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On global non-degeneracy conditions for chaotic behavior for a class of dynamical systems

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Abstract

For about 25 years, global methods from the calculus of variations have been used to establish the existence of chaotic behavior for some classes of dynamical systems. Like the analytical approaches that were used earlier, these methods require nondegeneracy conditions, but of a weaker nature than their predecessors. Our goal here is study such a nondegeneracy condition that has proved useful in several contexts including some involving partial differential equations, and to show this condition has an equivalent formulation involving stable and unstable manifolds.

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1. Introduction

The study of chaotic behavior in dynamical systems goes back to Poincaré [12]. Such behavior was studied originally using qualitative arguments. On the more analytical level, the first results were for small time dependent perturbations of autonomous systems where tools like the Melnikov function showed there was a transversal intersection of stable and unstable manifolds at a hyperbolic equilibrium point. See e.g. Kirchgraber and Stoffer [10]. More recently, starting from work of Séré, [16], [17], global methods based on tools from the calculus of variations have been used to obtain chaotic behavior.

In general, global variational methods such as minimization or mountain pass arguments, allow one to find an initial set of solutions that are homoclinic or heteroclinic to the equilibria. E.g. for the model case that will be studied in this paper, there are two such solutions together with their integer phase shifts. Then under some nondegeneracy conditions on this initial set of solutions that are of a milder nature than those that are employed in perturbation

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settings, the existence of infinitely many heteroclinic and/or homoclinic solutions, as well as chaotic dynamics can then be established for the system. See e.g. Coti Zelati and Rabinowitz [5], Rabinowitz [14], [15], Montecchiari, Nolasco and Terracini [8], [9], Bessi [1], Buffoni and Séré [3], and Cieliebak and Séré [4]. In particular for one dimensional Lagrangian equations these conditions allow one to obtain the existence of chaotic dynamics whenever the stable and unstable manifolds do not coincide. See [1] where this has been done for a perturbation of a nonlinear pendulum equation or [9] for the Duffing equation. A generalization of this property for a class of C^∞ Hamiltonian systems was carried out by Cieliebak and Séré in [4] where the chaotic behavior was obtained provided that the set of homoclinic solutions has compact connected components with respect to the uniform metric on $\mathbb{R}$. A different but related connectedness condition on the set of minimal homoclinic solutions was used in Rabinowitz [14] (see also [13]) for singular Lagrangian systems on $\mathbb{R}^2$.

Some recent papers generalize the nondegeneracy condition introduced in [13], [14]. They employ assumptions that have also proved useful in several contexts including some involving partial differential equations such as in Montecchiari and Rabinowitz [6], [7] and Byeon, Montecchiari and Rabinowitz [2]. Our goal in this paper is to further study such nondegeneracy conditions and establish their equivalence to other such conditions. For definiteness, the condition of [13], [6], as generalized in [7], will be treated in the context of a Hamiltonian system for a double well potential having equilibrium points at $a^-, a^+ \in \mathbb{R}^m$. We will show that this condition has an equivalent formulation involving the stable and unstable manifolds associated with these two points. We then show that, when the potential is smooth, that condition is equivalent to an analogue of the assumption made in Cieliebak and Séré [4], see Proposition 5.32.

The paper is organized as follows. In §2, we recall some earlier results from e.g. [6], [2] and [7] on the existence of a large family of local minimizers of our Hamiltonian system, (HS), as well as an infinitude of mountain pass solutions. These results require nondegeneracy conditions. Let $\mathcal{W}^u(a^\sigma)$ denote the unstable manifold of (HS), at $\sigma \in \{-,+\}$, i.e. the global continuation of the local unstable manifold as given by the implicit function theorem. Likewise let $\mathcal{W}^s(a^\sigma)$ denote the corresponding stable manifold. In §4, the equivalence of the above mentioned nondegeneracy conditions to ones involving $\mathcal{W}^u(a^\sigma)$ and $\mathcal{W}^s(a^\sigma)$ will be shown. Some technical results required for this purpose will be given in §3. Lastly some more degenerate situations will be studied in §5.

2. Some preliminaries

We consider the Hamiltonian system:

$$-\ddot{q} + V_q(t,q) = 0, \quad t \in \mathbb{R},\ q \in \mathbb{R}^m \tag{HS}$$

where

(V_1) $V \in C^1(\mathbb{R} \times \mathbb{R}^m, \mathbb{R})$ and is 1-periodic in $t \in \mathbb{R}$.
(V_2) There are points $a^- \neq a^+ \in \mathbb{R}^m$ such that $V(t,q) > V(t,a^\pm) = 0$ for any $q \in \mathbb{R}^m \setminus \{a^\pm\}$.
(V_3) There is a constant, $V_0 > 0$, such that $\liminf_{|q|\to+\infty} V(t,q) \geq V_0$.

By (V_2), (HS) is a double well potential system. It is not difficult to prove that there are heteroclinic solutions of (HS) from a^- to a^+ as well as from a^+ to a^-, the former being obtained as minima of

$$I(q) = \int_{\mathbb{R}} L(q)\,dt \equiv \int_{\mathbb{R}} (\tfrac{1}{2}|\dot{q}|^2 + V(t,q))\,dt$$

defined on

$$\Gamma(a^-,a^+) = \{q \in W^{1,2}_{loc}(\mathbb{R},\mathbb{R}^m) \mid I(q) < \infty \; and \; \|q - a^\pm\|_{L^2([j,j+1])} \to \infty,\ j \to \pm\infty\}$$

and the latter is obtained in a similar fashion. See e.g. [13]. Define

$$c(a^-,a^+) = \inf_{q \in \Gamma(a^-,a^+)} I(q),$$

and set

$$\mathcal{M}(a^-,a^+) \equiv \{Q \in \Gamma(a^-,a^+) \mid I(q) = c(a^-,a^+)\}.$$

Then if $q \in \mathcal{M}(a^-,a^+)$, so is $q(\cdot - k)$ for any $k \in \mathbb{Z}$. Interchanging the roles of $-$ and $+$ gives us $\Gamma(a^+,a^-)$, $c(a^+,a^-)$, and $\mathcal{M}(a^+,a^-)$.

In addition to the minimizers of I in $\Gamma(a^-,a^+)$ and $\Gamma(a^+,a^-)$, I possesses a large family of local minimizers provided that $\mathcal{M}(a^-,a^+)$ and $\mathcal{M}(a^+,a^-)$ satisfy mild nondegeneracy conditions. To formulate them, set

$$\mathcal{S}(a^-,a^+) = \{u|_{[0,1]} \mid u \in \mathcal{M}(a^-,a^+)\}.$$

Thus the members of $\mathcal{S}(a^-,a^+)$ are unit time snapshots of heteroclinic solutions of (HS) that minimize I on $\Gamma(a^-,a^+)$. Define $\mathcal{C}_{a^-}(a^-,a^+)$ to be the component of $\bar{\mathcal{S}}(a^-,a^+)$ in $W^{1,2}([0,1],\mathbb{R}^m)$ containing a^- and $\mathcal{C}_{a^+}(a^-,a^+)$ to be the component of $\bar{\mathcal{S}}(a^-,a^+)$ containing a^+. Then from e.g. [15] or [2], we have a sharp alternative for these sets: Either

1^o $\mathcal{C}_{a^-}(a^-,a^+) = \{a^-\}$ and $\mathcal{C}_{a^+}(a^-,a^+) = \{a^+\}$, or
2^o $\mathcal{C}_{a^-}(a^-,a^+) = \mathcal{C}_{a^+}(a^-,a^+)$.

Using the natural notation, a similar alternative holds for $\mathcal{C}_{a^+}(a^+,a^-)$ and $\mathcal{C}_{a^-}(a^+,a^-)$. Together the pair of conditions in 1^o represent the nondegeneracy conditions required for chaotic behavior here. They allow us to "variationally glue" the members of $\mathcal{M}(a^-,a^+)$ and $\mathcal{M}(a^+,a^-)$ to obtain, for any $k \in \mathbb{N}$, infinitely many solutions of (HS) that undergo k transitions between a^- and a^+ and that are local minima of I. See [2] for details.

To continue, assume

(V_4) $V \in C^2(\mathbb{R} \times \mathbb{R}^m, \mathbb{R})$ and for each $t \in [0,1]$, the matrix

$$V_{qq}(t,a^\pm) = (\frac{\partial^2 V}{\partial q_i \partial q_j}(t,a^\pm))$$

is positive definite.

Condition (V_4) together with some further nondegeneracy conditions on the set of solutions of (HS) leads to the existence of solutions of mountain pass type. To be more precise, let $d > 0$ and let $\mathcal{D}^d$ denote the set of heteroclinic and homoclinic solutions, q of (HS) with $I(q) \le d$ and define

$$\mathcal{S}^d = \{u|_{[0,1]} \mid u \in \mathcal{D}^d\}.$$

Note that $\mathcal{S}^d$ differs from $\mathcal{S}(a^-,a^+)$ in that the former set includes snapshots of all heteroclinics or homoclinics, q, of (HS) with $I(q) \le d$. By Proposition 3.29 of [7], $\bar{\mathcal{S}^d} = \mathcal{S}^d \cup \{a^-\} \cup \{a^+\}$ is a compact subset of $W^{1,2}([0,1],\mathbb{R}^m)$. As above, let $\mathcal{C}^d(a^\pm)$ denote the component of $\overline{\mathcal{S}}^d$ to which $a^\pm$ belongs and for $\xi_1, \xi_2 \in \{a^-,a^+\}$, set

$$\mathcal{S}^d(\xi_1,\xi_2) = \{u|_{[0,1]} \mid u \in \mathcal{D}^d \cap \Gamma(\xi_1,\xi_2)\}.$$

Then we have a stronger variant of our earlier alternatives: One of the following mutually exclusive possibilities holds:

3^o $\mathcal{C}^d(a^\pm) = \{a^\pm\}$,
4^o $\mathcal{C}^d(a^\pm) = \mathcal{C}^d(a^\mp)$,
5^o $\mathcal{C}^d(a^\pm) \ne \{a^\pm\}$ and $\mathcal{C}^d(a^\pm) \subset \mathcal{S}^d(a^\pm,a^\pm) \cup \{a^\pm\}$.

Assuming the new nondegeneracy conditions 3^o for all large d leads to the existence of an infinitude of mountain pass solutions of (HS) that are distinct from the local minima obtained above.

For both the local minimum and the mountain pass settings, an important consequence of assuming that alternatives 1^o or 3^o hold is that they imply the corresponding sets of solution snapshots can be split into two pieces. More precisely for $\overline{\mathcal{S}}^d$, there exist closed nonempty disjoint subsets, K_1^-, K_2^- of $\overline{\mathcal{S}}^d$ such that $K_1^- \cup K_2^- = \overline{\mathcal{S}}^d$. Moreover $a^- \in K_1^-$ and for any $z \in K_1^- \setminus \{a^-\}$, there exists a $p \in \mathbb{Z}$ such that $g^p(z) \in K_2^-$ where for $j \in \mathbb{Z}$,

$$g^j : \overline{\mathcal{S}^d} \to \overline{\mathcal{S}^d}, \quad g^j(q|_{[0,1]}) = q(\cdot + j)|_{[0,1]}.$$

Similarly there exist closed nonempty disjoint subsets, K_1^+, K_2^+ of $\overline{\mathcal{S}}^d$ such that $K_1^+ \cup K_2^+ = \overline{\mathcal{S}}^d$, $a^+ \in K_1^+$ and for any $z \in K_1^+ \setminus \{a^+\}$, there exists a $p \in \mathbb{Z}$ such that $g^p(z) \in K_2^+$.

With the aid of these decompositions of $\bar{\mathcal{S}}^d$, classes of functions that shadow the 4 sets, $K_i^{\pm}$, $i = 1, 2$, in an appropriate fashion are introduced and minimization or minimax arguments then lead to critical points of I and corresponding solutions of (HS). For the (local) minimization setting, the classes are distinguished by the number of transitions between being near $\mathcal{S}(a^-, a^+)$ and near $\mathcal{S}(a^+, a^-)$ and by the amount of time it takes for each transition. Minimizing I over these classes then produces the local minima of I that were mentioned earlier, see e.g. [2]. The presence of these local minima causes the geometric structure of the Mountain Pass Theorem to occur at higher and higher level sets of the functional and alternative 3^o above is used to gain the compactness needed to obtain related mountain pass type solutions. See [7].

Our goal in this paper is to show that this splitting or separation property, which is a consequence of 3^o is in fact equivalent to 3^o. In §3, a more precise formulation of the separation property and its consequences will be made. Then in §4, the equivalence theorem will be stated and proved.

3. The separation property

We will show that 3^o is equivalent to a separation property that in turn is related to the stable and unstable manifolds of (HS) at the equilibrium points, a^- and a^+. For $\sigma \in \{-, +\}$, we are interested in certain subsets of $\mathcal{W}^u(a^\sigma), \mathcal{W}^s(a^\sigma)$, that lie in $\mathcal{D}^d$, possess an invariance property under integer phase shifts, and also possess a uniformity property. To be more explicit, for $\sigma, \tau \in \{-, +\}$, let $\mathcal{D}^d(a^\sigma, a^\tau)$ denote the class of solutions of (HS) that are heteroclinic from a^σ to a^τ (or homoclinic if $\sigma = \tau$) and have $I(q) \le d$. Assume for each such σ,

$(\Phi^u(a^\sigma))$ There exists a compact set $K^u(a^\sigma) \subset \overline{\mathcal{S}^d}$ and an $r^- > 0$ such that

i) $K^u(a^\sigma) \cap \{a^-, a^+\} = \{a^\sigma\}$ and

ii) For each $U \in \mathcal{D}^d(a^\sigma, a^-) \cup \mathcal{D}^d(a^\sigma, a^+)$, there is a unique $j^u(U) \in \mathbb{Z}$ such that $U(\cdot + j^u(U) + i)|_{[0,1]} \in K^u(a^\sigma)$ for any $i \le 0$ and

$$\|U(\cdot + j^u(U) + 1) - K^u(a^\sigma)\|_{W^{1,2}([0,1],\mathbb{R}^m)} \ge 3r^-$$

A word about notation. For a set, A, the notation $\|u(\cdot) - A\|_{W^{1,2}([0,1],\mathbb{R}^m)}$ as appears in the above definition of $(\Phi^u(a^\sigma))$ will be used repeatedly, but also sometimes we write $\text{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(u(\cdot), A)$ which has the same meaning.

Each function, $U \in K^u(a^\sigma)$ is uniquely determined by its initial conditions, $U(0), U'(0)$. With this identification, the compact set, $K^u(a^\sigma)$, can be viewed as a subset of the unstable manifold of (HS) at a^σ and as such consists of unit time snapshots of heteroclinic and homoclinic solutions emanating from a^σ having $I \le d$ and normalized via ii). Of course we could work directly in $\mathbb{R}^{2m}$ with sets of such initial conditions and avoid dealing with our sets involving snapshots of solutions, but such an approach would not be applicable to the existence of heteroclinic and homoclinic solutions of nonlinear elliptic partial differential equations such as were treated in [6]. We believe the approach taken here will readily generalize to such settings. The set $K^u(a^\sigma)$ is not uniquely determined by the above assumption. For example, for any $k \in \mathbb{Z}$, the set $g^k(K^u(a^\sigma))$ also satisfies the assumption for a different value of r^-. Later in this section a choice will be made from this class of admissible sets, $K^u(a^\sigma)$. First some consequences of $(\Phi^u(a^\sigma))$ will be studied.

Consider the set

$$K_0^u(a^-) = \{U(\cdot + j^u(U))|_{[0,1]} \mid U \in \mathcal{D}^d(a^-, a^-) \cup \mathcal{D}^d(a^-, a^+)\}.$$

Equivalent definitions of $K_0^u(a^-)$ are:

$$K_0^u(a^-) = \{U|_{[0,1]} \mid U \in \mathcal{D}^d(a^-, a^-) \cup \mathcal{D}^d(a^-, a^+),\ j^u(U) = 0\}$$
$$K_0^u(a^-) = \{U \in K^u(a^-) \setminus \{a^-\} \mid j^u(U) = 0\}$$

Thus $K_0^u(a^-) \subset K^u(a^-)$ and

Lemma 3.1. *$K_0^u(a^-)$ is compact.*

Proof. If $(u_n) \subset K_0^u(a^-)$, it has a subsequence (still denoted (u_n)) that converges to a point $u_0 \in K^u(a^-)$. Then $g^j(u_n) \to g^j(u_0)$ for any $j \le 0$. Since $K^u(a^-)$ is compact and $g^j(u_n) \in K^u(a^-)$, by $(\Phi^u(a^-))$, $g^j(u_0) \in K^u(a^-)$ for any $j \le 0$. Moreover, by the definition of $K_0^u(a^-)$ and (ii) of $(\Phi^u(a^-))$, it follows that $\|g(u_n) - K^u(a^-)\|_{W^{1,2}([0,1],\mathbb{R}^m)} \ge 3r^-$ so $\|g(u_0) - K^u(a^-)\|_{W^{1,2}([0,1],\mathbb{R}^m)} \ge 3r^-$. Thus $j^u(u_0) = 0$ and $u_0 \in K_0^u(a^-)$.

Now define

$$\mathcal{K}^u(a^-) = \{U \in \mathcal{D}^d(a^-, a^-) \cup \mathcal{D}^d(a^-, a^+) \mid U|_{[0,1]} \in K_0^u(a^-)\}.$$

Thus identifying $U \in \mathcal{K}^u(a^-)$ with its initial conditions, $\mathcal{K}^u(a^-)$ can be considered as the subset of the unstable manifold of (HS) at a^- consisting of heteroclinic and homoclinic solutions emanating from a^- having $I \le d$ and normalized by their behavior on [0, 1]. It has a compactness property:

Proposition 3.2. *$\mathcal{K}^u(a^-)$ is compact with respect to the $W^{1,2}((-\infty, 2], \mathbb{R}^m)$ metric.*

The proof of Proposition 3.2 requires some preparation. It was shown in [7] that under conditions (V_1)–(V_4), $\{q \in W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^m) \mid I(q) < \infty\}$ is a (C^2) Hilbert manifold, $\mathcal{E}$, modeled on $E = W^{1,2}(\mathbb{R}, \mathbb{R}^m)$. Choosing any $\psi_1 \in \mathcal{M}(a^-, a^+)$ and $\psi_2 \in \mathcal{M}(a^+, a^-)$, $\mathcal{E}$ consists of four components: $a^\pm + E$, $\psi_1 + E$, $\psi_2 + E$. For $q \in \Gamma(a^-, a^+)$ and $u = q - \psi_1 \in E$, set

$$J(u) = I(\psi_1 + u) = I(q). \tag{3.3}$$

Then $J \in C^2(E, \mathbb{R})$. The functional, J, is defined similarly on the other components of $\mathcal{E}$. From [7], we have for $\xi_- \ne \xi_+ \in \{a^-, a^+\}$ and ψ the associated choice of ψ_1, ψ_2,

$$c(\xi_-, \xi_+) = \inf_{u \in E} I(\psi + u) = \inf_{u \in E} J(u) > 0. \tag{3.4}$$

Again from [7], for $\xi_- = \xi_+ \in \{a^-, a^+\}$,

$$c(\xi_-, \xi_+) = \inf_{u \in E} I(\xi_- + u) = \inf_{u \in E} J(u) > 0. \tag{3.5}$$

Therefore by (3.4)–(3.5),

$$c_0 = \min_{\xi_1, \xi_2 \in \{a^-, a^+\}} c(\xi_1, \xi_2) > 0. \tag{3.6}$$

To continue, a result about Palais–Smale (PS) sequences for J is needed. By Proposition 3.10 of [13] or Proposition 3.27 in [7] – see also [5] for similar arguments – (PS) sequences of J on $E = \Gamma(a^-, a^+) - \psi_1$ are characterized by

Proposition 3.7. *Suppose that $(q_n) \subset \Gamma(a^-, a^+)$ where $q_n = \psi_1 + u_n$ with $u_n \in E$. If further $J(u_n) \to b \ge c_0$ and $J'(u_n) \to 0$ as $n \to \infty$, then there exists*

- *a $\kappa_0 = \kappa_0(b) \in \mathbb{N}$,*
- *an $l_0 \in \mathbb{N} \cap [1, \kappa_0]$,*
- *$\zeta_1, \zeta_2, ..., \zeta_{l_0+1} \in \{a^-, a^+\}$ with $\zeta_1 = a^-$, $\zeta_{l_0+1} = a^+$,*
- *$U_j \in \mathcal{D}^d(\zeta_j, \zeta_{j+1})$ for $j \in \{1, \dots, l_0\}$,*
- *$(t_n^j) \subset \mathbb{Z}$, $j \in \{1, \dots, l_0\}$, such that $t_n^{j+1} - t_n^j \to +\infty$ as $n \to +\infty$*

having the property that along a subsequence of $n \to \infty$,

$$\|q_n - U_1(\cdot - t_n^1)\|_{W^{1,2}((-\infty,\, t_n^1 + \frac{t_n^2 - t_n^1}{2}), \mathbb{R}^m)} \to 0$$

$$\|q_n - U_j(\cdot - t_n^j)\|_{W^{1,2}((t_n^j - \frac{t_n^j - t_n^{j-1}}{2},\, t_n^j + \frac{t_n^{j+1} - t_n^j}{2}), \mathbb{R}^m)} \to 0 \quad j = 2, \dots, l_0 - 1$$

$$\|q_n - U_{l_0}(\cdot - t_n^{l_0})\|_{W^{1,2}((t_n^{l_0} - \frac{t_n^{l_0} - t_n^{l_0-1}}{2},\, +\infty), \mathbb{R}^m)}$$

and

$$J(u_n) = I(q_n) \to \sum_{j=1}^{l_0} I(U_j).$$

Remark 3.8. Let ξ_1, ξ_2 be any pair in $\{a^-, a^+\}$ and let $\Gamma(\xi_1, \xi_2)$ An analogous result holds for (PS) sequences of J on $\Gamma(\xi_1, \xi_2)$. The statement changes by replacing a^- by ξ_1, a^+ by ξ_2, and ψ_1 by the appropriate member of $\{a^\pm, \psi_1, \psi_2\}$.

Now the proof of Proposition 3.2 can be given.

Proof of Proposition 3.2. Let $(q_j) \subset \mathcal{K}^u(a^-)$. By $(\Phi^u(a^-))$ and the definition of $\mathcal{K}^u(a^-)$, $\|q_j - \mathcal{K}^u(a^-)\|_{W^{1,2}([1,2],\mathbb{R}^m)} \geq 3r_-$ for any $j \in \mathbb{N}$. Since $a^- \in \mathcal{K}^u(a^-)$,

$$\|q_j - a^-\|_{W^{1,2}([1,2],\mathbb{R}^m)} \geq 3r_- \text{ for any } j \in \mathbb{N}. \tag{3.9}$$

Now by Proposition 3.7, with $\zeta_2 \in \{a^-, a^+\}$, $U_1 \in \mathcal{D}^d(a^-, \zeta_2)$ and sequences $(t_j^1), (t_j^2) \subset \mathbb{Z}$ as given by that result, along a subsequence,

$$\|q_j - U_1(\cdot - t_j^1)\|_{W^{1,2}((-\infty,\, t_j^1 + \frac{t_j^2 - t_j^1}{2}),\mathbb{R}^m)} \to 0, \quad \text{as } j \to \infty. \tag{3.10}$$

Three cases must be considered: either along a further subsequence, as $j \to \infty$, (i) $t_j^1 \to \infty$; (ii) $t_j^1 \to -\infty$; or (iii) t_j^1 is bounded. If (i) occurs, $\|U_1(\cdot - t_j^1) - a^-\|_{W^{1,2}([1,2],\mathbb{R}^m)} \to 0$ and so $\|q_j - a^-\|_{W^{1,2}([1,2],\mathbb{R}^m)} \to 0$, contrary to (3.9). Thus (i) is not possible. Next suppose that (ii) occurs. By (3.10), for any $T > 0$, as $j \to \infty$ along the subsequence,

$$\|q_j(\cdot + t_j^1 + j^u(U_1)) - U_1(\cdot + j^u(U_1))\|_{W^{1,2}((-\infty,\, T],\mathbb{R}^m)} \to 0. \tag{3.11}$$

Since $t_j^1 \to -\infty$, $q_j(\cdot + t_j^1 + j^u(U_1))|_{[1,2]} \in K^u(a^-)$ for large j, while by definition $\|U_1(\cdot + j^u(U_1)) - K^u(a^-)\|_{W^{1,2}([1,2],\mathbb{R}^m)} \geq 3r_-$. This contradicts (3.11) showing that (ii) is also impossible. Hence (iii) occurs and (t_j^1) is a bounded sequence. Hence along a subsequence it is a constant, κ. Then defining $q_0 = U_1(\cdot - \kappa)$, by (3.10) we obtain in particular that along this subsequence

$$\|q_j - q_0\|_{W^{1,2}((-\infty,2],\mathbb{R}^m)} \to 0, \quad \text{as } j \to \infty. \tag{3.12}$$

Since $q_j(\cdot + i) \in K^u(a^-)$ for any $i \leq 0$ and $\|q_j(\cdot + 1) - K^u(a^-)\|_{W^{1,2}([0,1],\mathbb{R}^m)} \geq 3r_-$, the compactness of $K^u(a^-)$ and (3.12) show $q_0(\cdot + i) \in K^u(a^-)$ for any $i \leq 0$ and $\|q_0(\cdot + 1) - K^u(a^-)\|_{W^{1,2}([0,1],\mathbb{R}^m)} \geq 3r_-$. Thus $j^u(q_0) = 0$, $q_0 \in \mathcal{K}^u(a^-)$ and Proposition 3.2 follows from (3.12).

Proposition 3.2 implies

$$\sup_{q \in \mathcal{K}^u(a^-)} \|q\|_{W^{1,2}((-\infty,T),\mathbb{R}^m)} \to 0, \;\; T \to -\infty$$

which immediately yields:

Corollary 3.13. *For any $\epsilon > 0$, there exists a $j_\epsilon > 0$ such that*

$$\|g^{-j}(u) - a^-\|_{W^{1,2}([0,1],\mathbb{R}^m)} \leq \epsilon \quad \forall j \geq j_\epsilon \text{ and } u \in K_0^u(a^-).$$

A further consequence of Proposition 3.2 is

Proposition 3.14. *For any $\ell \in \mathbb{Z}$ with $\ell \geq 0$, the set $\{a^-\} \cup (\cup_{j \geq \ell} g^{-j}(K_0^u(a^-)))$ is compact.*

Proof. If $\ell \geq 0$ and $(u_n) \subset \cup_{j \geq \ell} g^{-j}(K_0^u(a^-))$, there is a $v_n \in K_0^u(a^-)$ and $j_n \in \mathbb{N} \cap [\ell, +\infty)$ such that $u_n = g^{-j_n}(v_n)$. If along a subsequence, $j_n \to \infty$ as $n \to \infty$, by Corollary 3.13, $u_n \to a^-$ as $n \to \infty$. On the other hand, if (j_n) is bounded, along a subsequence it is constant with say $j_n \equiv j_0 \geq \ell$. Since $K_0^u(a^-)$ is compact, along a further subsequence we have $v_n \to v_0 \in K_0^u(a^-)$. The map g is a homeomorphism so $u_n = g^{-j_0}(v_n) \to g^{-j_0}(v_0)$. This shows that (u_n) always has an accumulation point in $\{a^-\} \cup (\cup_{j \geq \ell} g^{-j}(K_0^u(a^-)))$ and the Proposition follows.

Another property of the phase shifts of $K_0^u(a^-)$ is:

Proposition 3.15. *For any* $i, j \in \mathbb{Z}, i \neq j \geq 0$, $g^{-j}(K_0^u(a^-))$ *is compact and* $g^{-i}(K_0^u(a^-)) \cap g^{-j}(K_0^u(a^-)) = \emptyset$.

Proof. The map g is a homeomorphism on $\mathcal{S}^d$ and $K_0^u(a^-)$ is compact so $g^{-j}(K_0^u(a^-))$ is compact for any $j \in \mathbb{Z}$. The second statement reduces to showing that

$$g^{-p}(K_0^u(a^-)) \cap K_0^u(a^-) = \emptyset \text{ for any } p \in \mathbb{N}.$$

Otherwise there exists a $p \in \mathbb{N}$ and $u, v \in K_0^u(a^-)$ such that $g^{-p}(u) = v$. By definition $g(v) \notin K^u(a^-)$ while $g(g^{-p}(u)) \in K^u(a^-)$ since $-p + 1 \leq 0$, a contradiction.

Next let $\ell_0 \in \mathbb{N}$ and set

$$\tilde{K}_{\ell_0}^u(a^-) = \{a^-\} \cup (\cup_{j \geq \ell_0} g^{-j}(K_0^u(a^-))).$$

As an immediate consequence of Proposition 3.14 and Proposition 3.15, we have:

Corollary 3.16. $\tilde{K}_{\ell_0}^u(a^-)$ *is compact and satisfies (i) and (ii) of* $(\Phi^u(a^-))$ *(with a different choice of* r^-*) for any* $\ell_0 > 0$.

By (V_4) – see e.g. [7] – there exists an $r_0 \in (0, |a^+ - a^-|/10)$ such that if $\xi_1, \xi_2 \in \{a_-, a_+\}$ and $q \in \mathcal{D}^d(\xi_1, \xi_2)$, then

$$\sup_{j \in \mathbb{Z}} \|q - \zeta\|_{W^{1,2}([j,j+1],\mathbb{R}^m)} \geq 2r_0 \text{ for } \zeta \in \{a_-, a_+\}. \tag{3.17}$$

Due to Corollary 3.13, ℓ_0 can be chosen so that

$$\sup_{j \geq \ell_0} \sup_{u \in K_0^u(a^-)} \|g^{-j}(u) - a^-\|_{W^{1,2}([0,1],\mathbb{R}^m)} \leq r_0$$

or, equivalently,

$$\sup_{u \in \tilde{K}_{\ell_0}^u(a^-)} \|u - a^-\|_{W^{1,2}([0,1],\mathbb{R}^m)} \leq r_0.$$

We make this choice of ℓ_0. Since $(\Phi^u(a^\sigma))$ is still meaningful if the constant r^- made smaller, it can be assumed that

$$3r^- < r_0. \tag{3.18}$$

These observations allow us to eliminate the non-uniqueness associated with $K^u(a^-)$ by replacing it by $\tilde{K}_{\ell_0}^u(a^-)$ or equivalently replacing $K_0^u(a^-)$ by $g^{-l_0}(K_0^u(a^-))$. Thus abusing notation slightly, we can write:

$$K^u(a^-) = \{a^-\} \cup (\cup_{j \geq 0} g^{-j}(K_0^u(a^-))) \tag{3.19}$$

and satisfies:

$$\sup_{u \in K^u(a^-)} \|u - a^-\|_{W^{1,2}([0,1],\mathbb{R}^m)} \leq r_0. \tag{3.20}$$

Replacing u by s in $(\Phi^u(a^\sigma))$, we also assume:

$(\Phi^s(a^\sigma))$ There exists a compact set $K^s(a^\sigma) \subset \overline{\mathcal{S}^d}$ and $r^- > 0$ such that

i) $K^s(a^\sigma) \cap \{a^-, a^+\} = \{a^\sigma\}$ and

ii) For each $U \in \mathcal{D}^d(a^\sigma, a^-) \cup \mathcal{D}^d(a^\sigma, a^+)$, $U(\cdot + i)|_{[0,1]} \in K^s(a^+)$ for any $i \geq 0$ and

$$\|U(\cdot - 1) - K^s(a^+)\|_{W^{1,2}([0,1],\mathbb{R}^m)} \geq 3r^-.$$

Replacing unstable by stable, the sets, $K^s(a^\pm)$, have a geometric interpretation as subsets of solution segments in $\mathcal{W}^s(a^\pm)$. Applying similar reasoning to the sets $K^u(a^+)$, $K^s(a^\pm)$ to that just employed above, we find compact sets $K_0^u(a^-)$, $K_0^s(a^\pm)$ in $\overline{\mathcal{S}}^d$ for which

$$K^u(a^+) = \{a^+\} \cup (\cup_{j\geq 0} g^{-j}(K_0^u(a^+))),\ \ K^s(a^\pm) = \{a^\pm\} \cup (\cup_{j\geq 0} g^{j}(K_0^s(a^\pm))), \tag{3.21}$$

and

$$\sup_{u\in K^u(a^+)} \|u - a^+\|_{W^{1,2}([0,1],\mathbb{R}^m)} \leq r_0, \tag{3.22}$$

$$\sup_{u\in K^s(a^-)} \|u - a^-\|_{W^{1,2}([0,1],\mathbb{R}^m)} \leq r_0, \tag{3.23}$$

$$\sup_{u\in K^s(a^+)} \|u - a^+\|_{W^{1,2}([0,1],\mathbb{R}^m)} \leq r_0. \tag{3.24}$$

Since $10r_0 < \|a^- - a^+\|_{W^{1,2}([0,1],\mathbb{R}^m)}$, by (3.20), (3.22)–(3.24) and the triangle inequality:

$$\text{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(K^u(a^\pm) \cup K^s(a^\pm),\ K^u(a^\mp) \cup K^s(a^\mp)) \geq 8r_0. \tag{3.25}$$

Define

$$\mathcal{K}^u(a^\pm) = \{U \in \mathcal{D}^d(a^\pm, a^\mp) \cup \mathcal{D}^d(a^\pm, a^\pm) \mid U|_{[0,1]} \in K_0^u(a^\pm)\},$$
$$\mathcal{K}^s(a^\pm) = \{U \in \mathcal{D}^d(a^\mp, a^\pm) \cup \mathcal{D}^d(a^\pm, a^\pm) \mid U|_{[0,1]} \in K_0^s(a^\pm)\},$$

so as earlier, $\mathcal{K}^s(a^\pm)$ can be interpreted as the subset of the stable manifold of (HS) at $a^\pm$ consisting of heteroclinic and homoclinic solutions emanating from $a^\pm$ having $I \leq d$ and normalized by their behavior on [0, 1]. Proposition 3.2 leads to:

Proposition 3.26. *$\mathcal{K}^u(a^\pm)$ is compact with respect to the $W^{1,2}((-\infty, 2], \mathbb{R}^m)$ metric and $\mathcal{K}^s(a^\pm)$ is compact with respect to the $W^{1,2}([-1, +\infty), \mathbb{R}^m)$ metric.*

Next observe that the argument described in the proof of Proposition 3.15 can be adapted to show that there exists $\bar{r} \in (0, r^-)$ such that

$$\text{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(K_0^u(a^\pm),\ K^u(a^-) \setminus K_0^u(a^\pm)) \geq 3\bar{r} \tag{3.27}$$

Similarly it can be assumed that $\bar{r}$ is so small that

$$\text{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(K_0^s(a^\pm),\ K^s(a^-) \setminus K_0^s(a^\pm)) \geq 3\bar{r}. \tag{3.28}$$

The inequalities (3.18), (3.25) imply

$$\text{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(K_0^s(a^\pm) \cup K_0^u(a^\pm),\ K^s(a^\mp) \cup K^u(a^\mp)) \geq 3\bar{r}. \tag{3.29}$$

Assume for the moment that

$$\begin{aligned} &K_0^u(a^-) \cap K^s(a^-) = K_0^u(a^+) \cap K^s(a^+) = \\ &= K_0^s(a^-) \cap K^u(a^-) = K_0^s(a^+) \cap K^u(a^+) = \emptyset. \end{aligned} \tag{3.30}$$

All the sets involved in (3.30) are compact. Therefore by taking $\bar{r}$ smaller if necessary, by (3.30) it can be assumed that

$$\text{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(K_0^u(a^-), K^s(a^-)),\ \ \text{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(K_0^s(a^-), K^u(a^-)) \geq 3\bar{r} \tag{3.31}$$

$$\text{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(K_0^u(a^+), K^s(a^+)),\ \ \text{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(K_0^s(a^+), K^u(a^+)) \geq 3\bar{r} \tag{3.32}$$

To verify (3.30), consider the representative case of $K_0^u(a^-) \cap K^s(a^-) = \emptyset$. Arguing indirectly, assume that there exists $\xi \in K_0^u(a^-) \cap K^s(a^-)$. By definition $g^j(\xi) \in K^u(a^-)$ for all $j \leq 0$ and $g^j(\xi) \in K^s(a^-)$ for all $j \geq 0$. Then (3.20) and (3.23) show $\|g^j(\xi) - a^-\|_{W^{1,2}([0,1],\mathbb{R}^m)} \leq r_0$ for any $j \in \mathbb{Z}$. But this is contrary to (3.17). Hence $K_0^u(a^-) \cap K^s(a^-) = \emptyset$. The other equalities in (3.30) can be obtained in an analogous way and (3.30) is proved.

Due to the assumptions $(\Phi^u(a^\sigma))$ and $(\Phi^s(a^\sigma))$ for $\sigma \in \{-,+\}$ and the choice of $\bar{r}$, it follows that

$$\sup_{i\in\mathbb{Z}} \mathrm{dist}_{W^{1,2}([i,i+1],\mathbb{R}^m)}(q,\{a^-,a^+\}) \geq 3\bar{r} \text{ for any } q \in \mathcal{D}^d. \tag{3.33}$$

4. The equivalence result

Our main result can now be stated:

Theorem 4.1. $\mathcal{C}^d(a^\pm) = \{a^\pm\}$ *if and only if for* $\sigma \in \{+,-\}$, *conditions* $(\Phi^u(a^\sigma))$ *and* $(\Phi^s(a^\sigma))$ *hold.*

Before proving Theorem 4.1, Proposition 3.7 will be used to obtain some further technical properties of the sets $\mathcal{K}^u(a^\pm)$ and $\mathcal{K}^s(a^\pm)$.

Proposition 4.2. *Assume that* $(\Phi^u(a^\sigma))$ *holds for a* $\sigma \in \{-,+\}$ *and let* $(q_i) \subset \mathcal{D}^d$, $(t_i) \subset \mathbb{N}$, *be such that*

$$t_i \to +\infty \text{ and } \mathrm{dist}_{W^{1,2}([-t_i,1],\mathbb{R}^m)}(q_i,\mathcal{K}^u(a^\sigma)) \leq \bar{r} \text{ as } i \to \infty.$$

Then there exists a $U \in \mathcal{K}^u(a^\sigma)$ *such that, along a subsequence,*

$$\|q_i - U\|_{W^{1,2}([-T,1],\mathbb{R}^m)} \to 0 \text{ as } i \to +\infty \text{ for any } T > 0.$$

Proof. Since the proofs are the same for either choice of σ, suppose that $\sigma = -$. It can be assumed that

$$\mathrm{dist}_{W^{1,2}([-t_i,1],\mathbb{R}^m)}(q_i,\mathcal{K}^u(a^-)) \leq \bar{r} \text{ for any } i \in \mathbb{N}. \tag{4.3}$$

Since the functions, q_i, are solutions of (HS) with $I(q_i) \leq d$, the sequence (q_i) is bounded in $W^{1,2}_{loc}$ [2]. Therefore (q_i) converges in $W^{1,2}_{loc}$ along a subsequence to a function $U \in \mathcal{D}^d$ and by (4.3),

$$\mathrm{dist}_{W^{1,2}((-\infty,1],\mathbb{R}^m)}(U,\mathcal{K}^u(a^-)) \leq \bar{r}. \tag{4.4}$$

Recall that $\bar{r} < r^- < \frac{r_0}{3} < \frac{|a^- - a^+|}{30}$. Therefore (4.4) and arguments from [2] or [7] imply $U \in \mathcal{D}^d(a^-,a^+) \cup \mathcal{D}^d(a^-,a^-)$. Thus the Proposition follows once we show that $U \in \mathcal{K}^u(a^-)$ or equivalently that $j^u(U) = 0$. Arguing indirectly, suppose first that $j^u(U) < 0$. Then $j^u(U)+1 \leq 0$. By $(\Phi^u(a^-))$,

$$\mathrm{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(U(\cdot + j^u(U)+1), K^u(a^-)) \geq 3r_-$$

and so, since $\{q(\cdot - i)|_{[0,1]} \mid q \in \mathcal{K}^u(a^-)\} = g^{-i}(K^u_0(a^-))$ for any $i \geq 0$, by (3.19),

$$\begin{aligned}\mathrm{dist}_{W^{1,2}((-\infty,1],\mathbb{R}^m)}(U,\mathcal{K}^u(a^-)) &\geq \mathrm{dist}_{W^{1,2}([j^u(U)+1,j^u(U)+2],\mathbb{R}^m)}(U,\mathcal{K}^u(a^-)) \\ &= \mathrm{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(U(\cdot + j^u(U)+1), g^{(j^u(U)+1)}(K^u_0(a^-))) \\ &\geq \mathrm{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(U(\cdot + j^u(U)+1), K^u(a^-)) \geq 3r_-\end{aligned}$$

which is in contradiction with (4.4). Next if $j^u(U) > 0$, then $U|_{[0,1]} \in g^{-j^u(U)}(K^u_0(a^-))$ so by (3.27),

$$3\bar{r} \leq \mathrm{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(U, K^u_0(a^-)) = \mathrm{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(U,\mathcal{K}^u(a^-))$$

again in contradiction with (4.4).

Then $j^u(U) = 0$ and the proof is complete.

There is a similar result for $\mathcal{K}^s(a^\sigma)$:

Proposition 4.5. *Assume that* $(\Phi^u(a^\sigma))$ *holds for a* $\sigma \in \{-,+\}$ *and let* $(q_i) \subset \mathcal{D}^d$, $(t_i) \subset \mathbb{N}$ *be such that*

$$t_i \to +\infty \text{ and } \mathrm{dist}_{W^{1,2}([0,t_i],\mathbb{R}^m)}(q_i,\mathcal{K}^s(a^\sigma)) \leq \bar{r} \text{ as } i \to \infty.$$

Then there exists a $U \in \mathcal{K}^s(a^\sigma)$ *such that, along a subsequence,*

$$\|q_i - U\|_{W^{1,2}([0,T],\mathbb{R}^m)} \to 0 \text{ as } i \to +\infty \text{ for any } T > 0.$$

The proof is similar to that of Proposition 4.2 and will be omitted.

The last two results and Proposition 3.7 are crucial for the proof of the next rather technical result which is of interest for small r and will play a role in the proof of Theorem 4.1. Geometrically it roughly says if a heteroclinic or homoclinic solution of (HS) is near the corresponding part of $\mathcal{D}^d$ in a certain sense depending on $\bar{r}$, in fact it is close in a stronger sense depending on r.

Proposition 4.6. *Suppose that* $(\Phi^u(a^\sigma))$*–*$(\Phi^s(a^\sigma))$ *hold for* $\sigma \in \{+, -\}$*. Given any* $r > 0$*, there exists an* $\bar{n}(r) \in \mathbb{N}$ *such that whenever* $L \in [\bar{n}(r), +\infty) \cap \mathbb{N}$*,* $q \in \mathcal{D}^d$*,* $\sigma \in \{+, -\}$ *and* $n_1, n_2 \in \mathbb{Z}$ *satisfy the conditions*

(h₁) $n_2 - n_1 \geq 3L$,
(h₂) $\mathrm{dist}_{W^{1,2}([-L,1],\mathbb{R}^m)}(q(\cdot + n_2), \mathcal{K}^u(a^\sigma)) \leq \bar{r}$,
(h₃) $\mathrm{dist}_{W^{1,2}([0,L],\mathbb{R}^m)}(q(\cdot + n_1), \mathcal{K}^s(a^\sigma)) \leq \bar{r}$,
(h₄) $\sup_{i \in [n_1+L, n_2-L-1],\mathbb{R}^m)} \|q - a^\sigma\|_{W^{1,2}([i,i+1],\mathbb{R}^m)} \leq \bar{r}$,

then in fact

(1) $\mathrm{dist}_{W^{1,2}([-L,1],\mathbb{R}^m)}(q(\cdot + n_2), \mathcal{K}^u(a^\sigma)) \leq r$,
(2) $\mathrm{dist}_{W^{1,2}([0,L],\mathbb{R}^m)}(q(\cdot + n_1), \mathcal{K}^s(a^\sigma)) \leq r$,
(3) $\|q - a^\sigma\|_{W^{1,2}([n_1+L,n_2-L],\mathbb{R}^m)} \leq r$.

Proof. Arguing indirectly, suppose there exists an $r > 0$ such that for any $i \in \mathbb{N}$, there is an $L_i \in \mathbb{N}$ such that $L_i \geq i$, a $q_i \in \mathcal{D}^d$ and numbers $n_{2,i}, n_{1,i} \in \mathbb{Z}$ that satisfy (h_1)–(h_4) but violate at least one of the conditions

(1_i) $\mathrm{dist}_{W^{1,2}([-L_i,1],\mathbb{R}^m)}(q_i(\cdot + n_{2,i}), \mathcal{K}^u(a^\sigma)) \leq r$,
(2_i) $\mathrm{dist}_{W^{1,2}([0,L_i],\mathbb{R}^m)}(q_i(\cdot + n_{1,i}), \mathcal{K}^s(a^\sigma)) \leq r$,
(3_i) $\|q - a^\sigma\|_{W^{1,2}([n_{1,i}+L_i,n_{2,i}-L_i],\mathbb{R}^m)} \leq r$.

Since $\mathcal{K}^u(a^\sigma)$ is compact with respect to the $W^{1,2}((-\infty, 1], \mathbb{R}^m)$ metric, there exists $T_0 \in \mathbb{N}$ such that

$$\sup_{l \geq T_0} \|q - a^\sigma\|_{W^{1,2}([-l,-l+1],\mathbb{R}^m)} \leq \bar{r} \text{ for any } q \in \mathcal{K}^u(a^\sigma). \tag{4.7}$$

Due to the analogous property of $\mathcal{K}^s(a^\sigma)$, it can also be assumed that

$$\sup_{l \geq T_0} \|q - a^\sigma\|_{W^{1,2}([l,l+1],\mathbb{R}^m)} \leq \bar{r} \text{ for any } q \in \mathcal{K}^s(a^\sigma). \tag{4.8}$$

We claim that

$$\sup_{n_{1,i}+T_0 \leq l \leq n_{2,i}-T_0} \|q_i - a^\sigma\|_{W^{1,2}([l,l+1],\mathbb{R}^m)} \leq 2\bar{r} \text{ for any } i \geq T_0. \tag{4.9}$$

Indeed first suppose that $n_{2,i} - L_i \leq l \leq n_{2,i} - T_0$. By (h_2) and the compactness of $\mathcal{K}^u(a^\sigma)$ with respect to the $W^{1,2}((-\infty, 1], \mathbb{R}^m)$ metric, there is a $\varphi_i \in \mathcal{K}^u(a^\sigma)$ such that

$$\begin{aligned} &\mathrm{dist}_{W^{1,2}([-L_i,1],\mathbb{R}^m)}(q_i(\cdot + n_{2,i}), \mathcal{K}^u(a^\sigma)) = \|q_i(\cdot + n_{2,i}) - \varphi_i\|_{W^{1,2}([-L_i,1],\mathbb{R}^m)} \\ &= \|q_i - \varphi_i(\cdot - n_{2,i})\|_{W^{1,2}([n_{2,i}-L_i,n_{2,i}+1],\mathbb{R}^m)} \leq \bar{r}. \end{aligned} \tag{4.10}$$

Next note that by (4.7), for $l \geq T_0$,

$$\|\varphi_i(\cdot - n_{2,i}) - a^\sigma\|_{W^{1,2}([n_{2,i}-l,n_{2,i}-l+1],\mathbb{R}^m)} = \|\varphi_i - a^\sigma\|_{W^{1,2}([-l,-l+1],\mathbb{R}^m)} \leq \bar{r}. \tag{4.11}$$

Thus (4.10)–(4.11) and the triangle inequality show (4.9) for the restricted range of $n_{2,i} - L_i \leq l \leq n_{2,i} - T_0$. A similar argument using (h_3) and (4.8) then gives (4.9) for $n_{1,i} + T_0 \leq l \leq n_{1,i} + L_i$. Lastly (h_4) then yields (4.9) for the remaining region, $n_{1,i} + L_i \leq l \leq n_{2,i} - L_i$.

By (h_2), (h_3) and Propositions 4.2, 4.5, there exist functions $U_- \in \mathcal{K}^u(a^\sigma)$ and $U_+ \in \mathcal{K}^s(a^\sigma)$ such that, along a subsequence, still denoted by (q_i), we have

$$\|q_i - U_-(\cdot - n_{2,i})\|_{W^{1,2}([n_{2,i}-T, n_{2,i}+1],\mathbb{R}^m)} \to 0 \text{ as } i \to +\infty. \tag{4.12}$$

$$\|q_i - U_+(\cdot - n_{1,i})\|_{W^{1,2}([n_{1,i}, n_{1,i}+T],\mathbb{R}^m)} \to 0 \text{ as } i \to +\infty \tag{4.13}$$

for any $T > 0$. Proposition 3.7 implies q_i is asymptotic to a chain of functions consisting of translates of members of $\mathcal{D}^d$. By (4.12) and (4.13), the chain contains at least the two functions U_- and U_+. We will show that U_+ is the left neighbor of U_- in the chain. To be more precise, by Proposition 3.7, there is a further subsequence of (q_i) that we continue to denote by (q_i), an $l_0 \in \mathbb{N}$, $\zeta_1, \zeta_2, ..., \zeta_{l_0+1} \in \{a^-, a^+\}$ with $\zeta_1 = \xi_1$, $\zeta_{l_0+1} = \xi_2$, $U_j \in \mathcal{D}(\zeta_j, \zeta_{j+1})$ for $j \in \{1, \dots, l_0\}$, and sequences $(t_i^j) \subset \mathbb{Z}$, such that $t_i^{j+1} - t_i^j \to +\infty$ as $i \to \infty$, the above having the property that (setting $t_i^0 = -\infty$ and $t_i^{l_0+1} = +\infty$),

$$\|q_i - U_j(\cdot - t_i^j)\|_{W^{1,2}((\frac{t_i^{j-1}+t_i^j}{2}, \frac{t_i^j+t_i^{j+1}}{2}),\mathbb{R}^m)} \to 0 \text{ for } j = 1, \dots, l_0. \tag{4.14}$$

By (h_1), $n_{2,i} - n_{1,i} \to \infty$ as $i \to \infty$, so comparing (4.12) to (4.14) shows there is $p \in [2, l_0] \cap \mathbb{N}$ such that, along the subsequence, $t_i^p - n_{2,i}$ must be bounded. Thus taking a further subsequence if necessary, it can be assumed that $t_i^p - n_{2,i}$ is a constant and $U_p \equiv U_-(\cdot - n_{2,i} + t_i^p)$. Now (4.14) shows as $i \to \infty$,

$$\|q_i - U_-(\cdot - n_{2,i})\|_{W^{1,2}((\frac{t_i^{p-1}+t_i^p}{2}, n_{2,i}+1),\mathbb{R}^m)} \to 0. \tag{4.15}$$

Since $U_- \in \mathcal{K}^u(a^\sigma)$, it follows that $\zeta_p = a^\sigma$. Setting $U \equiv U_{p-1}$, then $U \in \mathcal{D}^d(\xi, a^\sigma)$ with $\xi \in \{a^-, a^+\}$. By (4.14) again, for any $T > 0$

$$\|q_i - U(\cdot - t_i^{p-1})\|_{W^{1,2}((t_i^{p-1}-T, \frac{t_i^{p-1}+t_i^p}{2}),\mathbb{R}^m)} \to 0 \tag{4.16}$$

as $i \to \infty$. Since for large i,

$$\begin{aligned}
&\|q_i - a^\sigma - (U(\cdot - t_i^{p-1}) - a^\sigma) - (U_-(\cdot - n_{2,i}) - a^\sigma)\|_{W^{1,2}((t_i^{p-1}-T,\, n_{2,i}+1),\mathbb{R}^m)} \le \\
&\le \|q_i - a^\sigma - (U(\cdot - t_i^{p-1}) - a^\sigma) - (U_-(\cdot - n_{2,i}) - a^\sigma)\|_{W^{1,2}((t_i^{p-1}-T, \frac{t_i^{p-1}+t_i^p}{2}),\mathbb{R}^m)} + \\
&+ \|q_i - a^\sigma - (U(\cdot - t_i^{p-1}) - a^\sigma) - (U_-(\cdot - n_{2,i}) - a^\sigma)\|_{W^{1,2}((\frac{t_i^{p-1}+t_i^p}{2}, n_{2,i}+1),\mathbb{R}^m)} \le \\
&\le \|q_i - (U(\cdot - t_i^{p-1})\|_{W^{1,2}((t_i^{p-1}-T, \frac{t_i^{p-1}+t_i^p}{2}),\mathbb{R}^m)} + \\
&+ \|(U_-(\cdot - n_{2,i}) - a^\sigma)\|_{W^{1,2}((t_i^{p-1}-T, \frac{t_i^{p-1}+t_i^p}{2}),\mathbb{R}^m)} + \\
&+ \|q_i - U_-(\cdot - n_{2,i})\|_{W^{1,2}((\frac{t_i^{p-1}+t_i^p}{2}, n_{2,i}+1),\mathbb{R}^m)} + \\
&+ \|U(\cdot - t_i^{p-1}) - a^\sigma)\|_{W^{1,2}((\frac{t_i^{p-1}+t_i^p}{2}, n_{2,i}+1),\mathbb{R}^m)},
\end{aligned}$$

by (4.16) and (4.15), the first and third terms on the right above go to 0 as $i \to \infty$. The remaining two terms are essentially tails of convergent integrals and therefore also go to 0 as $i \to \infty$. Thus we have shown for any $T > 0$, as $i \to \infty$,

$$\|q_i - a^\sigma - (U(\cdot - t_i^{p-1}) - a^\sigma) - (U_-(\cdot - n_{2,i}) - a^\sigma)\|_{W^{1,2}((t_i^{p-1}-T,\, n_{2,i}+1),\mathbb{R}^m)} \to 0. \tag{4.17}$$

Next it will be shown that $(t_i^{p-1} - n_{1,i})$ is a bounded sequence. First observe if $t_i^{p-1} - n_{1,i} \to +\infty$ as $i \to \infty$, then for any $j_0 \in \mathbb{Z}$,

$$(t_i^{p-1} + j_0, t_i^{p-1} + j_0 + 1) \subset (n_{1,i} + T_0, n_{2,i} - T_0 + 1)$$

when $i = i(j_0)$ is large enough. Hence by (4.9),

$$\|q_i - a^\sigma\|_{W^{1,2}((t_i^{p-1}+j_0, t_i^{p-1}+j_0+1),\mathbb{R}^m)} \le 2\bar{r}. \tag{4.18}$$

By (4.14) we know

$$\|q_i - U(\cdot - t_i^{p-1})\|_{W^{1,2}((t_i^{p-1}+j_0, t_i^{p-1}+j_0+1),\mathbb{R}^m)} \to 0 \tag{4.19}$$

as $i \to \infty$. Combining (4.18)–(4.19) gives

$$\|U(\cdot - t_i^{p-1}) - a^\sigma\|_{W^{1,2}((t_i^{p-1}+j_0, t_i^{p-1}+j_0+1),\mathbb{R}^m)} \le 2\bar{r} + o(1),\ i \to \infty. \tag{4.20}$$

But by (3.33), there is a $j_0 \in \mathbb{Z}$ such that

$$\|U(\cdot + j_0) - a^\sigma\|_{W^{1,2}([0,1],\mathbb{R}^m)} > 2\bar{r},$$

contrary to (4.20). Thus $t_i^{p-1} - n_{1,i} \to +\infty$ as $i \to \infty$ (along a subsequence) is not possible.

To further exclude that $t_i^{p-1} - n_{1,i} \to -\infty$ (along a subsequence), first note that by (4.13), as $i \to \infty$,

$$\|q_i - U_+(\cdot - n_{1,i})\|_{W^{1,2}([n_{1,i}, n_{1,i}+1],\mathbb{R}^m)} \to 0. \tag{4.21}$$

Therefore for large i,

$$\begin{aligned}\|q_i - a^\sigma\|_{W^{1,2}([n_{1,i}, n_{1,i}+1],\mathbb{R}^m)} &\ge \|U_+(\cdot - n_{1,i}) - a^\sigma\|_{W^{1,2}([n_{1,i}, n_{1,i}+1],\mathbb{R}^m)} \\ &- \|q_i - U_+(\cdot - n_{1,i})\|_{W^{1,2}([n_{1,i}, n_{1,i}+1],\mathbb{R}^m)} \ge \bar{r}\end{aligned} \tag{4.22}$$

via $U_+ \in \mathcal{K}^s(a^\sigma)$ and (3.28). On the other hand,

$$\begin{aligned}&\|q_i - a^\sigma\|_{W^{1,2}([n_{1,i}, n_{1,i}+1],\mathbb{R}^m)} \le N_1 + N_2 + N_3 \equiv \\ &\equiv \|q_i - a^\sigma - (U(\cdot - t_i^{p-1}) - a^\sigma) - (U_-(\cdot - n_{2,i}) - a^\sigma)\|_{W^{1,2}([n_{1,i}, n_{1,i}+1],\mathbb{R}^m)} + \\ &+ \|U(\cdot - t_i^{p-1}) - a^\sigma\|_{W^{1,2}([n_{1,i}, n_{1,i}+1],\mathbb{R}^m)} + \|U_-(\cdot - n_{2,i}) - a^\sigma\|_{W^{1,2}([n_{1,i}, n_{1,i}+1],\mathbb{R}^m)}.\end{aligned} \tag{4.23}$$

Since $t_i^{p-1} - n_{1,i} \to -\infty$ as $i \to \infty$, $t_i^{p-1} < n_{2,i}$ for large i and (4.17) implies $N_1 \to 0$ as $i \to \infty$. Writing

$$N_2 = \|U - a^\sigma\|_{W^{1,2}([n_{1,i} - t_i^{p-1}, n_{1,i}+1-t_i^{p-1}],\mathbb{R}^m)}$$

and noting that $\|U - a^\sigma\|_{W^{1,2}([j,j+1],\mathbb{R}^m)} \to 0$ as $j \to \infty$ shows $N_2 \to 0$ as $i \to \infty$. Similarly

$$N_3 = \|U_- - a^\sigma\|_{W^{1,2}([n_{1,i} - n_{2,i}, n_{1,i}+1-n_{2,i}],\mathbb{R}^m)}$$

and $\|U_- - a^\sigma\|_{W^{1,2}([j,j+1],\mathbb{R}^m)} \to 0$ as $j \to -\infty$ gives $N_3 \to 0$ as $i \to \infty$. Combining these observations, we have

$$\|q_i - a^\sigma\|_{W^{1,2}([n_{1,i}, n_{1,i}+1],\mathbb{R}^m)} \to 0,\ \ i \to \infty. \tag{4.24}$$

Thus (4.24) is contrary to (4.22) and $t_i^{p-1} - n_{1,i} \to -\infty$ (along a subsequence) is not possible.

Now that we know $(t_i^{p-1} - n_{1,i})$ is a bounded sequence, along a subsequence it is a constant, κ. Hence again (4.13), (4.17) imply that $U = U_+(\cdot + \kappa)$ so by (4.17),

$$\|q_i - a^\sigma - (U_+(\cdot - n_{1,i}) - a^\sigma) - (U_-(\cdot - n_{2,i}) - a^\sigma)\|_{W^{1,2}((n_{i,1}, n_{2,i}+1),\mathbb{R}^m)} \to 0. \tag{4.25}$$

Note that

$$\|U_+(\cdot - n_{1,i}) - a^\sigma\|_{W^{1,2}((n_{1,i}+L_i, +\infty),\mathbb{R}^m)} = \|U_+ - a^\sigma\|_{W^{1,2}((L_i, +\infty),\mathbb{R}^m)} \to 0 \tag{4.26}$$

since $L_i \to +\infty$ as $i \to \infty$. Hence (h_1) and (4.25)–(4.26) show

$$\mathrm{dist}_{W^{1,2}([-L_i,1],\mathbb{R}^m)}(q_i(\cdot+n_{2,i}),\mathcal{K}^u(a^\sigma))\le\|q_i-U_-(\cdot-n_{2,i})\|_{W^{1,2}([n_{2,i}-L_i,n_{2,i}+1],\mathbb{R}^m)}$$
$$\le\|q_i-a^\sigma-(U_+(\cdot-n_{1,i})-a^\sigma)-(U_-(\cdot-n_{2,i})-a^\sigma)\|_{W^{1,2}([n_{2,i}-L_i,n_{2,i}+1],\mathbb{R}^m)}+$$
$$+\|U_+(\cdot-n_{1,i})-a^\sigma\|_{W^{1,2}([n_{2,i}-L_i,n_{2,i}+1],\mathbb{R}^m)}\to 0.$$

Thus q_i satisfies the property (1_i) whenever i is large.

Analogously to the above,

$$\|U_-(\cdot-n_{2,i})-a^\sigma\|_{W^{1,2}((-\infty,n_{2,i}-L_i),\mathbb{R}^m)}\to 0 \tag{4.27}$$

as $i\to\infty$ so (h_1), (4.25) and (4.27) imply

$$\mathrm{dist}_{W^{1,2}([0,L_i],\mathbb{R}^m)}(q_i(\cdot+n_{1,i}),\mathcal{K}^s(a^\sigma))\le\|q_i-U_+(\cdot-n_{1,i})\|_{W^{1,2}([n_{1,i},n_{1,i}+L_i],\mathbb{R}^m)}$$
$$\le\|q_i-a^\sigma-(U_+(\cdot-n_{1,i})-a^\sigma)-(U_-(\cdot-n_{2,i})-a^\sigma)\|_{W^{1,2}([n_{1,i},n_{1,i}+L_i],\mathbb{R}^m)}+$$
$$+\|U_-(\cdot-n_{1,i})-a^\sigma\|_{W^{1,2}([n_{1,i},n_{1,i}+L_i],\mathbb{R}^m)}\to 0.$$

Thus q_i also satisfies property (2_i) whenever i is large.

Finally by (h_1) and (4.25)–(4.27),

$$\|q-a^\sigma\|_{W^{1,2}([n_{1,i}+L_i,n_{2,i}-L_i],\mathbb{R}^m)}\le$$
$$\le\|q_i-a^\sigma-(U_+(\cdot-n_{1,i})-a^\sigma)-(U_-(\cdot-n_{2,i})-a^\sigma)\|_{W^{1,2}([n_{1,i}+L_i,n_{2,i}-L_i],\mathbb{R}^m)}+$$
$$+\|U_+(\cdot-n_{1,i})-a^\sigma\|_{W^{1,2}([n_{1,i}+L_i,n_{2,i}-L_i],\mathbb{R}^m)}+$$
$$+\|U_-(\cdot-n_{1,i})-a^\sigma\|_{W^{1,2}([n_{1,i}+L_i,n_{2,i}-L_i],\mathbb{R}^m)}\to 0$$

as $i\to\infty$ and q_i satisfies (3_i) when i is large. Therefore q_i satisfies properties (1_i), (2_i) and (3_i) for i large, contrary to the choice of the functions q_i. Thus Proposition 4.6 is proved.

There are three further preliminaries needed before giving the proof of Theorem 4.1. The first provides a useful estimate.

Proposition 4.28. *Suppose that* $K\in\{K_0^u(a^-),K_0^u(a^+),K_0^s(a^-),K_0^s(a^+)\}\equiv\mathfrak{K}$, $q\in\mathcal{D}^d$ *and*

$$\|g^j(q)-K\|_{W^{1,2}([0,1],\mathbb{R}^m)}\le\bar r \tag{4.29}$$

for some $j\in\mathbb{Z}$*. Then there exists a constant,* $\lambda>0$*, such that*

$$\int_j^{j+1}(\tfrac12|\dot q|^2+W(t,q))\,dt\ge\lambda. \tag{4.30}$$

Proof. By (3.27)–(3.28),

$$\mathrm{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(\mathfrak{K},\{a^-,a^+\})\ge 3\bar r$$

so by (4.29),

$$\mathrm{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(g^j(q),\{a^-,a^+\})\ge 2\bar r. \tag{4.31}$$

If there is no $\lambda=\lambda(\bar r)$ as above, there are sequences, $(q_n)\subset\mathcal{D}^d$ and $(j_n)\subset\mathbb{Z}$ such that $u_n\equiv g^{j_n}(q_n)$ satisfies (4.29) and (4.31), but as $n\to\infty$,

$$\int_0^1(\tfrac12|\dot u_n|^2+W(t,u_n))\,dt\to 0. \tag{4.32}$$

Then as $n\to\infty$,

$$\|\dot{u}_n\|^2_{L^2([0,1],\mathbb{R}^m)} \to 0 \text{ and } \int_0^1 W(t,u_n))\,dt \to 0. \tag{4.33}$$

By (4.29) and the compactness of K, $\|u_n\|_{L^\infty([0,1],\mathbb{R}^m)}$ is bounded. Since $\|\dot{u}_n\|^2_{L^2([0,1],\mathbb{R}^m)} \to 0$ as $n \to \infty$, there exists a $\xi_0 \in \mathbb{R}^m$ such that $u_n \to \xi_0 \in \mathbb{R}^m$ in $L^\infty([0,1],\mathbb{R}^m)$. By (4.33), $\int_0^1 W(t,\xi_0)\,dt = 0$. But then $\xi_0 \in \{a^-,a^+\}$ and (4.31) must hold with $g^j(q)$ replaced by ξ_0. Since this is impossible, there is a λ as claimed.

Define

$$\ell = [\frac{d}{\lambda}] + 1 \text{ and } r_\ell = \frac{\bar{r}}{30\,\ell}. \tag{4.34}$$

The next technical result is important for an inductive argument in the proof of Theorem 4.1.

Proposition 4.35. *Let $\xi \in \{a^-,a^+\}$, $0 < r_1 < r_2 \le \bar{r}$, $l_i \in \mathbb{N}$ with $l_i \to +\infty$ as $i \to \infty$, and $(v_i) \subset \mathcal{D}^d$ be such that*

$$r_1 \le \operatorname{dist}_{W^{1,2}([-l_i,1],\mathbb{R}^m)}(v_i, \mathcal{K}^u(\xi)) \le r_2 \quad \forall i \in \mathbb{N}. \tag{4.36}$$

Then there are integers $i_0, L_0 \in \mathbb{N}$ and $n_0 \in \mathbb{Z}$, with $L_0 > \bar{n}(r_\ell)$ (where $\bar{n}(r_\ell)$ is given by Proposition 4.6) and $n_0 \le -3L_0$ such that

- (i) $\operatorname{dist}_{W^{1,2}([-L_0,1],\mathbb{R}^m)}(v_{i_0}, \mathcal{K}^u(\xi)) \le r_\ell$,
- (ii) $\operatorname{dist}_{W^{1,2}([0,L_0],\mathbb{R}^m)}(v_{i_0}(\cdot + n_0), \mathcal{K}^s(\xi)) \le r_\ell$,
- (iii) $\|v_{i_0} - \xi\|_{W^{1,2}([n_0+L_0,-L_0],\mathbb{R}^m)} \le r_\ell$.
- (iv) $j^u(v_{i_0}) < n_0$.

Proof. By Proposition 4.2, there exists a function, $U^* \in \mathcal{K}^u(\xi)$ such that the sequence $v_i \to U^*$ weakly in $W^{1,2}(\mathbb{R},\mathbb{R}^m)$ as $i \to \infty$ (along a subsequence that still will be denoted by v_i). The sequence v_i does not converge strongly to U^* in $W^{1,2}((-\infty,1],\mathbb{R}^m)$; otherwise the constraint in (4.36) would be violated. This observation and Proposition 3.7 imply the sequence converges to a chain of homoclinic and/or heteroclinic solutions of (HS) and U^* is not the left end of the chain. Indeed the weak convergence of $v_i \to U^*$ in $W^{1,2}(\mathbb{R},\mathbb{R}^m)$ implies one of the sequences (t_i^j) given by Proposition 3.7 must be bounded. Hence there is a $p \in \mathbb{Z}$ such that, passing to a further subsequence, $t_i^j = p$ for all $i \in \mathbb{N}$. Thus

$$\|v_i - U_j(\cdot - p)\|_{W^{1,2}((\frac{t_i^{j-1}+p}{2}, \frac{t_i^{j+1}+p}{2}),\mathbb{R}^m)} \to 0$$

as $i \to \infty$ showing that $U^* = U_j(\cdot - p)$. Therefore by (4.14), as $i \to \infty$,

$$\|v_i - U^*\|_{W^{1,2}((\frac{t_i^{j-1}+p}{2}, 1],\mathbb{R}^m)} \to 0. \tag{4.37}$$

Knowing that v_i does not converge strongly to U_1 in $W^{1,2}((-\infty,1],\mathbb{R}^m)$ and recalling that $t_i^0 = -\infty$, (4.37) implies that $j > 1$. Set $s_i = t_i^{j-1} + p$ and $U = U_{j-1}(\cdot + p)$. By Proposition 3.7, $s_i \to -\infty$ as $i \to \infty$ so without loss of generality, $(s_i) \subset (-\infty,0) \cap \mathbb{Z}$. Moreover $U \in \mathcal{D}^d(\zeta,\xi)$ with $\zeta \in \{a^-,a^+\}$. By (4.14) again, for any $T > 0$, as $i \to \infty$,

$$\|v_i - U(\cdot - s_i)\|_{W^{1,2}((s_i - T,\, s_i/2),\mathbb{R}^m)} \to 0 \tag{4.38}$$

and

$$\|v_i - U^*\|_{W^{1,2}((s_i/2,\,1),\mathbb{R}^m)} \to 0 \tag{4.39}$$

By definition, $U(\cdot + j^s(U))|_{[0,1]} \in K_0^s(\xi)$ or equivalently $U(\cdot + j^s(U))|_{[0,+\infty)} \in \mathcal{K}^s(\xi)$. Since $U^* \in \mathcal{K}^u(\xi)$ and $U \in \mathcal{K}^s(\xi)$, an $L_0 \in \mathbb{N}$ with $L_0 \ge \bar{n}(r_\ell)$ can be chosen so large that

$$\|U(\cdot + j^s(U)) - \xi\|_{W^{1,2}([L_0,+\infty),\mathbb{R}^m)} \le \tfrac{r_\ell}{4}, \quad \|U^* - \xi\|_{W^{1,2}((-\infty,-L_0],\mathbb{R}^m)} \le \tfrac{r_\ell}{4}. \tag{4.40}$$

Choose $i_0 \in \mathbb{N}$ so large that $-\frac{s_{i_0}}{4} > \max\{L_0, L_0 + j^s(U)\}$. By (4.38)–(4.39), it can be further assumed that

$$\|v_{i_0} - U(\cdot - s_{i_0})\|_{W^{1,2}((s_{i_0}+j^s(U),\, s_{i_0}/2),\mathbb{R}^m)} \le r_\ell/4 \tag{4.41}$$

and

$$\|v_{i_0} - U^*\|_{W^{1,2}((s_{i_0}/2,\, 1),\mathbb{R}^m)} \le r_\ell/4. \tag{4.42}$$

Now we will show that (i)–(iv) follow on setting $n_0 = s_{i_0} + j^s(U)$. Indeed property (i) follows from (4.42):

$$\mathrm{dist}_{W^{1,2}([-L_0,1],\mathbb{R}^m)}(v_{i_0}, \mathcal{K}^u(\xi))) \le \|v_{i_0} - U^*\|_{W^{1,2}([-L_0,1],\mathbb{R}^m)} \le r_\ell/4.$$

Property (ii), is immediate from (4.41):

$$\begin{aligned}\mathrm{dist}_{W^{1,2}([0,L_0],\mathbb{R}^m)}&(v_{i_0}(\cdot + n_0), \mathcal{K}^s(\xi))\\ &\le \|v_{i_0}(\cdot + s_{i_0} + j^s(U)) - U(\cdot + j^s(U))\|_{W^{1,2}([0,L_0],\mathbb{R}^m)}\\ &= \|v_{i_0} - U(\cdot - s_{i_0})\|_{W^{1,2}([s_{i_0}+j^s(U), s_{i_0}+j^s(U)+L_0],\mathbb{R}^m)} \le r_\ell/4.\end{aligned}$$

Next by (4.41), (4.40), (4.42), and (4.40) again,

$$\begin{aligned}\|v_{i_0} - \xi\|_{W^{1,2}([n_0+L_0,-L_0],\mathbb{R}^m)} &\le \|v_{i_0} - \xi\|_{W^{1,2}([n_0+L_0,s_{i_0}/2],\mathbb{R}^m)} +\\ &+ \|q^1 - \xi\|_{W^{1,2}([s_{i_0}/2,-L_0],\mathbb{R}^m)}\\ &\le \|v_{i_0} - U(\cdot - s_{i_0})\|_{W^{1,2}([s_{i_0}+j^s(U)+L_0,s_{i_0}/2],\mathbb{R}^m)} +\\ &+ \|U(\cdot - s_{i_0}) - \xi\|_{W^{1,2}([s_{i_0}+j^s(U)+L_0,s_{i_0}/2],\mathbb{R}^m)} +\\ &+ \|v_{i_0} - U^*\|_{W^{1,2}([s_{i_0}/2,-L_0],\mathbb{R}^m)} + \|U^* - \xi\|_{W^{1,2}([s_{i_0}/2,-L_1],\mathbb{R}^m)} \le r_\ell,\end{aligned} \tag{4.43}$$

yielding (iii). Lastly, to verify (iv), observe that by (4.41),

$$\|v_{i_0}(\cdot + n_0) - K_0^s(\xi)\|_{W^{1,2}([0,1],\mathbb{R}^m)} \le r_l/4. \tag{4.44}$$

By (3.32),

$$\|K_0^s(\xi) - K^u(a^-)\|_{W^{1,2}([0,1],\mathbb{R}^m)}, \|K_0^s(\xi) - K^u(a^+)\|_{W^{1,2}([0,1],\mathbb{R}^m)} \ge 3\bar{r}. \tag{4.45}$$

Combining (4.44)–(4.45) gives

$$\mathrm{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(v_{i_0}(\cdot + n_0), K^u(a^-) \cup K^u(a^+)) \ge 2\bar{r}. \tag{4.46}$$

By the definition of $j^u(v_{i_0})$,

$$\mathrm{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(v_{i_0}(\cdot + j), K^u(a^-) \cup K^u(a^+)) = 0 \; for\ any\ j \le j^u(v_{i_0}) \tag{4.47}$$

and (4.46)–(4.47) yield $j^u(v_{i_0}) < n_0$, completing the proof of Proposition 4.35.

Our final preliminary result is:

Proposition 4.48. *Suppose that $(\Phi^u(a^\sigma))$–$(\Phi^s(a^\sigma))$ hold for each $\sigma \in \{-,+\}$ and $\mathcal{C}^d(a^\tau) \ne \{a^\tau\}$ for some $\tau \in \{-,+\}$. Then there exist*

1^o *$\ell + 1$ integers $p_0, p_1, \dots, p_\ell \in \mathbb{Z}$ such that $p_0 > p_1 > p_2 > \dots > p_l$,*
2^o *$\ell + 1$ sets $K_0, K_1, \dots, K_\ell \in \mathfrak{K}$,*
3^o *a point $q \in \mathcal{D}^d$*

such that

$$\|q - g^{-p_j}(K_j)\|_{W^{1,2}([p_j,p_j+1],\mathbb{R}^m)} \le \bar{r} \; for\ any\ j \in \{0, 1, \dots, \ell\}. \tag{4.49}$$

Assuming this proposition for the moment, the proof of Theorem 4.1 can be given.

Proof of Theorem 4.1. First it will be shown that if conditions $(\Phi^u(a^\sigma))$–$(\Phi^s(a^\sigma))$ hold for $\sigma \in \{a^-, a^+\}$, then $\mathcal{C}^d(a^\pm) = \{a^\pm\}$. The proofs being the same, only the $\sigma = -$ case will be treated. Let $q \in \mathcal{D}^d$ as given by 3^o of Proposition 4.48. Applying (4.30) to each of the sets, K_i, $1 \le i \le \ell + 1$, yields

$$d \ge I(q) > \Sigma_1^{\ell+1} \int_{p_i}^{p_i+1} (\tfrac{1}{2}|\dot{q}|^2 + W(t, q))\, dt \ge (l+1)\lambda. \tag{4.50}$$

But (4.50) is impossible via (4.34) and the first half of the equivalence has been proved.

For the converse statement, suppose that $\mathcal{C}^d(a^-) = \{a^-\}$. Recalling (3.17), set

$$\alpha = \frac{1}{3}\min(2r_0, |a^- - a^+|).$$

Let $B_r(x)$ denote the open ball in $W^{1,2}([0,1], \mathbb{R}^m)$ of radius r about x. Consider $\overline{B}_\alpha(a^-) \cap \overline{\mathcal{S}}^d$. By Proposition 3.29 of [7], it is a compact subset of $W^{1,2}([0,1], \mathbb{R}^m)$. If $\partial(\overline{B}_\alpha(a^-)) \cap \overline{\mathcal{S}}^d = \emptyset$, take $K_1^- = \overline{B}_\alpha(a^-) \cap \overline{\mathcal{S}}^d$ and $K_2^- = \overline{\mathcal{S}}^d \setminus K_1^-$. Then $K_1 \cup K_2 = \overline{\mathcal{S}}^d$, $K_1 \cap K_2 = \emptyset$, and K_1 and K_2 are each closed and nonempty. Next suppose that $\partial(\overline{B}_\alpha(a^-)) \cap \overline{\mathcal{S}}^d \ne \emptyset$. Since $\mathcal{C}^d(a^-) = \{a^-\}$, there does not exist a subcontinuum of $\overline{\mathcal{S}}^d$ joining a^- to $\partial(\overline{B}_\alpha(a^-)) \cap \overline{\mathcal{S}}^d$. By a separation result of Whyburn [18], there are closed nonempty disjoint subsets, G_1, G_2 of $\overline{B}_\alpha(a^-) \cap \overline{\mathcal{S}}^d$ such that $a^- \in G_1$, $\partial(\overline{B}_\sigma(a^-)) \cap \overline{\mathcal{S}}^d \subset G_2$, and $G_1 \cup G_2 = \overline{B}_\sigma(a^-) \cap \overline{\mathcal{S}}^d$. For this case, we obtain a splitting of $\overline{\mathcal{S}}^d$ by taking $K_1^- = G_1$ and $K_2^- = \overline{\mathcal{S}}^d \setminus K_1^-$. Now choose any $z \in K_1^- \setminus \{a^-\}$. If $z = q|_{T_0}$ with $q \in \mathcal{D}^d(a^-, a^-)$, by the choice of α – see (3.17) – there is a $p \in \mathbb{Z}$ such that $\|g_p(z) - a^-\|_{W^{1,2}([p,p+1],\mathbb{R}^m)} > \alpha$, i.e. $g^p(z) \in K_2^-$. Now we have to find a compact set, $K^u(a^-) \subset \overline{\mathcal{S}}^d$ and an $r^- > 0$ such that (i)–(ii) of $(\Phi^u(a^-))$ hold. The set, $K^u(a^-)$ will be obtained from (3.19) by making an appropriate choice of $K_0^u(a^-)$. Set

$$K_0^u(a^-) = \{q \in K_1^- \mid g^{-j}(u) \in K_1^- \ for\ j \in \mathbb{Z}, j \ge 0\, and\ g(u) \in K_2^- \}.$$

Due to (3.17), for each $U \in \mathcal{D}^d(a^\sigma, a^-) \cup \mathcal{D}^d(a^\sigma, a^+)$, there is a unique $p(U) \in \mathbb{Z}$ such that $U(\cdot + p(U) + i)|_{[0,1]} \in K_1^-$ for any $i \le 0$ and

$$\|U(\cdot + p(U) + 1) - K_1^-\|_{W^{1,2}([0,1],\mathbb{R}^m)} \ge \|K_1^- - K_2^-\|_{W^{1,2}([0,1],\mathbb{R}^m)}.$$

Thus an alternate characterization of $K_0^u(a^-)$ is

$$K_0^u(a^-) = \{U|_[0,1] \mid U \in \mathcal{D}^d(a^\sigma, a^-) \cup \mathcal{D}^d(a^\sigma, a^+)\ and\ p(U) = 0\}.$$

Clearly $K_0^u(a^-) \ne \emptyset$. It is also compact since if $(q_n) \subset K_0^u(a^-)$, (q_n) lies in the compact set K_1^-. Therefore there is a $q^* \in K_1^-$ such that along a subsequence, $q_n \to q^*$ as $n \to \infty$. Since these functions are all solutions of (HS), $q_n \to q^*$ in $C^2_{loc}(\mathbb{R}, \mathbb{R}^m)$. Thus $g(q^*) \in K_2^-$ so $p(q^*) \le 0$. Similarly, $g^{-j}(q^*) \in K_1^-$ for all $j \in \mathbb{N}$. It follows that $p(q^*) = 0$, $q^* \in K_0^u(a^-)$ and $K_0^u(a^-)$ is compact. Now (3.19) shows $K^u(a^-)$ and (i)–(ii) of $(\Phi^u(a^-))$ hold with $r^- = \frac{1}{3}\|K_1^- - K_2^-\|_{W^{1,2}([0,1],\mathbb{R}^m)}$.

Replacing $\mathcal{C}^d(a^-) = \{a^-\}$ by $\mathcal{C}^d(a^+) = \{a^+\}$, with minor changes, the above argument implies $(\Phi^u(a^+))$ holds. The remaining two cases are treated similarly and the proof of Theorem 4.1 is complete.

It remains to give the

Proof of Proposition 4.48. The proofs being the same for either choice of τ, we take $\tau = -$. Since $\mathcal{C}^d(a^-) \ne \{a^-\}$,

$$\exists\, a\ q^0 \in \mathcal{D}^d \text{ such that } \zeta^0 = q^0|_{[0,1]} \in \mathcal{C}^d(a^-). \tag{4.51}$$

For each $k \in \mathbb{N}$, we say property (P_k) is satisfied if the following 6 conditions are satisfied:

(P_k) (i) There exists a function $q^k \in \mathcal{D}^d$ such that $q^k|_{[0,1]} \in \mathcal{C}^d(a^-)$.

For each $j \in \{1, \dots, k\}$,

(ii) there are integers $L_j \in \mathbb{N}$ and $n_{1,j}, n_{2,j} \in \mathbb{Z}$, such that $L_j \geq \bar{n}(r_\ell)$ (with $\bar{n}(r_\ell)$ given by Proposition 4.6), $n_{2,j} - n_{1,j} > 3L_j$ and

$$j^u(q^k) < n_{1,k} < n_{2,k} < n_{1,k-1} < \ldots < n_{2,2} < n_{1,1} < n_{2,1} = 0;$$

(iii) there are points $\xi^j \in \{a^-, a^+\}$ and pairs of sets $\mathcal{H}_+^j, \mathcal{H}_-^j$ where

$$\mathcal{H}_+^j = \begin{cases} \mathcal{K}^s(a^-) & \text{if } \xi^j = a^- \\ \mathcal{K}^s(a^+) & \text{if } \xi^j = a^+ \end{cases} \text{ and } \mathcal{H}_-^j = \begin{cases} \mathcal{K}^u(a^-) & \text{if } \xi^j = a^- \\ \mathcal{K}^u(a^+) & \text{if } \xi^j = a^+ \end{cases}$$

such that

(iv) $\mathrm{dist}_{W^{1,2}([-L_j,1],\mathbb{R}^m)}(q^k(\cdot + n_{2,j}), \mathcal{H}_-^j) \leq r_\ell$,

(v) $\mathrm{dist}_{W^{1,2}([0,L_j],\mathbb{R}^m)}(q^k(\cdot + n_{1,j}), \mathcal{H}_+^j) \leq r_\ell$,

(vi) $\|q^k - \xi^j\|_{W^{1,2}([n_{1,j}+L_j, n_{2,j}-L_j],\mathbb{R}^m)} \leq r_\ell$.

We will show that (4.51) implies property (P_k) is valid for each $k \in \{1, \dots, \ell\}$.

However, before doing so, assuming property (P_ℓ), the proof of Proposition 4.48 can be completed. Using (ii) of (P_ℓ), by taking $p_0 = n_{2,1}, \dots, p_{\ell-1} = n_{2,\ell}$, and $p_\ell = j^u(q^\ell)$, 1^o of Proposition 4.48 is satisfied. To verify 2^o, take $K_j = \{q|_{[0,1]} \mid q \in \mathcal{H}_-^{j+1}\}$ for $j = 0, \dots, \ell$, where the sets $\mathcal{H}_-^{j+1}$ are given by (iii) of (P_l). Then $K_j \in \mathfrak{K}$ by the definition of the sets, $\mathcal{H}_-^{j+1}$. Lastly to prove 3^o, choose $q^\ell \in \mathcal{D}^d$ as given by (i) of (P_ℓ). Then by (iv) of (P_ℓ), for $j = 0, \dots, \ell - 1$,

$$\begin{aligned} &\|q^\ell - g^{-p_j}(K_j)\|_{W^{1,2}([p_j,p_j+1],\mathbb{R}^m)} = \|q^\ell(\cdot + p_j) - K_j\|_{W^{1,2}([0,1],\mathbb{R}^m)} = \\ &= \mathrm{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(q^\ell(\cdot + n_{2,j+1}), \mathcal{H}_-^{j+1}) \leq r_\ell < \bar{r} \end{aligned}$$

while for $j = \ell$ again by (iv) and the definition of $j^u(q^\ell)$,

$$\begin{aligned} &\|q^\ell - g^{-p_\ell}(K_\ell)\|_{W^{1,2}([p_\ell,p_\ell+1],\mathbb{R}^m)} = \|q^\ell(\cdot + p_\ell) - K_\ell\|_{W^{1,2}([0,1],\mathbb{R}^m)} = \\ &= \mathrm{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(q^\ell(\cdot + j^u(q^\ell)), \mathcal{H}_-^{\ell+1}) = 0 < \bar{r}. \end{aligned}$$

Thus Proposition 4.48 is proved.

To verify that (4.51) implies property (P_k) for each $k \in \{1, \dots, \ell\}$, a few further remarks are required. For any $i \in \mathbb{N} \cup \{0\}$, define the sets

$$\begin{aligned} &\mathcal{S}^{d,i} = \{q|_{[-i,1]} \mid q \in D^d\},\ \bar{\mathcal{S}}^{d,i} = \mathcal{S}^{d,i} \cup \{a^-, a^+\}, \\ &K_{0,i}^u(a^\sigma) = \{q|_{[-i,1]} \mid q \in D^d,\ q|_{[0,1]} \in K_0^u(a^\sigma)\}. \end{aligned}$$

In analogy with earlier results, the set $\bar{\mathcal{S}}^{d,i}$ is a compact metric space under the metric of $W^{1,2}([-i,1],\mathbb{R}^m)$. Moreover for $q \in \bar{\mathcal{S}}^d$, the map $\iota_i(q)|_{[0,1]} = q$, $\iota_i : \bar{\mathcal{S}}^d \to \bar{\mathcal{S}}^{d,i}$ is a homeomorphism.

Note that $K_{0,i}^u(a^\sigma) = \iota_i(K_0^u(a^\sigma))$. The shift map g induces the map $g_i : \bar{\mathcal{S}}^{d,i} \to \bar{\mathcal{S}}^{d,i}$ defined as $g_i = \iota_i \circ g \circ \iota_i^{-1}$. Even though they are formally different, for notational brevity, the maps g_i will still be denoted by g.

For $\sigma \in \{-, +\}$, let $\mathcal{C}^{d,i}(a^\sigma)$ be the component of $\bar{\mathcal{S}}^{d,i}$ containing a^σ. For any $i \in \mathbb{N} \cup \{0\}$, $K_{0,i}^u(a^\sigma)$ can also be characterized as

$$K_{0,i}^u(a^\sigma) = \{q|_{[-i,1]} \mid q \in \mathcal{K}^u(a^\sigma)\} \tag{4.52}$$

and since ι_i is a homeomorphism,

$$\mathcal{C}^{d,i}(a^\sigma) = \iota_i(\mathcal{C}^d(a^\sigma)). \tag{4.53}$$

Finally observe that

$$\begin{cases} K^u_{0,i}(a^\sigma) \text{is compact and} \\ \text{dist}_{W^{1,2}([-i+1,1],\mathbb{R}^m)}(K^u_{0,i}(a^\sigma), g(K^u_{0,i}(a^\sigma))) \geq 3r^-. \end{cases} \tag{4.54}$$

Now we are ready to show that (4.51) implies property (P_k) for each $k \in \{1, \dots, \ell\}$. This will be done via an inductive argument. First (P_1) will be established. Then assuming (P_k) holds for $1 \leq k < \ell$, (P_{k+1}) will be proved.

Proof of (P_1):

Let q^0 be given by (4.51) and let $\sigma \in \{+, -\}$ be such that

$$q^0 \in \mathcal{D}^d(a^\sigma, a^-) \cup \mathcal{D}^d(a^\sigma, a^+). \tag{4.55}$$

Equations (4.51) and (4.53) show $\zeta_i^0 = \iota_i(\zeta^0) \in \mathcal{C}^{d,i}(a^\sigma)$ for any $i \in \mathbb{N} \cup \{0\}$. Note that

$$g^j(\zeta_i^0) \subset \mathcal{C}^{d,i}(a^\sigma) \text{ for any } j \in \mathbb{Z}. \tag{4.56}$$

Using (4.55) and $(\Phi^u(a^\sigma))$ shows that for any $i \in \mathbb{N} \cup \{0\}$,

$$g^{j^u(q^0)}(\zeta_i^0) \in K^u_{0,i}(a^\sigma) \text{ and } \text{dist}_{W^{1,2}([-i,1],\mathbb{R}^m)}(g^{j^u(q^0)+1}(\zeta_i^0), \mathcal{K}^u(a^\sigma)) \geq 3r^-. \tag{4.57}$$

By the connectedness of $\mathcal{C}^{d,i}(a^\sigma)$, the continuity of the function

$$\text{dist}_{W^{1,2}([-i,1],\mathbb{R}^m)}(\cdot, \mathcal{K}^u(a^\sigma)),$$

and (4.56)–(4.57), for any $i \in \mathbb{N} \cup \{0\}$,

$$\text{there exists } q_i^0 \in \mathcal{C}^{d,i}(a^\sigma) \text{ such that } \text{dist}_{W^{1,2}([-i,1],\mathbb{R}^m)}(q_i^0, \mathcal{K}^u(a^\sigma)) = r_\ell. \tag{4.58}$$

By (4.58), the hypotheses of Proposition 4.35 with $v_i = q_i^0$, $\xi = a^\sigma$ and $l_i = i$ are satisfied. Taking i_0, L_0, and n_0 as given by Proposition 4.35, set $q^1 = v_{i_0} = q_{i_0}^0$, $L_1 = L_0$, $\mathcal{H}_+^1 = \mathcal{K}^s(a^\sigma)$, $\mathcal{H}_-^1 = \mathcal{K}^u(a^\sigma)$, $n_{1,1} = n_0$ and $n_{2,1} = 0$. Then properties (i)–(vi) of (P_1) follow from (4.58) and Proposition 4.35.

Finally it will be shown that:

If (P_k) holds for some k with $1 \leq k < \ell$, then (P_{k+1}) is also satisfied:

Suppose q^k satisfies (P_k) for a $k \in [1, \ell - 1] \cap \mathbb{N}$ so for some $\sigma \in \{-, +\}$,

$$q^k \in \mathcal{D}^d(a^\sigma, a^-) \cup \mathcal{D}^d(a^\sigma, a^+). \tag{4.59}$$

Then for each $j \in \{1, \dots, k\}$, there exist integers $n_{1,j}, n_{2,j}, L_j$, points $\xi_1, \dots, \xi_j \in \{a^-, a^+\}$ and sets $\mathcal{H}_+^j, \mathcal{H}_-^j$, for which the properties (ii)–(vi) of (P_k) are satisfied. In particular, by (iv)–(vi), the interval $[j^u(q^k), 1]$ contains the k intervals $[n_{1,j}, n_{2,j}]$ in each of which q^k enters and leaves a small neighborhood of a^- or a^+. Moreover by the definition of $j^u(q^k)$, the function $q^k(t)$ remains in a small neighborhood of a^- or a^+ for values of $t < j^u(q^k)$. The idea of the proof of (P_{k+1}) is to find a new function q^{k+1} near q^k with three properties. First, it possesses the same qualitative behavior as q^k on the interval $[j^u(q^k), 1]$. More explicitly, it continues to enter and leave a small neighborhood of a^- or a^+ on precisely the same intervals, $[n_{1,j}, n_{2,j}]$ for $j = 1, \dots, k$, as for q^k, and otherwise satisfies the properties (P_k). Secondly to the left of $j^u(q^k)$, there is a new interval $[n_{1,k+1}, n_{2,k+1}]$ where q^{k+1} again enters and leaves a small neighborhood of a^- or a^+ and further satisfies the remaining requirements of (P_{k+1}). Thirdly, as in (4.56)–(4.58), using in particular the connectedness of $\mathcal{C}^{d,i}(a^\sigma)$, and the continuity of the function, $f_{i,k}(q)$ (that will be introduced shortly and plays the role of

$$\text{dist}_{W^{1,2}([-i,1],\mathbb{R}^m)}(\cdot, \mathcal{K}^u(a^\sigma)),$$

in the proof of (P_1)) shows q^{k+1} lies in $\mathcal{C}^{d,i}(a^\sigma)$.

To implement these ideas, intervals $[-i, 1]$ larger than $[j^u(q^k), 1]$ must be considered. Hence assume $i \in \mathbb{N}$ and $i \geq -j^u(q^k)$ for what follows. For such values of i, set

$$\zeta_i^k = q^k|_{[-i,1]} \text{ and note that } g^j(\zeta_i^k) \in \mathcal{C}^{d,i}(a^\sigma) \text{ for any } j \in \mathbb{Z}.$$

By definition $q^k(\cdot + j^u(q^k))|_{[0,1]} \in K_0^u(a^\sigma)$ and by $(\Phi^u(a^\sigma))$,

$$g^{j^u(q^k)}(\zeta_i^k) \in K_{0,i}^u(a^\sigma) \text{ and } \mathrm{dist}_{W^{1,2}([-i-j^u(q^k),1],\mathbb{R}^m)}(g^{j^u(q^k)+1}(\zeta_i^k), \mathcal{K}^u(a^\sigma)) \geq 3\bar{r}.$$

For $q \in \mathcal{S}^{d,i}$, define

$$\begin{aligned} f_{i,k}(q) = &\mathrm{dist}_{W^{1,2}([-i,j^u(q^k)+1],\mathbb{R}^m)}(q, g^{-j^u(q^k)}(\mathcal{K}^u(a^\sigma))+ \\ &+\sum_{j=1}^{k} \mathrm{dist}_{W^{1,2}([-L_j,1],\mathbb{R}^m)}(q(\cdot+n_{2,j}), \mathcal{H}_-^j)+ \\ &+\sum_{j=1}^{k} \mathrm{dist}_{W^{1,2}([0,L_j],\mathbb{R}^m)}(q(\cdot+n_{1,j}), \mathcal{H}_+^j)+ \\ &+\sum_{j=1}^{k} \|q-\xi^j\|_{W^{1,2}([n_{1,j}+L_j,n_{2,j}-L_j],\mathbb{R}^m)} \end{aligned}$$

where $g^{-j^u(q^k)}(\mathcal{K}^u(a^\sigma)) = \{q(\cdot - j^u(q^k)) \mid q \in \mathcal{K}^u(a^\sigma)\}$. Properties (iv)–(vi) of (P_k) show

$$f_{i,k}(\zeta_i^k) \leq 3kr_\ell,\ \ f_{i,k}(g(\zeta_i^k)) \geq 3r^- \text{ and } \zeta_i^k,\ g(\zeta_i^k) \in \mathcal{C}^{d,i}(a^\sigma). \tag{4.60}$$

From the choice of r_ℓ in (4.34), for any $k \in \{1,\dots,\ell\}$, $3kr_\ell < \bar{r}/4$. Therefore as in the argument establishing (P_1),

$$\text{there exists } q_i^k \in \mathcal{C}^{d,i}(a^\sigma) \text{ such that } f_{i,k}(q_i^k) = \bar{r}/2. \tag{4.61}$$

Consequently for any $j \in \{1,\dots,k\}$

$$\begin{cases} \mathrm{dist}_{W^{1,2}([-L_j,1],\mathbb{R}^m)}(q_i^k(\cdot+n_{2,j}), \mathcal{H}_-^j) \leq \bar{r}/2 < \bar{r}, \\ \mathrm{dist}_{W^{1,2}([0,L_j],\mathbb{R}^m)}(q_i^k(\cdot+n_{1,j}), \mathcal{H}_+^j) \leq \bar{r}/2 < \bar{r}, \\ \|q^k-\xi^j\|_{W^{1,2}([n_{1,j}+L_j,n_{2,j}-L_j],\mathbb{R}^m)} \leq \bar{r}/2 < \bar{r}. \end{cases} \tag{4.62}$$

But $L_j \geq \bar{n}(r_l)$ and $n_{2,j} - n_{1,j} \geq 3L_j$ for any $j \in \{1,\dots,k\}$ so by Proposition 4.6, (4.62) can be improved to

$$\begin{cases} \mathrm{dist}_{W^{1,2}([-L_j,1],\mathbb{R}^m)}(q_i^k(\cdot+n_{2,j}), \mathcal{H}_-^j) \leq r_\ell, \\ \mathrm{dist}_{W^{1,2}([0,L_j],\mathbb{R}^m)}(q_i^k(\cdot+n_{1,j}), \mathcal{H}_+^j) \leq r_\ell, \\ \|q_i^k-\xi^j\|_{W^{1,2}([n_{1,j}+L_j,n_{2,j}-L_j],\mathbb{R}^m)} \leq r_\ell. \end{cases} \tag{4.63}$$

Moreover, by (4.61), (4.63) and since $3kr_\ell < \bar{r}/4$, we find that for any $i \geq -j^u(q^k)$

$$\bar{r}/2 \geq \mathrm{dist}_{W^{1,2}([-i,j^u(q^k)+1],\mathbb{R}^m)}(q_i^k, g^{-j^u(q^k)}(\mathcal{K}^u(a^\sigma))) \geq \bar{r}/4$$

or equivalently that

$$\bar{r}/2 \geq \mathrm{dist}_{W^{1,2}([-i-j^u(q^k),1],\mathbb{R}^m)}(q_i^k(\cdot + j^u(q^k)), \mathcal{K}^u(a^\sigma)) \geq \bar{r}/4. \tag{4.64}$$

By (4.64), for $i \geq -j^u(q^k)$, the hypotheses of Proposition 4.35 are satisfied by taking $v_i = q_i^k(\cdot + j^u(q^k))$, $\xi = a^\sigma$ and $l_i = -i - j^u(q^k)$. Then with $i_0 > -j^u(q^k)$, $L_0 > \bar{n}(r_\ell)$ and $n_0 \leq -3L_0$ obtained from Proposition 4.35, set

$$\begin{aligned} &q^{k+1} = v_{i_0}(\cdot - j^u(q^k)) = q_{i_0}^k,\ \xi^{k+1} = a^\sigma,\ L_{k+1} = L_0 \\ &\mathcal{H}_+^{k+1} = \mathcal{K}^s(a^\sigma),\ \mathcal{H}_-^{k+1} = \mathcal{K}^u(a^\sigma), \\ &n_{1,k+1} = n_0 + j^u(q^k) \text{ and } n_{2,k+1} = j^u(q^k). \end{aligned} \tag{4.65}$$

By (4.61), $q^{k+1}|_{[0,1]} \in \mathcal{C}^d(a^\sigma)$. By definition, $n_{2,k+1} < n_{1,k+1} < j^u(q^k)$ so from (P_k),

$$n_{1,k+1} < n_{2,k+1} < n_{1,k} < \dots < n_{2,2} < n_{1,1} < n_{2,1} = 0.$$

By Proposition 4.35, $j^u(v_{i_0}) < n_0$ and by (P_k), $j^u(q^k) < n_{1,k}$. Hence

$$j^u(q^{k+1}) = j^u(v_{i_0}(\cdot - j^u(q^k))) < n_0 + j^u(q^k) = n_0 + n_{1,k}$$

and q^{k+1} verifies (i)–(iii) of (P_{k+1}). Moreover (4.63) shows $q^{k+1} = q^k_{i_0}$ satisfies (iv)–(vi) of (P_{k+1}) on the intervals $(n_{1,j}, n_{2,j} + 1)$ for the same set of values: ξ_j, L_j, $\mathcal{H}^j_-$ and $\mathcal{H}^j_-$ for $j = 1, \dots, k$, as for q^k. To complete the proof it, remains to verify that properties (iv)–(vi) for q^{k+1} also hold on the new interval $(n_{1,k+1}, n_{2,k+1} + 1)$.

By (4.65) and (i) of Proposition 4.35,

$$\mathrm{dist}_{W^{1,2}([-L_{k+1},1],\mathbb{R}^m)}(q^{k+1}(\cdot + n_{2,k+1}), \mathcal{H}^{k+1}_-) = \mathrm{dist}_{W^{1,2}([-L_{k+1},1],\mathbb{R}^m)}(v_{i_0}, \mathcal{H}^{k+1}_-) \le \bar{r}_\ell,$$

i.e. (iv). Similarly by (4.65) and (ii) of Proposition 4.35,

$$\mathrm{dist}_{W^{1,2}([0,L_{k+1}],\mathbb{R}^m)}(q^{k+1}(\cdot + n_{2,k+1}), \mathcal{H}^{k+1}_+) = \mathrm{dist}_{W^{1,2}([0,L_{k+1}],\mathbb{R}^m)}(v_{i_0}(\cdot + n_0), \mathcal{H}^{k+1}_+) \le \bar{r}_\ell,$$

i.e. (v). Lastly property (vi) is a consequence of

$$\|q^k - \xi^{k+1}\|_{W^{1,2}([n_{1,k+1}+L_{k+1}, n_{2,k+1}-L_{k+1}],\mathbb{R}^m)} = \|v_{i_0} - \xi^{k+1}\|_{W^{1,2}([n_0+L_{k+1}, -L_{k+1}],\mathbb{R}^m)} \le \bar{r}_\ell.$$

Thus (P_{k+1}) follows and Theorem 4.1 is proved.

5. Some degenerate cases

In this section, it will be shown how the methods introduced earlier in §3–4 can be applied to study the behavior of $\mathcal{D}^d$ when $\mathcal{C}^d(a^-) \neq \{a^-\}$ or $\mathcal{C}^d(a^+) \neq \{a^+\}$, i.e. when we are in a degenerate situation. It will also be shown that condition 3^o of §2 is equivalent to a non degeneracy condition used by Cieliebak and Séré in [4].

To be more precise, by 4^o–5^o or from [7] we know if $\mathcal{C}^d(a^\pm) \neq \{a^\pm\}$, $\overline{\mathcal{S}}^d$ contains a connected set, $\mathcal{F}$, to which $a^\pm$ belongs and $\mathcal{F}$ consists of snapshots of heteroclinic or homoclinic solutions of (HS). Given $\xi_1, \xi_2 \in \{a^-, a^+\}$, the space $\mathcal{D}^d(\xi_1, \xi_2)$ under the $W^{1,2}(\mathbb{R}, \mathbb{R}^m)$ metric will be considered. Define the projection map, i by

$$i : \mathcal{D}^d \to \mathcal{S}^d, \quad i(q) = q|_{[0,1]} \text{ for } q \in \mathcal{D}^d$$

and note that it is continuous and invertible. The map i^{-1} associates to any element $\zeta \in \mathcal{S}^d$, the unique element $i^{-1}(\zeta) = q \in \mathcal{D}^d$ such that $q|_{[0,1]} = \zeta$. In general, i^{-1} is not continuous on $\mathcal{S}^d$ although Proposition 3.7 gives us some information about it. Therefore when $\mathcal{C}^d(a^\pm) \neq \{a^\pm\}$, the connectedness properties of $\overline{\mathcal{S}}^d$ given by 4^o–5^o do not provide similar information for $\mathcal{D}^d$. If it were continuous, $i^{-1}(\mathcal{F})$ would be a connected subset of $\mathcal{D}^d$. (Note that i^{-1} is continuous if e.g. $\mathcal{D}^d$ and $\mathcal{S}^d$ are replaced by $\mathcal{D}^d(a^-, a^+)$, $\mathcal{S}^d(a^-, a^+)$ and $d = c(a^-, a^+)$.) Our first result shows that in fact $\mathcal{D}^d$ does possess some connectivity properties,

Proposition 5.1. *If $\mathcal{C}^d(a^-) \neq \{a^-\}$ or $\mathcal{C}^d(a^+) \neq \{a^+\}$, then there exists a $q \in \mathcal{D}^d$ whose component is different from $\{q\}$.*

When the potential V is sufficiently smooth, a stronger result obtains:

Proposition 5.2. *If $V \in C^{2m}(\mathbb{R} \times \mathbb{R}^m)$ and $\mathcal{C}^d(a^-) \neq \{a^-\}$ or $\mathcal{C}^d(a^+) \neq \{a^+\}$, then there exists points $\xi_-, \xi_+ \in \{a^-, a^+\}$ such that $D^d(\xi_-, \xi_+)$ contains a point U whose component with respect to the $W^{1,2}(\mathbb{R}, \mathbb{R}^m)$ metric is not compact.*

Proof of Proposition 5.1. An indirect argument will be used. For a given $q \in \mathcal{D}^d(\xi_1, \xi_2)$, and $\xi_1, \xi_2 \in \{a^-, a^+\}$, let $\mathcal{C}^d_q$ denote the component of q in $\mathcal{D}^d(\xi_1, \xi_2)$. Suppose that

$$\mathcal{C}^d_q = \{q\} \text{ for each } q \in \mathcal{D}^d. \tag{5.3}$$

This has an important consequence:

(P1) Suppose that $\mathcal{C}^d(a^-) \neq \{a^-\}$ and (5.3) holds. Let $\zeta_0 \in \mathcal{C}^d(a^-) \setminus \{a^-\}$ and $\rho > 0$. Set

$$B_\rho(\zeta_0, \overline{\mathcal{S}}^d) \equiv \{\zeta \in \overline{\mathcal{S}}^d \mid \|\zeta - \zeta_0\|_{W^{1,2}([0,1],\mathbb{R}^m)} < \rho\}$$

and

$$\mathcal{U} \equiv (\mathcal{C}^d(a^-) \setminus \{a^-\}) \cap \bar{B}_\rho(\zeta_0, \overline{\mathcal{S}}^d).$$

Then the map $i^{-1}|_{\mathcal{C}^d(a^-)}$ is not continuous on $\mathcal{U}$.

To prove (P1), assume to the contrary that there exists a $\zeta_0 \in \mathcal{C}^d(a^-) \setminus \{a^-\}$ and a $\rho > 0$ for which the map $i^{-1}|_{\mathcal{C}^d(a^-)}$ is continuous on $\mathcal{U}$. Suppose $i(z_0) = \zeta_0$ for $z_0 \in \mathcal{D}^d$. Let $g \equiv g^1$ be as in the Introduction. Since $\zeta_0 \in \mathcal{C}^d(a^-) \setminus \{a^-\}$,

$$g(\zeta_0) \in \mathcal{C}^d(a^-) \setminus \{a^-\} \text{ with } \|\zeta_0 - g(\zeta_0)\|_{W^{1,2}([0,1],\mathbb{R}^m)} \equiv r_{\zeta_0} > 0. \tag{5.4}$$

Choose $r < \min\{r_{\zeta_0}, \rho\}$. By (5.4) and a result from [18], since $\mathcal{C}^d(a^-)$ is compact and connected, there exists a connected set $\mathcal{C}_0 \subset \mathcal{U}$ containing ζ_0 and a point $\zeta_r \in \mathcal{C}^d(a^-) \setminus \{a^-\}$ such that $\|\zeta_0 - \zeta_r\|_{W^{1,2}([0,1],\mathbb{R}^m)} = r$. Since $i^{-1}|_{\mathcal{C}^d(a^-)}$ is continuous on $\mathcal{U}$, the set $i^{-1}(\mathcal{C}_0)$ is connected and so by (5.3), $i^{-1}(\mathcal{C}_0) \subset \mathcal{C}^d_{q_0} = \{q_0\}$. But this is contrary to (5.3) since $q_0 = i^{-1}(\zeta_0) \in i^{-1}(\mathcal{C}_0)$, $q_r = i^{-1}(\zeta_r) \in i^{-1}(\mathcal{C}_0)$ and $q \neq q_r$. Thus (P1) follows.

Set

$$\lambda_0 = \inf_{q \in \mathcal{D}^d} I(q)$$

and note that, as observed in [7], $\lambda_0 > 0$. As a consequence of (P1), we have:

(P2) Under the hypotheses of (P1), for any $q_0 \in \mathcal{D}^d$ such that $\xi_0 = q_0|_{[0,1]} \in \mathcal{C}^d(a^-) \setminus \{a^-\}$, there exists a $q_1 \in \mathcal{D}^d$ such that $\xi_1 = q_1|_{[0,1]} \in \mathcal{C}^d(a^-) \setminus \{a^-\}$ and $I(q_1) \geq I(q_0) + \frac{1}{2}\lambda_0$.

Assuming (P2) for the moment, it will be shown that it leads to a contradiction so Proposition 5.1 holds. Indeed using (P2), chose any point $q_0 \in \mathcal{D}^d$ such that $\xi_0 = q_0|_{[0,1]} \in \mathcal{C}^d(a^-) \setminus \{a^-\}$. Iterating this application of (P2) k times, with $k > 2d/\lambda_0$, yields k points $q_1, \ldots, q_k \in \mathcal{D}^d$ such that $\xi_j = q_j|_{[0,1]} \in \mathcal{C}^d(a^-) \setminus \{a^-\}$ and $I(q_j) \geq I(q_{j-1}) + \frac{1}{2}\lambda_0$ for $j = 1, \ldots, k$. But then

$$I(q_k) \geq \sum_{j=1}^{k} \frac{1}{2}\lambda_0 = k\lambda_0/2 > d$$

contrary to the fact that $q_k \in \mathcal{D}^d$.

Now to conclude the proof of Proposition 5.1, (P2) will be derived from (P1). Let $q_0 \in \mathcal{D}^d$ be such that $\zeta_0 = q_0|_{[0,1]} \in \mathcal{C}^d(a^-) \setminus \{a^-\}$. Since $q_0 \in \mathcal{D}^d$,

$$\min_{\xi \in \{a^-, a^+\}} \|q_0 - \xi\|_{W^{1,2}([0,1],\mathbb{R}^m)} \equiv r(q_0) > 0. \tag{5.5}$$

Chose $R_0 > 0$ so large that

$$I_{[-R_0,R_0]}(q_0) \equiv \int_{-R_0}^{R_0} \tfrac{1}{2}|\dot{q}_0|^2 + V(t, q_0)\, dt \geq I(q_0) - \tfrac{1}{8}\lambda_0. \tag{5.6}$$

Due to the continuous dependence on the initial data of the Cauchy problem for (HS), $r_0 \in (0, r(q_0))$ can be taken so small that if $q \in \mathcal{D}^d$ is such that $\|q - q_0\|_{W^{1,2}([0,1],\mathbb{R}^m)} < r_0$, then

$$I_{[-R_0,R_0]}(q_0) - \tfrac{1}{8}\lambda_0 \leq I_{[-R_0,R_0]}(q) \leq I_{[-R_0,R_0]}(q_0) + \tfrac{1}{8}\lambda_0. \tag{5.7}$$

By (5.6) and (5.7),

$$I(q) \geq I(q_0) - \tfrac{1}{4}\lambda_0 \text{ for any } q \in \mathcal{D}^d \text{ such that } \zeta = i(q) \in B_{r_0}(\zeta_0, \overline{\mathcal{S}}^d). \tag{5.8}$$

Since by (P1), the map $i^{-1}|_{\mathcal{C}^d(a^-)}$ is not continuous on $(\mathcal{C}^d(a^-)\setminus\{a^-\})\cap B_{r_0/2}(\zeta_0,\overline{\mathcal{S}}^d)$, there exists a $\zeta_1\in\mathcal{C}^d(a^-)\setminus\{a^-\}$ with $\|\zeta_1-\zeta_0\|_{W^{1,2}([0,1],\mathbb{R}^m)}\le r_0/2$ and a sequence $(\eta_n)\in(\mathcal{C}^d(a^-)\setminus\{a^-\})\cap B_{r_0/2}(\zeta_0,\overline{\mathcal{S}}^d)$ such that

$$\eta_n\to\zeta_1 \text{ in } W^{1,2}([0,1],\mathbb{R}^m) \text{ but} \tag{5.9}$$

$$v_n=i^{-1}(\xi_n)\not\to\bar q_1=i^{-1}(\zeta_1) \text{ in the } W^{1,2}(\mathbb{R},\mathbb{R}^m) \text{ metric.} \tag{5.10}$$

Note that v_n may belong to a component of $\Gamma(\xi_1,\xi_2)$ different from the one containing q_1. Taking a subsequence if needed, it can be assumed that for some pair $\xi_1,\xi_2\in\{a^-,a^+\}$,

$$v_n\in\mathcal{D}^d(\xi_1,\xi_2) \text{ for any } n\in\mathbb{N}. \tag{5.11}$$

The sequence (v_n) is a (PS) sequence for I since, by (5.11), it consists of elements of $\mathcal{D}^d(\xi_1,\xi_2)$. By Proposition 3.7, there exists an $l_0\in\mathbb{N}$, $\bar\xi_1=\xi_1,\bar\xi_2,\dots,\bar\xi_{l_0+1}=\xi_2\in\{a^-,a^+\}$, $U_j\in\mathcal{D}^d(\bar\xi_j,\bar\xi_{j+1})$ for $j\in\{1,\dots,l_0\}$, and sequences $(t_n^j)\subset\mathbb{Z}$, $j\in\{1,\dots,l_0\}$, such that $t_n^{j+1}-t_n^j\to+\infty$ as $n\to+\infty$ having the property that along a subsequence, setting $t_n^0\equiv-\infty$ and $t_n^{l_0+1}\equiv+\infty$, as $n\to\infty$,

$$\|v_n-U_j(\cdot-t_n^j)\|_{W^{1,2}(\frac12(t_n^j+t_n^{j-1}),\,\frac12(t_n^{j+1}+t_n^j)),\mathbb{R}^m)}\to 0 \text{ for } j=1,\dots,l_0 \tag{5.12}$$

and

$$I(v_n)\to\sum_{j=1}^{l_0}I(U_j). \tag{5.13}$$

Since $v_n=i^{-1}(\eta_n)$ and $\bar q_1\equiv i^{-1}(\zeta_1)$, by the definition of i, $\eta_n(x)=v_n(x)$ and $\zeta_1(x)=\bar q_1(x)$ for any $x\in[0,1]$. Thus by (5.9), as $n\to\infty$,

$$\|\eta_n-\zeta_1\|_{W^{1,2}([0,1],\mathbb{R}^m)}=\|v_n-\bar q_1\|_{W^{1,2}([0,1],\mathbb{R}^m)}\to 0. \tag{5.14}$$

We claim that (5.12) and (5.14) imply

(W) for some $j_0\in\{1,\dots,l_0\}$, along a subsequence, $t_n^{j_0}$ is constant, t_0, so $t_n^{j_0}=t_0\in\mathbb{Z}$ independently of $n\in\mathbb{N}$, and $U_{j_0}=\bar q_1(\cdot+t_0)$.

To prove (W), arguing indirectly suppose first that as $n\to\infty$, $|t_n^j|\to+\infty$ for any $j\in\{1,\dots,l_0\}$. Since $t_n^{j+1}-t_n^j\to+\infty$ for any $j\in\{1,\dots,l_0-1\}$, there exists a $p_0\in\{0,1,\dots,l_0\}$ for which

$$t_n^j\to-\infty \text{ for } j\le p_0 \text{ and } t_n^j\to+\infty \text{ for } j>p_0 \tag{5.15}$$

as $n\to\infty$. Set $U_0\equiv\xi_1$. Since $U_j\in\mathcal{D}^d(\bar\xi_j,\bar\xi_{j+1})$ for any $j\in\{1,\dots,l_0\}$, taking $j=p_0$ or $j=p_0+1$ shows

$$\|\bar\xi_{p_0+1}-U_j(\cdot-t_n^j)\|_{W^{1,2}([0,1],\mathbb{R}^m)}\to 0 \tag{5.16}$$

as $n\to\infty$. Since $r_0\in(0,r(q_0))$ and $\|\zeta_1-q_0\|_{W^{1,2}([0,1],\mathbb{R}^m)}=\|\zeta_1-\zeta_0\|_{W^{1,2}([0,1],\mathbb{R}^m)}\le r_0/2$ by (5.5), it follows that

$$\|\zeta_1-\bar\xi_{p_0+1}\|_{W^{1,2}([0,1],\mathbb{R}^m)}>0. \tag{5.17}$$

Writing the interval $[0,1]$ as the union of the intervals $[0,\frac12(t_n^{p_0+1}+t_n^{p_0})]$ and $[\frac12(t_n^{p_0+1}+t_n^{p_0}),1]$ with the understanding that the first or the second of these intervals is the empty set whenever respectively $\frac12(t_n^{p_0+1}+t_n^{p_0})\le 0$ or $\frac12(t_n^{p_0+1}+t_n^{p_0})\ge 1$ leads to the estimates:

$$\begin{aligned}\|\zeta_1-\bar\xi_{p_0+1}\|_{W^{1,2}([0,1],\mathbb{R}^m)}&\le\|\zeta_1-v_n\|_{W^{1,2}([0,1],\mathbb{R}^m)}+\|v_n-\bar\xi_{p_0+1}\|_{W^{1,2}([0,1],\mathbb{R}^m)}\\&\le\|\zeta_1-v_n\|_{W^{1,2}([0,1],\mathbb{R}^m)}+\\&\quad+\|v_n-U_{p_0}(\cdot-t_n^{p_0})\|_{W^{1,2}([0,\frac12(t_n^{p_0+1}+t_n^{p_0})],\mathbb{R}^m)}+\\&\quad+\|\bar\xi_{p_0+1}-U_{p_0}(\cdot-t_n^{p_0})\|_{W^{1,2}([0,\frac12(t_n^{p_0+1}+t_n^{p_0})],\mathbb{R}^m)}+\end{aligned}$$

$$+ \|v_n - U_{p_0+1}(\cdot - t_n^{p_0+1})\|_{W^{1,2}([\frac{1}{2}(t_n^{p_0+1}+t_n^{p_0}),1],\mathbb{R}^m)}$$
$$+ \|\bar{\xi}_{p_0+1} - U_{p_0+1}(\cdot - t_n^{p_0+1})\|_{W^{1,2}([\frac{1}{2}(t_n^{p_0+1}+t_n^{p_0}),1],\mathbb{R}^m)}.$$

Applying (5.14), (5.12) and (5.16) yields $\|\zeta_1 - \bar{\xi}_{p_0}\|_{W^{1,2}([0,1],\mathbb{R}^m)} = 0$, in contradiction with (5.17).

This argument shows there is a $j_0 \in \{1, \dots, l_0\}$ such that along a subsequence, $t_n^{j_0}$ is bounded and therefore can be assumed to be constant on a further subsubsequence. Reindexing this latter subsequence with $n \in \mathbb{N}$, let $t_n^{j_0} = t_0 \in \mathbb{Z}$ for any $n \in \mathbb{N}$, and set $U_{j_0} = \bar{q}_1(\cdot + t_0)$. Since by (5.12), as $n \to \infty$,

$$\|v_n - U_{j_0}(\cdot - t_0)\|_{W^{1,2}([0,1],\mathbb{R}^m)} \le \|v_n - U_{j_0}(\cdot - t_0)\|_{W^{1,2}(\frac{1}{2}(t_n^{j_0-1}+t_0),\frac{1}{2}(t_0+t_n^{j_0+1})),\mathbb{R}^m)} \to 0$$

and by (5.14),

$$\|v_n - \bar{q}_1\|_{W^{1,2}([0,1],\mathbb{R}^m)} \to 0$$

as $n \to \infty$, we conclude that $U_{j_0}(x - t_0) = \bar{q}_1(x)$ for $x \in [0, 1]$. Since both $U_{j_0}(x - t_0)$ and $\bar{q}_1(x)$ satisfy (HS), they then coincide on $\mathbb{R}$ and (W) follows.

Combining (W) and (5.12) shows that as $n \to \infty$, $v_n \to U_{j_0}(\cdot - t_0)$ and $v_n \to \bar{q}_1$ in $W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^m)$. By (5.10), $v_n \not\to \bar{q}_1$ with respect to the $W^{1,2}(\mathbb{R}, \mathbb{R}^m)$ metric. Hence (5.12) implies that $l_0 > 1$ and so, by the definition of λ_0, as $n \to \infty$,

$$I(v_n) \to \sum_{j=1}^{l_0} I(U_j) \ge I(U_{j_0}) + \sum_{j \ne j_0} I(U_j) \ge I(U_{j_0}) + \lambda_0.$$

Since $I(U_{j_0}) = I(\bar{q}_1)$ and $\bar{q}_1 \in B_{r_0/2}(\zeta_0, \overline{\mathcal{S}}^d)$, by (5.8), we conclude that

$$\lim_{n\to+\infty} I(v_n) \ge I(\bar{q}_1) + \lambda_0 > I(q_0) + \tfrac{3}{4}\lambda_0$$

and (P2) follows by choosing $q_1 = v_n$ for n sufficiently large.

The proof of the above proposition contains the following result which will also be useful in the arguments that follow:

Lemma 5.18. *Let $\xi_1, \xi_2 \in \{a^-, a^+\}$, $v_n = i^{-1}(\eta_n) \in \mathcal{D}^d(\xi_1, \xi_2)$ with $\eta_n \in \mathcal{S}^d$ for $n \in \mathbb{N}$ and $\bar{q}_1 = i^{-1}(\zeta_1) \in \mathcal{D}^d$ for a $\zeta_1 \in \mathcal{S}^d$, such that as $n \to \infty$,*

$\eta_n \to \zeta_1$ in $W^{1,2}([0, 1], \mathbb{R}^m)$ but
$v_n = i^{-1}(\xi_n) \not\to \bar{q}_1 = i^{-1}(\zeta_1)$ with respect to the $W^{1,2}(\mathbb{R}, \mathbb{R}^m)$ metric.

Then along a subsequence

$$\lim_{n\to+\infty} I(v_n) \ge I(q_1) + \lambda_0.$$

Now we are ready for the

Proof of Proposition 5.2. The role of the smoothness condition $V \in C^{2m}(\mathbb{R} \times \mathbb{R}^m)$ is to show that

(P3) For any pair, $\xi_1, \xi_2 \in \{a_-, a_+\}$, the set of critical values of I on $\Gamma(\xi_1, \xi_2)$ is of measure 0.

Property (P3) is obtained from a version of the Sard–Smale Theorem. Indeed let $\xi_1, \xi_2 \in \{a_-, a_+\}$ with ψ_{ξ_1,ξ_2} the appropriate normalizing function and $J(u) = I(u + \psi_{\xi_1,\xi_2})$ for $u \in E = W^{1,2}(\mathbb{R}, \mathbb{R}^m)$. Then $J'(u)$ is a nonlinear Fredholm operator for every $u \in E$. To see this, consider the matrix function $\alpha : \mathbb{R} \to \mathbb{R}^{m^2}$ such that $\alpha(x) = V_{q,q}(x, a^+)$ for $x > 1$ and $\alpha(x) = V_{q,q}(x, a^-)$ for $x < -1$ and

$$\alpha(x) = \tfrac{1}{2}(1-x)V_{q,q}(-1,a^-) + \tfrac{1}{2}(1+x)V_{q,q}(1,a^+) \text{ for } x \in [-1,1].$$

Then α is continuous, bounded, and by (V_4) it is positive definite uniformly for $x \in \mathbb{R}$. Note that

$$J''(u)h \cdot k = \int_{\mathbb{R}} (\dot h \dot k + \alpha h k)\, dx + \int_{\mathbb{R}} (V_{q,q}(x, u+\psi_{\xi_1,\xi_2}) - \alpha) hk dx$$
$$\equiv \mathcal{L}(h)(k) + \int_{\mathbb{R}} (V_{q,q}(x, u+\psi_{\xi_1,\xi_2}) - \alpha) hk dx \text{ for } u, h, k \in E.$$

Since α is bounded and uniformly positive definite, the operator $\mathcal{L}$ is a linear homeomorphism between E and $W^{-1,2}(\mathbb{R},\mathbb{R}^m)$. Moreover the multiplication operator $h \to (V_{q,q}(x, u+\psi_{\xi_1,\xi_2}) - \alpha)h$ is compact from $W^{1,2}(\mathbb{R},\mathbb{R}^m)$ to $L^2(\mathbb{R},\mathbb{R}^m)$ (since $V_{q,q}(x, u+\psi_{\xi_1,\xi_2}) - \alpha \to 0$ as $|x| \to +\infty$). Thus $J''(u)$ is a Fredhlom operator. The condition $V \in C^{2m}(\mathbb{R} \times \mathbb{R}^m)$ together with (V_4) implies that $J \in C^{2m}(E)$ and (P3) then follows by the application of a version of the Sard–Smale Lemma as given e.g. in [11].

An immediate consequence of (P3) is:

(P4) If $\xi_1, \xi_2 \in \{a^-, a^+\}$ and $\mathcal{C} \subset \mathcal{D}^d(\xi_1,\xi_2)$ is connected with respect to the $W^{1,2}(\mathbb{R},\mathbb{R}^m)$ metric, then I is constant on $\mathcal{C}$.

The remainder of the proof of Proposition 5.2 uses an indirect argument. Assume to the contrary that

(CH) For every $\xi_1, \xi_2 \in \{a^-, a^+\}$ and $q \in \mathcal{D}^d(\xi_1,\xi_2)$, that $\mathcal{C}^d_q$ is compact.

For $q \in \mathcal{D}^d$ and $k \in \mathbb{Z}$, set $\tau_k(q) = q(\cdot + k)$. Note that by (CH), if $\xi_1, \xi_2 \in \{a^-, a^+\}$,

$$\tau_1(q) \in \mathcal{D}^d(\xi_1,\xi_2) \setminus \mathcal{C}^d_q \text{ for any } q \in \mathcal{D}^d(\xi_1,\xi_2). \tag{5.19}$$

Indeed the set $\tau_1(\mathcal{C}^d_q) = \{\tau_1(v) \mid v \in \mathcal{C}^d_q\}$ is connected since τ_1 is continuous with respect to the $W^{1,2}(\mathbb{R},\mathbb{R}^m)$ metric. If $\tau_1(q) \in \mathcal{C}^d_q$ for some $q \in \mathcal{D}^d$, then $\tau_1(\mathcal{C}^d_q) \subset \mathcal{C}^d_q$ and similarly $\tau_k(q) \in \mathcal{C}^d_q$ for any $k \in \mathbb{N}$. But the sequence $(\tau_k(q))$ converges weakly and not strongly to ξ_2. Hence $\mathcal{C}^d_q$ is not compact, contrary to (CH).

Suppose $\mathcal{C}^d(a^-) \neq \{a^-\}$ and $\zeta_0 \in \mathcal{C}^d(a^-) \setminus \{a^-, a^+\}$. Then $g(\zeta_0) \in \mathcal{C}^d(a^-) \setminus \{a^-, a^+\}$ and there exists $\xi_1, \xi_2 \in \{a^-, a^+\}$ such that $q_0 = i^{-1}(\zeta_0) \in \mathcal{D}^d(\xi_1,\xi_2)$. By (CH), $\mathcal{C}^d_{q_0}$ is compact in $\mathcal{D}^d(\xi_1,\xi_2)$ and the continuity of i implies $i(\mathcal{C}^d_{q_0})$ is connected and compact in $\overline{\mathcal{S}}^d$. Moreover the compactness of $\mathcal{C}^d_{q_0}$ in $\mathcal{D}^d(\xi_1,\xi_2)$ shows

$$\bar r_0(q_0) = \inf_{q \in \mathcal{C}^d_{q_0}} \min_{\xi \in \{a^-,a^+\}} \|q - \xi\|_{W^{1,2}([0,1],\mathbb{R}^m)} > 0. \tag{5.20}$$

Indeed, if (5.20) fails, there is a sequence $(q_n) \subset \mathcal{C}^d_{q_0}$ and a $\xi \in \{a^-, a^+\}$ such that $\|q_n - \xi\|_{W^{1,2}([0,1],\mathbb{R}^m)} \to 0$. The compactness of $\mathcal{C}^d_{q_0}$ in $\mathcal{D}^d(\xi_1,\xi_2)$ gives a $\bar q \in \mathcal{C}^d_{q_0}$ such that along a subsequence $q_n - \bar q \to 0$ in $W^{1,2}(\mathbb{R},\mathbb{R}^m)$. Then $\|\bar q - \xi\|_{W^{1,2}([0,1],\mathbb{R}^m)} = 0$ so $\bar q(x) = \xi$ for $x \in [0,1]$. Hence, since both $\bar q$ and ξ are solutions of (HS) on $\mathbb{R}$, the uniqueness of the solution of the Cauchy problem for (HS) gives $\bar q(x) = \xi$ for any $x \in \mathbb{R}$. But this is not possible since by definition $\mathcal{D}^d \cap \{a^-, a^+\} = \emptyset$ and (5.20) follows.

By (5.20),

$$\operatorname{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(i(\mathcal{C}^d_{q_0}), \{a^-, a^+\}) \geq r_0(q_0) \tag{5.21}$$

and by (5.19), $\tau_1(q_0) \notin \mathcal{C}^d_{q_0}$. Moreover $g(\zeta_0) = i(\tau_1(q_0)) \in \mathcal{C}^d(a^-) \setminus \{a^-, a^+\}$ and since $i(\mathcal{C}^d_{q_0})$ is compact in $\overline{\mathcal{S}}^d$, there exists an $r_{\zeta_0} \in (0, \bar r_0(q_0)/2)$ such that

$$\operatorname{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(g(\zeta_0), i(\mathcal{C}^d_{q_0})) \equiv 2r_{\zeta_0} > 0. \tag{5.22}$$

Since $\mathcal{C}^d(a^-)$ is a compact connected set, contains the point $g(\zeta_0)$ and the compact set $i(\mathcal{C}^d_{q_0})$, by (5.22) and [18], there is a closed subcontinuum, $\mathcal{C}$ of $\mathcal{C}^d(a^-)$ joining $g(\zeta_0)$ to $i(\mathcal{C}^d_{q_0})$. These observations show:

(P5) Let $\zeta_0 \in \mathcal{C}^d(a^-) \setminus \{a^-\}$ and set $q_0 = i^{-1}(\zeta_0)$. Then there exists a point $\bar{\zeta}_0 \in i(\mathcal{C}^d_{q_0})$ and a closed connected set $\mathcal{C}(\zeta_0)$ such that $g(\zeta_0), \bar{\zeta}_0 \in \mathcal{C}(\zeta_0)$.

Property (P5) implies

(P6) Let $\zeta_0 \in \mathcal{C}^d(a^-) \setminus \{a^-, a^+\}$ and $q_0 = i^{-1}(\zeta_0)$. If $\bar{\zeta}_0 \in i(\mathcal{C}^d_{q_0})$ and $\mathcal{C}(\zeta_0)$ are as in (P5), then for any $\rho \in (0, r_{\zeta_0}/2)$, where r_{ζ_0} is given by (5.22), we have

$$i^{-1}|_{\mathcal{C}^d(a^-)} \text{ is not continuous on } \mathcal{C}(\zeta_0) \cap N^{\overline{\mathcal{S}}^d}_\rho(i(\mathcal{C}^d_{q_0}))$$

where $N^{\overline{\mathcal{S}}^d}_\rho(i(\mathcal{C}^d_{q_0})) = \{\zeta \in \overline{\mathcal{S}}^d \mid \mathrm{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(\zeta, i(\mathcal{C}^d_{q_0})) \le \rho\}$.

To prove (P6), note that since $\rho \in (0, r_{\zeta_0}/2)$, $\|g(\zeta_0) - \bar{\zeta}_0\|_{W^{1,2}([0,1],\mathbb{R}^m)} \ge 2\rho$ via (5.22). The set $\mathcal{C}(\zeta_0)$ is closed, connected and contains the points $g(\zeta_0)$ and $\bar{\zeta}_0$. We claim that there exists a connected set $\mathcal{C}_0 \subset \mathcal{C}(\zeta_0) \cap N^{\overline{\mathcal{S}}^d}_\rho(i(\mathcal{C}^d_{q_0}))$ containing $\bar{\zeta}_0$ and a point ζ_ρ such that

$$\mathrm{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(\zeta_\rho, i(\mathcal{C}^d_{q_0})) \ge \rho.$$

Indeed the set $\mathcal{C}(\zeta_0) \cap N^{\overline{\mathcal{S}}^d}_\rho(i(\mathcal{C}^d_{q_0}))$ is compact. Consider its subsets

$$\mathcal{C}(\zeta_0) \cap \partial N^{\overline{\mathcal{S}}^d}_\rho(i(\mathcal{C}^d_{q_0})) \text{ and } \{\bar{\zeta}_0\}.$$

They are compact, disjoint and nonempty. If our claim is false, by a separation lemma from [18], $N^{\overline{\mathcal{S}}^d}_\rho(i(\mathcal{C}^d_{q_0})) \cap \mathcal{C}(\zeta_0)$ is the union of two disjoint compact sets K_1 and K_2, the first containing $\mathcal{C}(\zeta_0) \cap \partial N^{\overline{\mathcal{S}}^d}_\rho(i(\mathcal{C}^d_{q_0}))$ and the second $\bar{\zeta}_0$. Then

$$\mathcal{C}(\zeta_0) = (\mathcal{C}(\zeta_0) \setminus N^{\overline{\mathcal{S}}^d}_\rho(i(\mathcal{C}^d_{q_0})) \cup K_1 \cup K_2$$

is not connected, a contradiction which proves our claim.

Since $\mathrm{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(\zeta_\rho, i(\mathcal{C}^d_{q_0})) \ge \rho$,

$$\zeta_\rho \notin i(\mathcal{C}^d_{q_0}). \tag{5.23}$$

If $i^{-1}|_{\mathcal{C}^d_{a^-}}$ is continuous on $\mathcal{C}(\zeta_0) \cap N^{\overline{\mathcal{S}}^d}_\rho(i(\mathcal{C}^d_{q_0}))$, then $i^{-1}(\mathcal{C}_0)$ is connected and contains $i^{-1}(\bar{\zeta}_0)$. Since $\bar{\zeta}_0 \in i(\mathcal{C}^d_{q_0})$, $i^{-1}(\bar{\zeta}_0) \in \mathcal{C}^d_{q_0}$ and so $i^{-1}(\mathcal{C}_0) \subset \mathcal{C}^d_{q_0}$. By construction, $\zeta_\rho \in \mathcal{C}_0$ and so we also have

$$i^{-1}(\zeta_\rho) \in \mathcal{C}^d_{q_0}. \tag{5.24}$$

But it is not possible for both (5.23) and (5.24) to hold. Thus (P6) follows.

Now as the final step in the proof of Proposition 5.2, as in the proof of Proposition 5.1, we will show

(P7) For any $q_0 \in \mathcal{D}^d$ such that $\xi_0 = q_0|_{[0,1]} \in \mathcal{C}^d(a^-) \setminus \{a^-, a^+\}$, there exists a $q_1 \in \mathcal{D}^d$ such that $\xi_1 = q_1|_{[0,1]} \in \mathcal{C}^d(a^-) \setminus \{a^-\}$ and $I(q_1) \ge I(q_0) + \frac{1}{2}\lambda_0$.

As earlier, (P7) leads to a contradiction and Proposition 5.2 follows.

To verify (P7), choose any $q_0 \in \mathcal{D}^d$ such that $\zeta_0 = q_0|_{[0,1]} \in \mathcal{C}^d(a^-) \setminus \{a^-, a^+\}$. Recall that by (P4) we have $I(q) = I(q_0)$ for any $q \in \mathcal{C}^d(a^-)$. Then an argument similar to the one which establishes (5.8) in the proof of Proposition 5.1 shows that there exists an $\bar{r} > 0$ such that

$$I(q) \ge I(q_0) - \tfrac{1}{4}\lambda_0 \text{ for any } q \in \mathcal{D}^d \text{ such that } \zeta = i(q) \in N^{\overline{\mathcal{S}}^d}_\rho(i(\mathcal{C}^d_{q_0})). \tag{5.25}$$

Indeed since $\mathcal{C}^d_{q_0}$ is compact, there exists an $R_0 > 0$ such that for any $\bar{q}_0 \in \mathcal{C}^d_{q_0}$,

$$I_{[-R_0,R_0]}(\bar{q}_0)=\int_{-R_0}^{R_0}\tfrac{1}{2}|\dot{\bar{q}}_0|^2+V(t,\bar{q}_0)\,dt\ge I(\bar{q}_0)-\tfrac{1}{8}\lambda_0=I(q_0)-\tfrac{1}{8}\lambda_0. \tag{5.26}$$

By the continuous dependence of solutions of (HS) on the data of the Cauchy problem, and the compactness of $\mathcal{C}^d_{q_0}$, an $\bar{r}>0$ can be chosen so that if $q\in\mathcal{D}^d$ is such that $\|q-\bar{q}_0\|_{W^{1,2}([0,1],\mathbb{R}^m)}<\bar{r}$ for a $\bar{q}_0\in\mathcal{C}^d_{q_0}$, then

$$I_{[-R_0,R_0]}(\bar{q}_0)-\tfrac{1}{8}\lambda_0\le I_{[-R_0,R_0]}(q)\le I_{[-R_0,R_0]}(\bar{q}_0)+\tfrac{1}{8}\lambda_0. \tag{5.27}$$

Thus (5.25) follows by (5.26) and (5.27).

If r_{ζ_0} is given by (5.22) and $\rho\in(0,\min\{r_{\zeta_0},\bar{r}\}/2)$, by (P6) the map $i^{-1}|_{\mathcal{C}^d(a^-)}$ is not continuous on $\mathcal{C}(\zeta_0)\cap N^{\overline{\mathcal{S}}^d}_\rho(i(\mathcal{C}^d_{q_0}))$. In particular there exists a $\zeta_1\in\mathcal{C}(\zeta_0)$ with

$$\mathrm{dist}_{W^{1,2}([0,1],\mathbb{R}^m)}(\zeta_1,i(\mathcal{C}^d_{q_0}))<\rho \tag{5.28}$$

and a sequence $(\eta_n)\in\mathcal{C}^d(a^-)$ such that

$$\eta_n\to\zeta_1 \text{ in } W^{1,2}([0,1],\mathbb{R}^m) \text{ but}$$
$$v_n=i^{-1}(\eta_n)\not\to\bar{q}_1=i^{-1}(\zeta_1) \text{ in } W^{1,2}(\mathbb{R},\mathbb{R}^m).$$

Taking a subsequence if needed, it can be assumed that for a pair $\xi_1,\xi_2\in\{a^-,a^+\}$,

$$v_n\in\mathcal{D}^b(\xi_1,\xi_2) \text{ for any } n\in\mathbb{N}. \tag{5.29}$$

Then the sequences (v_n) and (η_n) satisfy the assumptions of Lemma 5.18. Hence along a subsequence

$$\lim_{n\to+\infty}I(v_n)\ge I(\bar{q}_1)+\lambda_0. \tag{5.30}$$

Since by definition $\bar{q}_1(x)=\zeta_1(x)$ and $\bar{q}_0(x)=\bar{\zeta}_0(x)$ for $x\in[0,1]$, by (5.28), $\|\bar{q}_1-\bar{q}_0\|_{W^{1,2}([0,1],\mathbb{R}^m)}=\|\zeta_1-\bar{\zeta}_0\|_{W^{1,2}([0,1],\mathbb{R}^m)}\le\rho\le\bar{r}/2$. Then by (5.25), $I(\bar{q}_1)\ge I(q_0)-\frac{1}{4}\lambda_0$ and (P7) follows from (5.30) by choosing $q_1=v_n$ for n sufficiently large.

Remark 5.31. Since the component, $\mathfrak{K}$, of U in Proposition 5.2 is not compact, there is a sequence, $(U_n)\subset\mathfrak{K}$ which does not have a convergent subsequence. Therefore either along a subsequence (a) $U_n|_{[0,1]}\to\xi\in\{\xi_1,\xi_2\}$ or (b) $U_n|_{[0,1]}\to Q\in\mathcal{S}^d$. Hence along a subsequence, if (a) occurs, U_n converges weakly to ξ while if (b), U_n converges to a nontrivial chain of solutions.

For our final result, suppose as in Proposition 5.2, that $V\in C^{2N}(\mathbb{R}\times\mathbb{R}^N)$. Then the following result gives the equivalence of 3^o of §2 with an analogue in the present context of the assumption made by Cieliebak and Séré in [4].

Proposition 5.32. *If $V\in C^{2m}(\mathbb{R}\times\mathbb{R}^m)$, then $\mathcal{C}^d(a^-)=\{a^-\}$ and $\mathcal{C}^d(a^+)=\{a^+\}$ if and only if for any pair $\xi_1,\xi_2\in\{a^-,a^+\}$, the components of $\mathcal{D}^d(\xi_1,\xi_2)$ are compact with respect to the $W^{1,2}(\mathbb{R},\mathbb{R}^m)$ metric.*

Proof. Let $\xi_1,\xi_2\in\{a_-,a_+\}$ with ψ_{ξ_1,ξ_2} being the appropriate normalizing function and $J(u)=I(u+\psi_{\xi_1,\xi_2})$ for $u\in E=W^{1,2}(\mathbb{R},\mathbb{R}^m)$. By Proposition 3.32 in [7], we know that if $\mathcal{C}^d(a^-)=\{a^-\}$ and $\mathcal{C}^d(a^+)=\{a^+\}$ then there exist constants $\nu>0$, $r_0>0$ and a countable family of sets $\{\mathcal{A}_j\subset E\mid j\in\mathbb{N}\}$ such that

- **(i)** $\{u\in E\mid \|J'(u)\|\le\nu,\ J(u)\le d\}\subseteq\cup_j\mathcal{A}_j$,
- **(ii)** if $i\neq j$ then $\mathrm{dist}_E(\mathcal{A}_i,\mathcal{A}_j)\ge r_0$,
- **(iii)** the Palais–Smale condition, (PS), holds in each set $\mathcal{A}_j$.

By (i) and (ii), any component of $\mathcal{D}^d(\xi_1,\xi_2)$ lies in one of these sets $\mathcal{A}_j$. Since by (iii), the PS condition holds in $\mathcal{A}_j$, it follows that any component of $\mathcal{D}^d(\xi_1,\xi_2)$ is compact. This shows that if $\mathcal{C}^d(a^-)=\{a^-\}$ and $\mathcal{C}^d(a^+)=\{a^+\}$, then any component of $\mathcal{D}^d(\xi_1,\xi_2)$ is compact with respect to the $W^{1,2}(\mathbb{R},\mathbb{R}^m)$ metric.

To show that the reverse implication is true, it is sufficient to use Proposition 5.2 and a contradiction argument. Indeed if $\mathcal{C}^d(a^-) \neq \{a^-\}$ or $\mathcal{C}^d(a^+) \neq \{a^+\}$, Proposition 5.2 guarantees that there exists a pair $\xi_1, \xi_2 \in \{a^-, a^+\}$ and a point $U \in \mathcal{D}^d(\xi_1, \xi_2)$ whose component is not compact and the proof is complete.

Conflict of interest statement

The authors declare they do not have a conflict of interest.

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