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Hessian estimates for convex solutions to quadratic Hessian equation

Matt McGonagle [a], Chong Song [b,∗], Yu Yuan [a]

[a] *University of Washington, Department of Mathematics, Box 354350, Seattle, WA 98195, United States of America*
[b] *Xiamen University, School of Mathematical Sciences, Xiamen, 361005, PR China*

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Abstract

We derive Hessian estimates for convex solutions to quadratic Hessian equation by compactness argument.

Résumé

Nous dérivons des estimations de Hessian pour des solutions convexes à l'équation de Hessian quadratique par argument de compacité.

Keywords: Nonlinear elliptic equations; A priori estimates; Sigma-2 equation; Legendre–Lewy transformation

1. Introduction

In this note, we prove a priori Hessian estimates for convex solutions to the Hessian equation

$$\sigma_k\left(D^2u\right)=\sigma_k\left(\lambda\right)=\sum_{1\le i_1<\cdots<i_k\le n}\lambda_{i_1}\cdots\lambda_{i_k}=1$$

with $k=2$. Here $\lambda_i s$ are the eigenvalues of the Hessian D^2u.

Theorem 1.1. *Let u be a smooth solution to $\sigma_2\left(D^2u\right)=1$ on $B_R\left(0\right)\subset\mathbb{R}^n$ with $D^2u\ge\left[\delta-\sqrt{2/\left[n\left(n-1\right)\right]}\right]I$ for any $\delta>0$. Then*

$$\left|D^2u\left(0\right)\right|\le g\left(\|Du\|_{L^\infty(B_R(0))}/R,n\right),$$

where $g\left(t,n\right)$ is a finite and positive function for each positive t and dimension n.

∗ Corresponding author.
E-mail addresses: mmcgonal@math.washington.edu (M. McGonagle), songchong@xmu.edu.cn (C. Song), yuan@math.washington.edu (Y. Yuan).

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By Trudinger's gradient estimates for σ_k equations [10], we can bound D^2u in terms of the solution u in $B_{2R}(0)$ as

$$\left|D^2u(0)\right| \le g\left(\|u\|_{L^\infty(B_{2R}(0))}/R^2, n\right).$$

Recall any solution to the Laplace equation $\sigma_1\left(D^2u\right) = \triangle u = 1$ enjoys a priori Hessian estimates; yet there are singular solutions to the three dimensional Monge–Ampère equation $\sigma_3\left(D^2u\right) = \det D^2u = 1$ by Pogorelov [8], which automatically generalize to singular solutions to $\sigma_k\left(D^2u\right) = 1$ with $k \ge 3$ in higher dimensions $n \ge 4$.

A long time ago, Heinze [6] achieved a Hessian bound for solutions to equation $\sigma_2\left(D^2u\right) = 1$ in dimension two by two dimension techniques. Not so long time ago, Hessian bound for $\sigma_2\left(D^2u\right) = 1$ in dimension three was obtained via the minimal surface feature of the "gradient" graph $(x, Du(x))$ in the joint work with Warren [13]. Along this "integral" way, Qiu [9] has proved Hessian estimates for solutions to the three dimensional quadratic Hessian equation with $C^{1,1}$ variable right hand side. Hessian estimates for convex solutions to general quadratic Hessian equations have also been obtained via a new pointwise approach by Guan and Qiu [5]. Hessian estimates for solutions to Monge–Ampère equation $\sigma_n\left(D^2u\right) = \det D^2u = 1$ and Hessian equations $\sigma_k\left(D^2u\right) = 1$ $(k \ge 2)$ in terms of the reciprocal of the difference between solutions and their boundary values, were derived by Pogorelov [8] and Chou–Wang [4], respectively, using Pogorelov's pointwise technique. Lastly, we also mention Hessian estimates for solutions to σ_k as well as σ_k/σ_n equations in terms of certain integrals of the Hessian by Urbas [11,12], Bao–Chen–Guan–Ji [1].

Our argument towards Hessian bound for a semiconvex solution to $\sigma_2\left(D^2u\right) = 1$ is through a compactness one. If the Hessian blows up at the origin, then the slope of the "gradient" graph $y = Du(x)$ or $(x, Du(x))$ already blows up everywhere. But one cannot see this impossible picture directly (Step 1). After a Legendre–Lewy transformation of the solution $u(x)$ so that the new solution $\bar{w}(y)$ has bounded nonnegative Hessian; the new corresponding equation is uniformly elliptic (for any large negative lower bound for the original Hessian D^2u); and the new equation is concave (only under the particular lower Hessian bound $D^2u \ge \left[\delta - \sqrt{2/[n(n-1)]}\right] I$) (Step 2). By the standard Evans–Krylov–Safonov theory, the smooth "gradient" graph $(D\bar{w}(y), y) = (x, Du(x))$ has a zero slope at the origin (Step 3). Employing the constant rank theorem of Caffarelli–Guan–Ma [2], the zero slope of the "gradient" graph $(D\bar{w}(y), y)$ propagates everywhere. The impossible picture of $(x, Du(x))$ with infinite slope everywhere becomes clear (Step 4). In passing, we remark that in dimension two, the solution is already convex and the new equation is just the Laplace equation, in turn, our compactness argument is elementary.

Finally, the Hessian estimates for general solutions to quadratic Hessian equation $\sigma_2\left(D^2u\right) = 1$ in higher dimension $n \ge 4$ still remain an issue to us.

2. Proof

We prove Theorem 1.1 by a compactness argument. By scaling $v(x) = u(Rx)/R^2$, we assume $R = 1$. Denote $K = \sqrt{2/[n(n-1)]}$.

Step 1. Otherwise, there exist a sequence of solutions u_k to $\sigma_2\left(D^2u\right) = 1$ such that

$$\|Du_k\|_{L^\infty(B_1)} \le \|Du\|_{L^\infty(B_1)},$$
$$(\delta - K)\, I \le D^2u_k,$$

and (convergence)

$$\left.\begin{array}{l} \left|D^2u_k(0)\right| \to \infty, \\ Du_k \to Du_\infty \ \text{ in } \ L^1\left(B_1^m\right), \end{array}\right. \quad \text{as } k \to \infty,$$

where $u_\infty \in W^{1,1}(B_1)$ and B_1^m denotes the m dimensional ball $B_1^m(0) \subset B_1 = B_1^n(0) \subset \mathbb{R}^n$ for all $m = 1, \cdots, n$. The L^1 convergence (possibly passing to a sub-convergent sequence, still denoted by u_k; we only need $m = 1$) comes from the compact Sobolev embedding for semiconvex $u_k \in W^{2,1}\left(B_1^m\right) \hookrightarrow W^{1,1}\left(B_1^m\right)$, as

$$\int_{B_1^m} \left|D^2u_k + K\right| dx \le \int_{B_1^m} (\triangle u_k + nK)\, dx \le C(n)\left[\|Du\|_{L^\infty(B_1)} + 1\right].$$

Remark. Another way to see the above L^1 convergence for H^{n-m} almost all $(x_{m+1}, \cdots, x_n)$ in $B_1 \cap \mathbb{R}^{n-m}$ is the following. From our equation, $\sqrt{|\lambda|^2+2} = \triangle u_k$, then $\int_{B_1} \left|D^2 u_k\right| dx < \int_{B_1} \triangle u_k dx \le C(n) \|Du_k\|_{L^\infty(\partial B_1)} \le C(n) \|Du\|_{L^\infty(B_1)}$. (We just assume $\|Du_k\|_{L^\infty(\partial B_1)} \le \|Du\|_{L^\infty(B_1)}$.) Now the compact Sobolev embedding coupled with Fubini theorem implies the "almost everywhere" L^1 convergence.

Step 2. As in [3], we make Legendre–Lewy transformation of solutions $u_k(x)$ to solutions $\bar{w}_k(y)$ of a new uniformly elliptic and concave equation with bounded Hessian from both sides, so that we can extract smoother convergent limit. The Legendre–Lewy transformation is the Legendre transformation of $w_k(x) = u_k(x) + K|x|^2/2$; see [7]. Geometrically we re-present the "gradient" graph $G: y = Dw_k(x)$, or $(x, Dw_k(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ over y-space as another "gradient" graph in $\mathbb{R}^{2n}$. Note that the (canonical) angles between the tangent planes of G and x-space are

$$\arctan(\lambda_i + K) \in [\arctan\delta, \frac{\pi}{2})$$

by the semiconvexity assumption $\lambda_i \ge \delta - K$. From this angle condition and the symmetry of $\left(D^2 w\right)^{-1}$, it follows that G can still be represented as a "gradient" graph $x = D\bar{w}_k(y)$, or $(D\bar{w}_k(y), y)$ over ball $B_\delta(0)$ in y-space, here we may and assume $Du_k(0) = 0$; further the (canonical) angles between the tangent planes of G and y-space are

$$\arctan\bar{\lambda}_i = \frac{\pi}{2} - \arctan(\lambda_i + K) \in (0, \frac{\pi}{2} - \arctan\delta],$$

where $\bar{\lambda}_i s$ are the eigenvalues of the Hessian $D^2\bar{w}_k$.

Therefore, the function $\bar{w}_k(y)$ satisfies in $B_\delta(0)$

$$0 \le D^2\bar{w}_k = \left(D^2 u + K\right)^{-1} < \frac{1}{\delta} I$$

and

$$q\left(\bar{\lambda}\left(D^2\bar{w}_k\right)\right) = \frac{\sigma_{n-1}\left(\bar{\lambda}\right)}{\sigma_{n-2}\left(\bar{\lambda}\right)} = \frac{1}{(n-1)K},$$

where $\bar{\lambda}_i s$ are the eigenvalues of the Hessian $D^2\bar{w}_k$.

As proved in [3, pp. 661–663], we have

i) the level set $\Gamma = \left\{\bar{\lambda} \middle| \; q\left(\bar{\lambda}\right) = 1/[(n-1)K]\right\}$ is convex;

ii) the normal vector Dq of the level set Γ is uniformly inside the positive cone for $\bar{\lambda}_i \in \left[0, \delta^{-1}\right]$;

iii) all but one among $\lambda_i s$ are uniformly bounded, equivalently all but one among $\bar{\lambda}_i s$ have a uniform positive lower bound, then $\sigma_{n-2}\left(\bar{\lambda}\right)$ has a uniform positive lower bound.

Thus $\bar{w}_k$ satisfies a uniformly elliptic and concave equation.

Step 3. By the Evans–Krylov–Safonov theory, there are a subsequence of $\bar{w}_k$, still denoted by $\bar{w}_k$, and $\bar{w}_\infty \in C^{2,\alpha}\left(B_{\delta/2}(0)\right)$ with $\alpha = \alpha(n, \delta) > 0$ such that

$$\bar{w}_k \to \bar{w}_\infty \text{ in } C^{2,\alpha}\left(B_{\delta/2}(0)\right),$$

then

$$q\left(\bar{\lambda}\left(D^2\bar{w}_\infty\right)\right) = 1/[(n-1)K],$$

$$D^2\bar{w}_\infty(y) \ge 0, \text{ also one and only one eigenvalue of } D^2\bar{w}_\infty, \text{ say, } D_{11}\bar{w}_\infty \text{ is } 0 \text{ at } 0.$$

Step 4. By the constant rank theorem of Caffarelli–Guan–Ma [2, Theorem 1.1 and Remark 1.7] (which leads to a qualitative lower Hessian bound for concave equations), $D_{11}\bar{w}_\infty(y) \equiv 0$ in a neighborhood of 0. Restrict to (x_1, y_1) space, the "gradient" graph of $(D\bar{w}_\infty(y), y)$ takes the form

$$\left(D_1\bar{w}_\infty\left(y_1, y'\right), y_1\right) = (c, y_1) = \left(x_1, D_1 u_\infty\left(x_1, x'\right) + Kx_1\right) \quad \text{near } (0,0).$$

This is impossible, as $\left(x_1, D_1 u_\infty\left(x_1, x'\right) + Kx_1\right)$ is an L^1 graph (for almost all $x' \in \mathbb{R}^{n-1}$ without using the semiconvexity assumption) from Step 1.

Conflict of interest statement

There is no conflict of interest.

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