf{e}, \mathbf{d}$. Consider the closed subvariety $\mathcal{L}_\mathbf{f}$ of $\prod_{i \in Q_0} \operatorname{Grass}(k^{e_i}, f_i)$ consisting of sequences $(V_i)_{i \in Q_0}$ such that $U(\alpha)(V_{t\alpha}) \subseteq V_{h\alpha}$, $\alpha \in Q_1$. In other words, $\mathcal{L}_\mathbf{f}$ is the variety of all subrepresentations V of U with dimension vector $\mathbf{f}$. Note that $\dim \mathcal{L}_\mathbf{f} \leqslant \eta(\mathbf{e})$ since $\dim \operatorname{Grass}(k^e, f) = (e - f) f \leqslant [e^2/4]$. The subset $\mathcal{F}_\mathbf{f}$ of $\mathcal{L}_\mathbf{f} \times \operatorname{rep}(Q, \mathbf{d})$ consisting of pairs (V, X) such that there is an epimorphism from X onto V is constructible. Indeed, the affine subvariety

$$\mathcal{H} = \{(\varphi = (\varphi_i), X);\ \varphi \in \operatorname{Hom}_Q(X, U)\} \subseteq \prod_{i \in Q_0} \operatorname{Hom}_k(k^{d_i}, k^{e_i}) \times \operatorname{rep}(Q, \mathbf{d})$$

is the disjoint union $\coprod_{\mathbf{f} \leqslant \mathbf{e}, \mathbf{d}} \mathcal{H}_\mathbf{f}$ of the locally closed subsets

$$\mathcal{H}_\mathbf{f} = \{(\varphi, X) \in \mathcal{H};\ \operatorname{rk} \varphi_i = f_i,\ i \in Q_0\},$$

and $\mathcal{L}_\mathbf{f}$ is the image of $\mathcal{H}_\mathbf{f}$ under the regular map sending (φ, X) to $((\operatorname{im} \varphi_i), X)$. Observe that $\mathcal{X}_U$ is the union $\bigcup_{0 \neq \mathbf{f} \leqslant \mathbf{d}, \mathbf{e}} \pi_2(\mathcal{F}_\mathbf{f})$ of the images under the second projection, and thus

$$\dim \mathcal{X}_U \leqslant \max_{0 \neq \mathbf{f} \leqslant \mathbf{d}, \mathbf{e}} \dim \mathcal{F}_\mathbf{f}.$$

Now consider the first projection $\pi_1 : \mathcal{F}_\mathbf{f} \to \mathcal{L}_\mathbf{f}$. For $V \in \mathcal{L}_\mathbf{f}$, we have $\pi_1^{-1}(V) = \{V\} \times \mathcal{A}_V$, so we know by Corollary 3.2 that

$$\dim \pi_1^{-1}(V) \leqslant \dim \operatorname{rep}(Q, \mathbf{d}) - 1 - \lambda,$$

as $V \subseteq U$ is a subrepresentation of τT with $\mathbf{dim}\, V \leqslant \mathbf{d}$. We conclude that

$$\begin{aligned} \dim \mathcal{X}_U &\leqslant \max_{0 \neq \mathbf{f} \leqslant \mathbf{d}, \mathbf{e}} \dim \mathcal{F}_\mathbf{f} \leqslant \big(\max_{0 \neq \mathbf{f} \leqslant \mathbf{d}, \mathbf{e}} \dim \mathcal{L}_\mathbf{f}\big) + \dim \operatorname{rep}(Q, \mathbf{d}) - 1 - \lambda \\ &\leqslant \eta(\mathbf{e}) - 1 - \lambda + \dim \operatorname{rep}(Q, \mathbf{d}), \end{aligned}$$

which implies our claim.

COROLLARY 3.4. – *Let* $c = \max\{\eta(\tau T_i);\ i = 1, \ldots, r\}$. *Then the set*

$$\mathcal{E}_{\mathbf{d}} = \left\{X \in \operatorname{rep}(Q, \mathbf{d});\ {}^1[T, X] > 0\right\}$$

is either empty or else $\operatorname{codim} \mathcal{E}_{\mathbf{d}} \geqslant 1 + \lambda - c$.

Proof. – If T is projective then the set $\mathcal{E}_{\mathbf{d}}$ is empty. Otherwise, any non-zero map in

$$\operatorname{Hom}_Q(X, \tau T) \simeq \operatorname{Ext}^1_Q(T, X)$$

induces a non-zero map $X \to \tau T_i$ for some non-projective T_i, and we see that

$$\mathcal{E}_{\mathbf{d}} = \bigcup_{T_i \text{ non-projective}} \mathcal{X}_{\tau T_i}.$$

The claim follows from Lemma 3.3.

Remark 3.5. – For $\lambda \geqslant c + n - r$, we have that either $\mathcal{E}_{\mathbf{d}}$ is empty or that $\operatorname{codim} \mathcal{E}_{\mathbf{d}} \geqslant 1 + n - r$.

Now we concentrate on the set $\mathcal{Z}'_{\mathbf{d}} = \{X \in \mathcal{Z}_{Q,\mathbf{d}};\ {}^1[T, X] = 0\}$.

LEMMA 3.6. – *For* $X \in \mathcal{Z}'_{\mathbf{d}}$, *there exists an epimorphism* $X \to S = \bigoplus S_j$, *where the sum is taken over the* $n - r$ *simple objects of* $T^{\perp}$.

Proof. – We choose a basis $\{f_1, \ldots, f_s\}$ of $\operatorname{Hom}_Q(T, X)$ and we put

$$f = (f_1, \ldots, f_s) : T^s \to X.$$

Then any homomorphism from T to X factors through f. Let $X' = \operatorname{im} f$ and $\overline{X} = \operatorname{coker} f$. The exact sequence

$$0 \to X' \to X \to \overline{X} \to 0 \tag{3.1}$$

induces the following long exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}_Q(T, X') &\xrightarrow{g} \operatorname{Hom}_Q(T, X) \to \operatorname{Hom}_Q(T, \overline{X}) \\ &\to \operatorname{Ext}^1_Q(T, X') \to \operatorname{Ext}^1_Q(T, X) \to \operatorname{Ext}^1_Q(T, \overline{X}) \to 0. \end{aligned}$$

Since there is an epimorphism $T' \to X'$ and since ${}^1[T, T'] = 0$, we have ${}^1[T, X'] = 0$. Moreover, g is bijective by the universality of f and, together with our assumption ${}^1[T, X] = 0$, this implies that $\overline{X} \in T^{\perp}$.

Recall that $[X, Y] \neq 0$ for all non-zero $Y \in T^{\perp}$ as X lies in $\mathcal{Z}_{Q,\mathbf{d}}$. In particular, $[X, S_j] \neq 0$ for $j = 1, \ldots, n - r$. Mapping the sequence (3.1) to S_j and using that $[X', S_j] \leqslant [T', S_j] = 0$, we find that $[\overline{X}, S_j] \neq 0$ for all j. But any non-zero morphism $\overline{X} \to S_j$ is surjective, because $\overline{X} \in T^{\perp}$ and $S_j \in T^{\perp}$ is simple. We obtain the required epimorphism by composing the projection $X \to \overline{X}$ with a surjective map $\overline{X} \to S = \bigoplus_{j=1}^{n-r} S_j$. □

Since the set $\mathcal{Z}_{Q,\mathbf{d}}$ is given by $(n - r)$ equations, each irreducible component of $\mathcal{Z}_{Q,\mathbf{d}}$ has codimension at most $(n - r)$. Thus Theorem 1.1 follows from Remark 3.5 and the following fact.

PROPOSITION 3.7. – *If* $\mathcal{Z}'_{\mathbf{d}}$ *is not empty, then it is irreducible and* $\operatorname{codim} \mathcal{Z}'_{\mathbf{d}} = n - r$.

Proof. – If $\mathcal{Z}'_{\mathbf{d}}$ is non-empty, Lemma 3.6 tells us that $\mathbf{d} \geqslant \mathbf{dim}\, S$ and that $\mathcal{Z}'_{\mathbf{d}}$ lies in

$$\mathcal{A}_S = \{X \in \operatorname{rep}(Q, \mathbf{d});\ \exists \text{ epimorphism } X \to S\}.$$

By Lemma 3.1, $\mathcal{A}_S$ is irreducible, and

$$\operatorname{codim} \mathcal{A}_S \geqslant n - r - \langle \mathbf{d}, \mathbf{d}'' \rangle = n - r - [T, S] + {}^1[T, S] = n - r.$$

As ${}^1[T, X] = 0$ is an open condition, $\mathcal{Z}'_{\mathbf{d}}$ is open in $\mathcal{Z}_{Q,\mathbf{d}}$, and therefore $\operatorname{codim} \mathcal{Z}'_{\mathbf{d}} \leqslant n - r$. Thus $\mathcal{Z}'_{\mathbf{d}}$ is open and dense in $\mathcal{A}_S$ and consequently irreducible. □

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