Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 36 (2019) 347–363

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Nonlinear responses from the interaction of two progressing waves at an interface

Maarten de Hoop [a], Gunther Uhlmann [b,*], Yiran Wang [c]

[a] *Computational and Applied Mathematics and Earth Science, Rice University, United States of America*
[b] *Department of Mathematics, University of Washington, Institute for Advanced Study, the Hong Kong University of Science and Technology*
[c] *Department of Mathematics, University of Washington, United States of America*

Received 26 August 2017; received in revised form 17 April 2018; accepted 26 April 2018
Available online 6 June 2018

Abstract

For scalar semilinear wave equations, we analyze the interaction of two (distorted) plane waves at an interface between media of different nonlinear properties. We show that new waves are generated from the nonlinear interactions, which might be responsible for the observed nonlinear effects in applications. Also, we show that the incident waves and the nonlinear responses determine the location of the interface and some information of the nonlinear properties of the media. In particular, for the case of a jump discontinuity at the interface, we can determine the magnitude of the jump.

Keywords: Nonlinear interaction; Progressing waves; Interface; Inverse problems

1. Introduction

Let g be a smooth Riemannian metric on $\mathbb{R}^3$. In local coordinates $x = (x^1, x^2, x^3)$, the (positive) Laplace–Beltrami operator is given by

$$\Delta_g = -\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{3} \frac{\partial}{\partial x^i} (\sqrt{\det g}\, g^{ij} \frac{\partial}{\partial x^j}).$$

We shall work with the associated wave operator

$$P = \partial_t^2 + \Delta_g.$$

However, one can consider P with lower order perturbations to which the results of this work apply as well. For example, one can consider wave operators with variable sound speed and density

* Corresponding author.
E-mail addresses: mdehoop@rice.edu (M. de Hoop), gunther@math.washington.edu (G. Uhlmann), wangy257@math.washington.edu (Y. Wang).

https://doi.org/10.1016/j.anihpc.2018.04.005

$$\widetilde{P} = \partial_t^2 - c^2(t,x)\rho(t,x)\nabla \cdot (\frac{1}{\rho(t,x)}\nabla u),$$

where $c(t,x)$ is the sound speed and $\rho(t,x)$ is the density of the medium.

Consider the following semilinear wave equation

$$\begin{aligned} Pu(t,x) + a(t,x)u^2(t,x) = 0, \quad &\text{in } (-\infty \times T) \times \mathbb{R}^3, \\ u(t,x) = u_0(t,x), \quad &\text{in } (-\infty, 0) \times \mathbb{R}^3, \end{aligned} \tag{1.1}$$

with $T > 0$. Suppose that the incident wave u_0 consists of progressing plane waves with conormal singularities to two characteristic surfaces S_1 and S_2 for P which do not intersect for $t < 0$. When a is smooth and the spatial dimension is two, the interaction of waves was studied in Bony [3], Melrose–Ritter [21] and others. In particular, as a special case of [21, Theorem 1], we know that the solution is conormal to S_1 and S_2 after the interaction and no new wave is produced. Melrose and Ritter [21, Theorem 2] showed that the interaction of three progressing waves could generate new waves. Explicit examples when the new waves are indeed produced have been constructed by various authors; see Rauch–Reed [24] and the text book by Beals [2]. For a smooth and spatial dimension three, such phenomena have also been analyzed and the newly generated waves have played an important role in the inverse problem for nonlinear hyperbolic equations in [17,18,20].

In this work, we are interested in the interactions of two progressive waves at an interface of media with difference nonlinear properties. In particular, we assume that $a(t,x)$ has conormal singularities at a co-dimension one submanifold S_0 (the interface) of $\mathbb{R}^4$ not characteristic for P. A useful example to keep in mind is $a(t,x) = a(x)$ conormal to some $Y \subset \mathbb{R}^3$ regarded as the interface. For example, $a(x)$ or its derivatives have jump discontinuities across Y. If S_1, S_2 and S_0 intersect in $t \in (0,T)$ for some $T > 0$ small, we show in Theorem 4.3 that a new wave is produced due to the nonlinear interactions; see Fig. 1 for an illustration of this interaction. In some sense, the nonlinear coefficient $a(t,x)$ plays the role of the third wave in the result mentioned above.

The main motivation of our analysis comes from the study of nonlinear interaction of waves related to conormal discontinuities ("interfaces") in the nonlinearities of the elastic moduli in sedimentary rocks. Nonlinear properties of such rocks are commonly associated with material damage. Nonlinear properties of solids have been extensively studied in the laboratory by Rollins, Taylor and Todd [25], Johnson, Shankland, O'Connell and Albricht [15], Johnson and Shankland [14], and many others. In the context of this paper, we are concerned with so-called fast nonlinear dynamics (Johnson and McCall [13]). Traditionally, the nonlinear interaction, in the absence of singularities in the nonlinearities of the elastic moduli, has been studied using monochromatic incident waves aiming to observe the generation of combined harmonics; for an early analysis, see Jones and Kobett [16]. (The experimental counterpart to our problem in some sense is the one of two incident non-collinear beams generating a new beam at their difference frequency.) This is also the underlying principle in the scalar-wave formulation – which we consider here – for vibro-acoustography [7,8] based on ultrasound-stimulated acoustic emission. However, the use of transient incident waves and the generation (emission) of a new transient wave that we analyze, here, has so far not been considered in applications and experiments.[1] Indeed, the generation of this wave opens new ways for nonlinear imaging in Earth's subsurface, which we elucidate here in the form of an inverse problem. Studying the interaction with conormal singularities in the nonlinearities of the elastic moduli was motivated by the work of Kuvshinov, Smit and Campman [19]. In a forthcoming paper, we extend the results of this paper pertaining to scalar waves to the elastic case.

We consider in Section 6 an inverse problem and we apply the results of the previous sections. We send two distorted plane waves concentrated along geodesics that meet at the interface. We observe the nonlinear response. We show that from this information we can determine the interface and the principal symbol of $a(t,x)$. In particular, in the case that $a(t,x)$ has a jump type singularity we can determine the magnitude of the jump. For a precise statement of the problem and the results see Theorem 6.1.

The paper is organized as follows. In Section 2, we review the theory for linear wave equations and construct distorted plane waves as in [17]. We establish local well-posedness of the nonlinear wave equation with a non-smooth nonlinear term in Section 3. In Section 4, we analyze the nonlinear responses after the interactions. In Section 5, we compare the linear and nonlinear responses in case the linear operator P also has conormal singularities. We

[1] P.A. Johnson, personal communication.

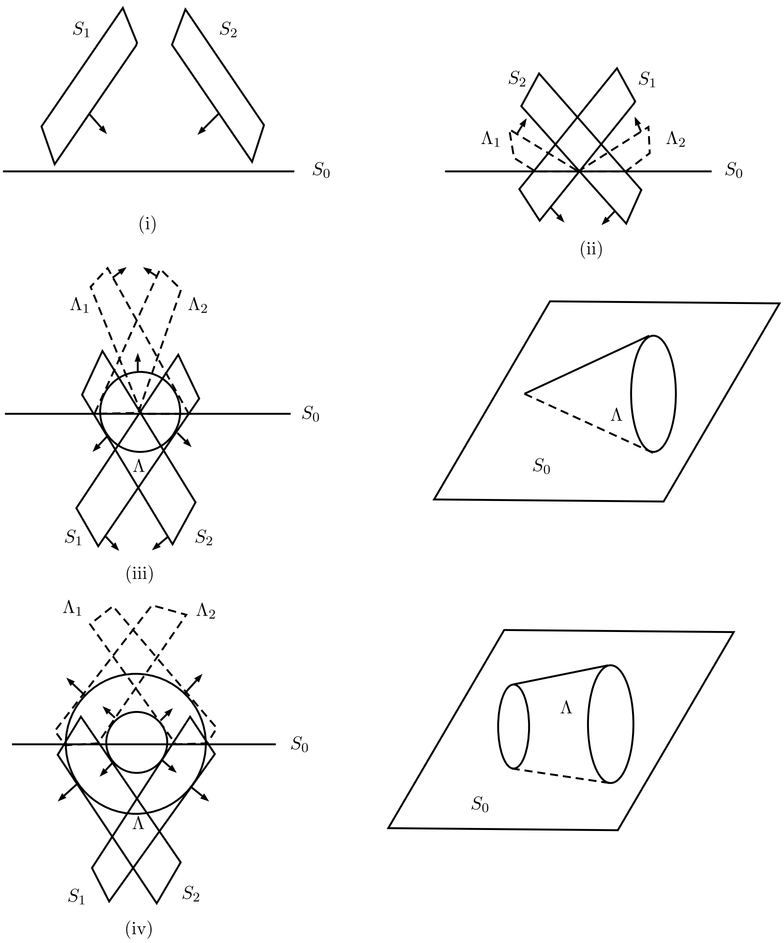

Fig. 1. Evolution of two plane waves interacting at an interface. In Figure (i), S_1, S_2 represent the wave fronts (singular supports) of two progressing waves in $\mathbb{R}^3$ and S_0 represents the singular support of $a(t,x)$. The picture shows the projective view on a plane $\mathbb{R}^2$ before the wave meets. The arrows indicate the directions of the wave propagation. Figure (ii) shows the intersection of the two waves at S_0 before they meet together. The dashed surfaces represent the reflected waves. Figure (iii) illustrates various waves during the interaction of the two waves at S_0. The wave front of the newly generated wave is demonstrated by the disk denoted by Λ. The figure to the right shows the wave front in $\mathbb{R}^3$ which is the surface of a cone. Figure (iv) shows the waves after the interaction is complete. The wave front Λ actually becomes the surface of a truncated cone in $\mathbb{R}^3$ (picture to the right).

demonstrate that the conic wave is a distinctive feature of the nonlinear response. Finally, in Section 6 we formulate and study the inverse problem.

Acknowledgement

Maarten V. de Hoop acknowledges and sincerely thanks the Simons Foundation under the MATH+X program for financial support. He was also partially supported by NSF under grant DMS-1559587 and the members of the Geo-Mathematical Imaging Group at Rice University. Gunther Uhlmann is partially supported by NSF grant DMS-1265958, a Si-Yuan Professorship at HKUST. The authors thank the anonymous referee for a careful reading of the paper and helpful suggestions.

2. The linear wave equation and distorted plane waves

We know (e.g. from [1]) that for the linear wave equation

$$Pv = (\partial_t^2 + \Delta_g)v = f,$$

there exists a fundamental solution (causal inverse) Q such that $QP = \mathrm{Id}$ on the space of distributions $\mathscr{D}'(\mathbb{R}^4)$. We review the structure of the Schwartz kernel of the causal inverse.

In the following, we use $x = (x^i)_{i=0}^3$ as the local coordinates of $\mathbb{R}^4$ with $x^0 = t$. The dual variables in the cotangent bundle are denoted by $\zeta = (\tau, \xi)$, $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}^3$. Let $p(x,\zeta) = -\tau^2 + |\xi|^2_{g^*}$ be the symbol of P, where $g^* = g^{-1} = (g^{ij})$ is the dual metric. We denote by $\Sigma_P = \{(x,\zeta) \in T^*\mathbb{R}^4 : p(x,\zeta) = 0\}$ the characteristic set for P and $\Sigma_{P,x} \doteq \Sigma_P \cap T^*_x\mathbb{R}^4$ for any $x \in \mathbb{R}^4$. The Hamilton vector field of $p(x,\zeta)$ is denoted by H_p and in local coordinates

$$H_p = \sum_{i=0}^{3}\left(\frac{\partial p}{\partial \zeta_i}\frac{\partial}{\partial x^i} - \frac{\partial p}{\partial x^i}\frac{\partial}{\partial \zeta_i}\right).$$

The integral curves of H_p in Σ_P are called null bicharacteristics. Sometimes it is convenient to view these curves on the Lorentzian manifold $(\mathbb{R}^4, \tilde{g} = -dt^2 + g)$. Then the set Σ_P consists of light-like vectors of $\tilde{g}$ and the projections of null bicharacteristics to $\mathbb{R}^4$ are light-like geodesics.

Let $\mathrm{Diag} = \{(x,x') \in \mathbb{R}^4 \times \mathbb{R}^4 : x = x'\}$ be the diagonal of the product manifold and

$$N^*\mathrm{Diag} = \{(x,\zeta,x',\zeta') \in T^*(\mathbb{R}^4 \times \mathbb{R}^4)\backslash 0 : x = x', \zeta' = -\zeta\}$$

be the conormal bundle of Diag minus the zero section. By abuse of notations, we let $\Sigma_P = \{(x,\zeta,x',\zeta') \in T^*\mathbb{R}^4 \times T^*\mathbb{R}^4 : p(x,\zeta) = p(x',\zeta') = 0\}$. Then we define Λ_P to be the Lagrangian submanifold of $T^*(\mathbb{R}^4 \times \mathbb{R}^4)$ obtained by flowing out $N^*\mathrm{Diag} \cap \Sigma_P$ under H_p. Here, we also regarded $p(z,\zeta)$ as a function on the product manifold $T^*(\mathbb{R}^4 \times \mathbb{R}^4)$.

For two Lagrangian submanifolds $\Lambda_0, \Lambda_1 \subset T^*(\mathbb{R}^4 \times \mathbb{R}^4)$ intersecting cleanly at a co-dimension k submanifold $\Omega \doteq \Lambda_0 \cap \Lambda_1$, the space of paired Lagrangian distributions associated with (Λ_0, Λ_1) is denoted by $I^{p,l}(\Lambda_0, \Lambda_1)$, see [5,22,12] for details. A useful fact is that for $u \in I^{p,l}(\Lambda_0, \Lambda_1)$, we have $u \in I^{p+l}(\Lambda_0\backslash\Omega)$ and $u \in I^p(\Lambda_1\backslash\Omega)$ as Lagrangian distributions which is recalled in the next paragraph. We know from the results of Melrose–Uhlmann [22] that the Schwartz kernel of the causal inverse $Q = P^{-1}$ is a paired Lagrangian distribution in $I^{-\frac{3}{2},-\frac{1}{2}}(N^*\mathrm{Diag}, \Lambda_P)$. From [5, Prop. 5.6], we also know that $Q : H^m_{\mathrm{comp}}(\mathbb{R}^4) \to H^{m+1}_{\mathrm{loc}}(\mathbb{R}^4)$ is continuous for $m \in \mathbb{R}$.

Let Λ be a smooth conic Lagrangian submanifold of $T^*\mathbb{R}^4\backslash 0$. Following the standard notation, we denote by $I^\mu(\Lambda)$ the space of Lagrangian distributions of order μ associated with Λ, see [11, Definition 25.1.1]. Such distributions can be represented locally as follows. For U open in X, let $\phi(x,\xi) : U \times \mathbb{R}^N \to \mathbb{R}$ be a smooth non-degenerate phase function that locally parametrizes Λ that is, $\{(x, d_x\phi) : x \in U, d_\xi\phi = 0\} \subset \Lambda$. Then $u \in I^\mu(\Lambda)$ can be locally written as a finite sum of oscillatory integrals

$$\int_{\mathbb{R}^N} e^{i\phi(x,\xi)} a(x,\xi) d\xi, \quad a \in S^{\mu + \frac{n}{4} - \frac{N}{2}}(U \times \mathbb{R}^N),$$

where $S^\bullet(\bullet)$ denotes the standard symbol class, see [10, Section 18.1]. For $u \in I^\mu(\Lambda)$, the wave front set $\mathrm{WF}(u) \subset \Lambda$ and $u \in H^s(\mathbb{R}^4)$ for any $s < -\mu - 1$. The principal symbol $\sigma(u)$ of $u \in I^\mu(\Lambda)$ is invariantly defined as section of a half-density bundle tensored with the Maslov bundle on Λ, see [11, Section 25.1]. In local coordinates, these bundles can be trivialized. We remark that we do not specify the order of the principal symbol in the notation but refer to the distribution space for the order.

A class of Lagrangian distributions especially important for our purpose is the one of conormal distributions. For a co-dimension k submanifold $Y \subset \mathbb{R}^4$, the conormal bundle

$$N^*Y = \{(y,\zeta) \in T^*\mathbb{R}^4\backslash 0 : y \in Y, \quad \zeta|_{T_yY} = 0\}$$

is a conic Lagrangian submanifold. The space of conormal distributions to Y of order μ are denoted by $I^\mu(N^*Y)$. An equivalent definition is that $I^\mu(N^*Y)$ consists of $u \in \mathscr{D}'(\mathbb{R}^4)$ such that

$$L_1 L_2 \cdots L_N u \in {}^\infty H^{\mathrm{loc}}_{-\mu-1}(\mathbb{R}^4),$$

where $L_i, i = 1, \cdots, N$ are first order differential operators with smooth coefficients tangential to Y and ${}^\infty H^{\mathrm{loc}}_\bullet(\mathbb{R}^4)$ denotes the Besov space, see [10, Definition 18.2.6] for details. Such distributions can be represented locally as oscillatory integrals as well. We know, e.g. from [12, Section 1], that $I^\mu(N^*Y) \subset L^p_{\mathrm{loc}}(\mathbb{R}^4)$ for $\mu < -\frac{k}{2} + \frac{k}{p} - 1$.

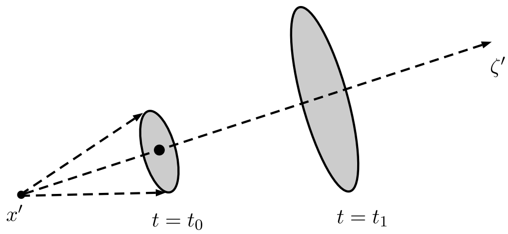

Fig. 2. Distorted plane waves in $\mathbb{R}^3$. The two shaded ovals represent the singular support of f at $t = t_0$ and of v at $t = t_1 > t_0$.

Examples of conormal distributions are the delta distribution δ_Y on Y, which is in $I^{\frac{k}{2}-1}(N^*Y)$, and a distribution with Heaviside type singularity at Y, which is in $I^{-\frac{k}{2}-1}(N^*Y)$.

We restate [12, Prop. 2.1] for the conormal case below.

Proposition 2.1. *Let Y be a submanifold of M such that N^*Y intersects Σ_P transversally and each bicharacteristics of P intersects N^*Y a finite number of times. For $f \in I^\mu(N^*Y)$, we have*

$$v = Q(f) \in I^{\mu-\frac{3}{2},-\frac{1}{2}}(N^*Y, \Lambda_1)$$

where $\Lambda_1 = \Lambda_P \circ N^*Y$ is the flow-out from $N^*Y \cap \Sigma_P$. Furthermore, for $(x, \zeta) \in \Lambda_1 \backslash N^*Y$,

$$\sigma(v)(x, \zeta) = \sum_j \sigma(Q)(x, \zeta, y_j, \eta_j)\sigma(f)(y_j, \eta_j)$$

where $(y_j, \eta_j) \in N^*Y$ is joined to (x, ζ) by bicharacteristics.

We use the above proposition to construct distorted plane waves. These are generalizations of progressing plane waves but supported near a fixed geodesic. The construction is based on that of [17]. For any $(x', \zeta') \in \Sigma_P$, we denote the bicharacteristics from (x', ζ') by $\Theta_{x',\zeta'}$. Their projections to $\mathbb{R}^4$ are denoted by $\gamma_{x',\zeta'}$, which are light-like geodesics on the Lorentzian manifold $(\mathbb{R}^4, \tilde{g})$. Here, by abuse of notations, we take ζ' to be the tangent vector at x' corresponding to $\zeta' \in T^*_{x'}\mathbb{R}^4$. This is valid because the non-degenerate metric g induces an isomorphism between $T_{x'}\mathbb{R}^4$ and $T^*_{x'}\mathbb{R}^4$. For $s_0 > 0$ a small parameter, we let

$$S(x', \zeta'; s_0) \doteq \{\gamma_{x',\zeta}(\theta) \in \mathbb{R}^4 : \zeta \in \Sigma_{P,x'}, \|\zeta - \zeta'\| < s_0, \theta > 0\},$$

where the norm is defined using the positive definite metric $\widehat{g} = dt^2 + g$ on $\mathbb{R}^4$. Notice that as $s_0 \to 0$, $S(x', \zeta'; s_0)$ tends to the geodesic $\gamma_{x',\zeta'}$. For $t_0 > 0$, we let

$$Y(x', \zeta'; t_0, s_0) \doteq S(x', \zeta'; s_0) \cap \{t = t_0\}, \tag{2.1}$$

which is a 2-dimensional surface. See Fig. 2. Then we let

$$\Lambda(x', \zeta'; t_0, s_0) \doteq \Lambda_P \circ (N^*S(x', \zeta'; s_0) \cap N^*Y(x', \zeta'; t_0, s_0)) \tag{2.2}$$

be the flow out. *For convenience, we assume that there is no conjugation point on* $(\mathbb{R}^3, g)$. We remark that since we essentially consider a local problem in this work, this is not restrictive. Then $S(x', \zeta'; s_0)$ is a co-dimension 1 submanifold near $\gamma_{x',\zeta'}$ and

$$\Lambda(x', \zeta'; t_0, s_0) = N^*S(x', \zeta'; s_0).$$

When it is clear from the background, we shall abbreviate the above notations by dropping the dependency on x', ζ', t_0, s_0. For $f \in I^\mu(N^*Y)$, using Proposition 2.1, we obtain that $v = Qf \in I^{\mu-\frac{3}{2}}(\Lambda)$ away from the submanifold Y. We conclude that v is conormal to S and we call v a *distorted plane wave*.

3. Local well-posedness of the nonlinear equation

For $T > 0$ fixed and $\epsilon > 0$ small, we consider the well-posedness of the inhomogeneous Cauchy problem

$$\begin{aligned} &Pu(t,x) + a(t,x)u^2(t,x) = \epsilon F(t,x), \quad (0,T)\times X \\ &\quad u(0,x) = \epsilon f(x), \quad \partial_t u(0,x) = \epsilon g(x). \end{aligned}$$

In this section, we use $x \in \mathbb{R}^3$ for spatial variables. There is an extensive literature on local and global well-posedness of semilinear wave equations, typically for smooth or power-type nonlinear terms, see e.g. Sogge [27]. Here, the problem is that we have a non-smooth nonlinear term. If $a(t,x)$ is sufficiently regular, e.g. in $H^3(\mathbb{R}^4)$ which is an algebra, it is relatively straightforward to prove the existence for f, g, F sufficiently regular and ϵ sufficiently small, see for example [18, Appendix B]. However, we would like to consider $a(t,x) \in L^\infty(\mathbb{R}^4)$ which includes the jump discontinuity. Then the solution is expected to be of only low regularity. We shall give a well-posedness result following the standard argument using Strichartz type estimates. We remark that we do not intend to pursue the optimal or general result here.

We recall the Strichartz estimates for the Cauchy problem from [23] valid for the wave operator on compact Riemannian manifolds without boundary. This is sufficient as we only consider the local problem. Consider the solution u to the Cauchy problem

$$\begin{aligned} &(\partial_t^2 + \Delta_g)u(t,x) = 0, \quad (0,T)\times\mathbb{R}^3 \\ &u(0,x) = f(x), \quad \partial_t u(0,x) = g(x). \end{aligned}$$

Assume that f, g are supported in a compact set $K \subset \mathbb{R}^3$. For $4 \le q < \infty$ and $2 \le r < \infty$, Corollary 3.3 of [23] tells that

$$\|u\|_{L^r((0,T);L^q(\mathbb{R}^3))} \le C_T(\|f\|_{H^\gamma(\mathbb{R}^3)} + \|g\|_{H^{\gamma-1}(\mathbb{R}^3)}), \tag{3.1}$$

with $\gamma = 3(1/2 - 1/q) - 1/r$ and C_T depending on $T > 0$. Here, the norm of the (inhomogeneous) Sobolev spaces are defined by

$$\|f\|_{H^\alpha(\mathbb{R}^3)} = (2\pi)^{-\frac{3}{2}}\Big(\int_{\mathbb{R}^3}(1+|\xi|^2)^\alpha|\hat f(\xi)|^2 d\xi\Big)^{\frac12}, \quad \alpha\in\mathbb{R},$$

where $\hat f$ denotes the Fourier transform of f. Below, we also need the homogeneous Sobolev space $\dot H^\alpha(\mathbb{R}^3)$ with norm

$$\|f\|_{\dot H^\alpha(\mathbb{R}^3)} = (2\pi)^{-\frac{3}{2}}\Big(\int_{\mathbb{R}^3}|\xi|^{2\alpha}|\hat f(\xi)|^2 d\xi\Big)^{\frac12}.$$

For our purpose, we shall take $q = r = 4$ in (3.1) so that $\gamma = \frac12$. Then we get

$$\|u\|_{L^4((0,T)\times\mathbb{R}^3)} \le C_T(\|f\|_{H^{\frac12}(\mathbb{R}^3)} + \|g\|_{H^{-\frac12}(\mathbb{R}^3)}). \tag{3.2}$$

It is known that the homogeneous Strichartz estimates imply inhomogeneous estimates from a lemma of Christ and Kiselev [4]. Consider

$$\begin{aligned} &(\partial_t^2 + \Delta_g)u(t,x) = F(t,x), \quad (0,T)\times\mathbb{R}^3 \\ &u(0,x) = f(x), \quad u_t(0,x) = g(x), \quad \text{at } t = 0 \end{aligned}$$

where f, g, F are supported in K. From [26, Theorem 3.2] and (3.2), we get

$$\|u\|_{L^4((0,T)\times\mathbb{R}^3)} \le C_T(\|f\|_{H^{\frac12}(\mathbb{R}^3)} + \|g\|_{H^{-\frac12}(\mathbb{R}^3)} + \|F\|_{L^{\frac43}((0,T)\times\mathbb{R}^3)}),$$

with C_T a generic constant depending on T. Together with the conservation of energy for linear wave equations

$$\|u(\cdot,T)\|_{\dot H^{\frac12}(\mathbb{R}^3)} + \|\partial_t u(\cdot,T)\|_{\dot H^{-\frac12}(\mathbb{R}^3)} = \|f\|_{\dot H^{\frac12}(\mathbb{R}^3)} + \|g\|_{\dot H^{-\frac12}(\mathbb{R}^3)},$$

we obtain

$$\|u\|_{L^4((0,T)\times\mathbb{R}^3)} + \|u(\cdot,T)\|_{\dot H^{\frac12}(\mathbb{R}^3)} + \|\partial_t u(\cdot,T)\|_{\dot H^{-\frac12}(\mathbb{R}^3)} \leq C_T(\|f\|_{H^{\frac12}(\mathbb{R}^3)} + \|g\|_{H^{-\frac12}(\mathbb{R}^3)} + \|F\|_{L^{\frac43}((0,T)\times\mathbb{R}^3)}). \tag{3.3}$$

Proposition 3.1. *Suppose that* $f(x)\in H^{\frac12}(\mathbb{R}^3)$, $g(x)\in H^{-\frac12}(\mathbb{R}^3)$, $F(t,x)\in L^{\frac43}((0,T)\times\mathbb{R}^3)$ *are supported in* $x\in K\subset\subset\mathbb{R}^3$. *Consider the Cauchy problem*

$$\begin{aligned} Pu(t,x)+a(t,x)u^2(t,x) &= \epsilon F(t,x), \quad (0,T)\times\mathbb{R}^3\\ u(0,x)=\epsilon f(x), &\quad \partial_t u(0,x)=\epsilon g(x), \end{aligned} \tag{3.4}$$

where $a\in L^\infty((0,T)\times\mathbb{R}^3)$, $\epsilon\geq 0$. *For* $T>0$ *fixed, there exists* $\epsilon_0>0$ *so that for* $\epsilon\in[0,\epsilon_0)$, *there is a unique solution* u *such that*

$$(u,\partial_t u)\in C^0((0,T);\dot H^{\frac12}(\mathbb{R}^3)\times\dot H^{-\frac12}(\mathbb{R}^3)) \text{ and } u\in L^4((0,T)\times\mathbb{R}^3).$$

Moreover, there exists a constant C *depending on* K,T *such that*

$$\|u\|_{L^4((0,T)\times\mathbb{R}^3)} \leq C\epsilon(\|f\|_{H^{\frac12}(\mathbb{R}^3)} + \|g\|_{H^{-\frac12}(\mathbb{R}^3)} + \|F\|_{L^{\frac43}((0,T)\times\mathbb{R}^3)}).$$

For later reference, we shall denote the solution space by

$$\mathscr{X} \doteq \{f\in L^4((0,T)\times\mathbb{R}^3) : (f,\partial_t f)\in C^0((0,T);\dot H^{\frac12}(\mathbb{R}^3)\times\dot H^{-\frac12}(\mathbb{R}^3))\}. \tag{3.5}$$

Proof. We follow a standard argument in the proof of [27, Theorem 4.1]. Consider the existence part. Let $u_{-1}=0$. We define a sequence $u_m, m=0,1,2,\cdots$ by

$$\begin{aligned} Pu_m(t,x)+a(t,x)u_{m-1}^2(t,x) &= \epsilon F(t,x), \quad (0,T)\times\mathbb{R}^3\\ u_m(0,x)=\epsilon f(x), &\quad \partial_t u_m(0,x)=\epsilon g(x). \end{aligned} \tag{3.6}$$

It follows from the finite speed of propagation that all u_m are compactly supported in $(0,T)\times\mathbb{R}^3$. Let

$$A_m(T)=\|u_m\|_{L^4((0,T)\times\mathbb{R}^3)}, \quad B_m(T)=\|u_m-u_{m-1}\|_{L^4((0,T)\times\mathbb{R}^3)}.$$

We claim that there exists $\epsilon_0>0$ so that

$$A_m(T)\leq 2A_0(T), \quad B_{m+1}(T)\leq \frac12 B_m(T) \text{ if } 2A_0(T)\leq\epsilon_0.$$

For $m,j=0,1,2,\cdots$, we obtain from (3.6) that

$$\begin{aligned} P(u_m(t,x)-u_j(t,x))+a(t,x)[u_{m-1}^2(t,x)-u_{j-1}^2(t,x)] &= 0, \quad (0,T)\times\mathbb{R}^3\\ u_m(0,x)-u_j(0,x)=0, &\quad \partial_t[u_m(0,x)-u_j(0,x)]=0. \end{aligned} \tag{3.7}$$

It follows from the Strichartz estimates (3.3) and Hölder's inequality that

$$\begin{aligned} \|u_m-u_j\|_{L^4((0,T)\times\mathbb{R}^3)} &\leq C\|u_{m-1}^2-u_{j-1}^2\|_{L^{\frac43}((0,T)\times\mathbb{R}^3)}\\ &\leq C\|u_{m-1}+u_{j-1}\|_{L^2((0,T)\times\mathbb{R}^3)}\|u_{m-1}-u_{j-1}\|_{L^4((0,T)\times\mathbb{R}^3)}\\ &\leq \frac12\|u_{m-1}-u_{j-1}\|_{L^4((0,T)\times\mathbb{R}^3)}, \end{aligned}$$

provided $C[\|u_{m-1}\|_{L^2((0,T)\times\mathbb{R}^3)}+\|u_{j-1}\|_{L^2((0,T)\times\mathbb{R}^3)}]\leq\frac12$. Hereafter, C denotes a generic constant. Suppose that the first part of the claim is true. Using the fact that u_m are compactly supported, we derive

$$\|u_m\|_{L^2((0,T)\times\mathbb{R}^3)}\leq C\|u_m\|_{L^4((0,T)\times\mathbb{R}^3)}=CA_m(T)\leq 2CA_0(T).$$

If we take $\epsilon_0=1/(4C)$, we proved that $B_m(T)\leq\frac12 B_{m-1}(T)$.

Next we prove by induction that $A_m(T)\leq 2A_0(T)$. Suppose this is true for $A_k(T), k\leq m-1$. Taking $j=0$ in (3.7), we obtain the estimate

$$\|u_m - u_0\|_{L^4((0,T)\times\mathbb{R}^3)} \le \frac{1}{2}\|u_{m-1}\|_{L^4((0,T)\times\mathbb{R}^3)} \le A_0(T). \tag{3.8}$$

It follows easily that $\|u_m\|_{L^4((0,T)\times\mathbb{R}^3)} \le 2A_0(T)$. This completes the proof of the claim.

Now we show that the sequence u_m converges to u in $L^4((0,T)\times\mathbb{R}^3)$. From the Strichartz estimates for u_0

$$\|u_0\|_{L^4((0,T)\times\mathbb{R}^3)} \le C_T\epsilon(\|f\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} + \|g\|_{H^{-\frac{1}{2}}(\mathbb{R}^3)} + \|F\|_{L^{\frac{4}{3}}((0,T)\times\mathbb{R}^3)}), \tag{3.9}$$

we can choose $\epsilon = \epsilon_0$ to satisfy the requirement in the claim. Then it follows that u_m converges to some u in L^4, hence in the sense of distribution. Next, it is straightforward to see that

$$\begin{aligned}\|au_m^2 - au_{m-1}^2\|_{L^{\frac{4}{3}}((0,T)\times\mathbb{R}^3)} &\le C\|u_m + u_{m-1}\|_{L^2((0,T)\times\mathbb{R}^3)}\|u_m - u_{m-1}\|_{L^4((0,T)\times\mathbb{R}^3)} \\ &\le C\epsilon_0\|u_m - u_{m-1}\|_{L^4((0,T)\times\mathbb{R}^3)} \le C\epsilon_0 2^{-m}.\end{aligned}$$

Thus au_m^2 converges to au^2 in $L^{\frac{4}{3}}$ hence also in the sense of distribution. Thus we proved that $u \in L^4((0,T)\times\mathbb{R}^3)$ is a weak solution to the Cauchy problem (3.4). It follows from (3.8) and (3.9) that

$$\|u_m\|_{L^4((0,T)\times\mathbb{R}^3)} \le C_T\epsilon(\|f\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} + \|g\|_{H^{-\frac{1}{2}}(\mathbb{R}^3)} + \|F\|_{L^{\frac{4}{3}}((0,T)\times\mathbb{R}^3)})$$

for all $m \ge 1$, so the estimates for $\|u\|_{L^4((0,T)\times\mathbb{R}^3)}$ follows.

For the regularity of u, observe that for $f, g \in C_0^\infty(\mathbb{R}^3)$, the u_m defined in (3.6) are all smooth and compactly supported. We can slightly modify the argument for the existence part to show that $(u_m, \partial_t u_m)$ is a Cauchy sequence in $C^0((0,T); \dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))$ converging to $(u, \partial_t u) \in C^0((0,T); \dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))$. Finally, for $f \in H^{\frac{1}{2}}(\mathbb{R}^3)$, $g \in H^{-\frac{1}{2}}(\mathbb{R}^3)$, we use approximation by compactly supported functions to conclude that the solution $u \in \mathscr{X}$.

At last, consider the uniqueness of the solution. Suppose that u, w are two solutions and let $U = u - w$. Then we have

$$\begin{aligned}PU + a(t,x)(u+w)U = 0, &\quad (0,T)\times\mathbb{R}^3 \\ U = 0, \quad \partial_t U = 0.&\end{aligned}$$

The Strichartz estimates (3.3) imply that

$$\|U\|_{L^4((0,T)\times\mathbb{R}^3)} \le C(\|u\|_{L^2} + \|v\|_{L^2})\|U\|_{L^4((0,T)\times\mathbb{R}^3)} \le C\epsilon_0\|U\|_{L^4((0,T)\times\mathbb{R}^3)}.$$

If $\|U\|_{L^4((0,T)\times\mathbb{R}^3)} \ne 0$, we reach a contradiction when ϵ_0 is sufficiently small. Thus the solution is unique in $L^4((0,T)\times\mathbb{R}^3)$. □

We also obtain the following asymptotic expansion.

Corollary 3.2. *Let $f \in I^\mu(N^*Y)$ be the (compactly supported) source function constructed in the end of Section 2. Also, take $\mu < -\frac{1}{2}$ so that $f(t,x) \in L^{\frac{4}{3}}((0,T)\times\mathbb{R}^3)$. Consider*

$$\begin{aligned}Pu(t,x) + a(t,x)u^2(t,x) = \epsilon f(t,x), &\quad (0,T)\times\mathbb{R}^3 \\ u(0,x) = 0, \quad \partial_t u(0,x) = 0, &\quad (-\infty,0)\times\mathbb{R}^3\end{aligned} \tag{3.10}$$

where $a \in L^\infty((0,T)\times\mathbb{R}^3)$, $\epsilon \ge 0$. For $T > 0$ fixed, there exists $\epsilon_0 > 0$ so that for $\epsilon \in [0,\epsilon_0)$, there is a unique solution $u \in L^4((0,T)\times\mathbb{R}^3)$ and u has the following expansion

$$u = \epsilon v + \epsilon^2 w + h,$$

where $v = Q(f) \in L^4((0,T)\times\mathbb{R}^3)$ is the distorted plane wave, $w = -Q(av^2) \in L^4((0,T)\times\mathbb{R}^3)$ and $h = O(\epsilon^3)$ in $L^4((0,T)\times\mathbb{R}^3)$.

Proof. The statements about v and w follow from the Strichartz estimates for P. Now let $h = u - \epsilon v - \epsilon^2 w$. We have that $Pv = f$ and $Pw = -av^2$ so the equation for h is

$$
\begin{aligned}
Ph &= Pu - \epsilon Pv - \epsilon^2 Pw = \epsilon f - a(h + \epsilon v + \epsilon^2 w)^2 - \epsilon f + \epsilon^2 av^2 \\
&\Longrightarrow Ph + 2(\epsilon v + \epsilon^2 w)h + ah^2 = \epsilon^3 H,
\end{aligned}
$$

where the term H is in $L^{\frac{4}{3}}((0,T) \times \mathbb{R}^3)$. Also, h satisfies the initial conditions $h(0,x) = 0$, $\partial_t h(0,x) = 0$. Now the proof is finished by following the same arguments in Proposition 3.1 with minor modifications to include the linear terms. □

4. The nonlinear responses

It is convenient to work with a more general setup which includes both the source problem and the Cauchy problem.

Setup 4.1. *Consider the semilinear wave equation*

$$
P(t,x)u + a(t,x)u^2 = 0, \quad \text{in } (0,T) \times \mathbb{R}^3. \tag{4.1}
$$

We make the following assumptions.

(1) $a \in I^{\mu_0}(N^*S_0) \cap L^\infty(\mathbb{R}^4)$ *for a co-dimension one submanifold* S_0 *of* $\mathbb{R}^4$ *non-characteristic for* P *and* a *is supported in* $t > 0$.

(2) $u = u(\epsilon; t, x) \in C^\infty((0, \epsilon_0); \mathscr{X})$ *is a smooth family of solutions to* (4.1) *and* u *possesses the following asymptotic expansion*

$$
u = \epsilon v + \epsilon^2 w + o(\epsilon^2), \tag{4.2}
$$

where the $o(\epsilon^2)$ *term is in* $L^4((0,T) \times \mathbb{R}^3)$. *We shall call* v *the* linear response *and* w *the* nonlinear response.

(3) *We assume that in* (4.2), $v = v_1 + v_2$ *where* v_i *satisfies* $Pv_i = 0$ *in* $(0,T) \times \mathbb{R}^3$ *and* $v_i \in I^{\mu_i}(N^*S_i)$, $\mu_i < -1$, $i = 1, 2$, *in which* S_i *are co-dimension one submanifolds of* $\mathbb{R}^4$ *characteristic for* P.

(4) *We assume that* S_i *intersects* S_j, $0 \le i < j \le 2$ *transversally at co-dimension* 2 *submanifolds* S_{ij}, *namely* $T_pS_i + T_pS_j = T_p\mathbb{R}^4, \forall p \in S_i \cap S_j$. *Also,* S_{12} *and* S_0 *intersect at a co-dimension* 3 *submanifold* $S_{012} \subset (0,T) \times \mathbb{R}^3$. *Roughly speaking, we assume that the singular supports of* a, v_1, v_2 *intersect at* S_{012} *in a transversal way.*

Observe that in the above setup, the interaction of two waves and the interface only appears in $t > 0$. Our main result Theorem 4.3 is to describe the singularities of w in $t > 0$. We assumed in (1) that a is supported in $t > 0$ for clarity. Then the equation (4.1) is linear for $t < 0$ and one has $u = \epsilon v$ for $t < 0$. Otherwise, the nonlinear response w may be non-trivial in $t < 0$. However, because the triple interaction appears only in $t > 0$ by (4), the singularities of w only come from the two wave interactions and one wave interacting with the interface in $t < 0$, which follow from our analysis below.

We remark that the above setup naturally arises from the source problem

$$
\begin{aligned}
Pu(t,x) + a(t,x)u^2(t,x) &= \epsilon f(t,x), \quad \text{in } (0,T) \times \mathbb{R}^3, \\
u &= 0, \quad (-\infty, 0) \times \mathbb{R}^3,
\end{aligned}
$$

with ϵ a small parameter and f constructed in Section 2 with compact support. In particular, if f is compactly supported in $(0,T_1) \subset (0,T)$, the solution u has the expansion in ϵ as shown in Corollary 3.2. So one has the setup of 4.1 for the equation in $(T_1, T_0) \times \mathbb{R}^3$. In this case, the linearized solution $v = v_1 + v_2$ where $v_i, i = 1, 2$ are distorted plane waves.

Now we write (4.2) as $u = \epsilon v + \epsilon^2 w + h$ and $h = o(\epsilon^2)$ in $L^4((0,4) \times \mathbb{R}^3)$. Using equation (4.1) and asymptotic expansion, we derive that

$$
\begin{aligned}
& P(\epsilon^2 w) = -\epsilon^2[av_1^2 + av_2^2 + 2av_1v_2] \\
\Longrightarrow\ & w = -[Q(av_1^2) + Q(av_2^2) + 2Q(av_1v_2)].
\end{aligned} \tag{4.3}
$$

We shall analyze the singularities in the nonlinear response w, which is a linear combination of

$$X_1 = Q(av_1^2), \quad X_2 = Q(av_2^2), \quad X_{12} = Q(av_1v_2).$$

We use some methods in [20] to analyze the singularities of these terms in two subsections.

4.1. Singularities in X_1, X_2

For these two terms, we claim that the waves can be split into transmitted waves and reflected waves, see Fig. 1. We start with

Lemma 4.2. *Let S be a co-dimension one submanifold of $\mathbb{R}^4$. For $v \in I^{\mu}(N^*S)$ with $\mu < -1$, we have $v^2 \in I^{2\mu+\frac{3}{2}}(N^*S)$.*

Proof. For any $p_0 \in N^*S$, we can choose local coordinates $x = (x^i)_{i=0}^3$ so that $S = \{x^0 = 0\}$ near p_0. Let $\xi = (\xi_i)_{i=0}^3$ be the dual coordinates on $T^*\mathbb{R}^3$. We have $N^*S = \{x^0 = 0, \xi_1 = \xi_2 = \xi_3 = 0\}$. Then we can write $v \in I^{\mu}(N^*S)$ near p_0 as an oscillatory integral

$$v(x) = \int_{\mathbb{R}} e^{ix^0\xi_0} a(x, \xi_0) d\xi_0$$

with $a(x, \xi_0) \in S^m(\mathbb{R}^4 \times \mathbb{R})$, $m = \mu + \frac{1}{2}$. Therefore, we get

$$v^2(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix^0\xi_0} e^{ix^0\eta_0} a(x, \xi_0) a(x, \eta_0) d\xi_0 d\eta_0 = \int_{\mathbb{R}} e^{ix^0\zeta_0} b(x, \zeta_0) d\zeta_0,$$

where $\zeta_0 = \eta_0 + \xi_0$ and

$$b(x, \zeta_0) = \int_{\mathbb{R}} a(x, \xi_0) a(x, \zeta_0 - \xi_0) d\xi_0.$$

Let $\eta = \xi_0/\langle\zeta_0\rangle$. We have

$$\partial_x^{\alpha} \partial_{\zeta_0}^{\beta} b(x, \zeta_0) = \langle\zeta_0\rangle^{2m+1-|\beta|} \sum_{\alpha_0+\alpha_1=\alpha} \int_{\mathbb{R}} \frac{\partial_x^{\alpha_0} a(x, \langle\zeta_0\rangle\eta)}{\langle\zeta_0\rangle^m} \cdot \frac{\partial_x^{\alpha_1} \partial_{\zeta_0}^{\beta} a(x, \zeta_0 - \langle\zeta_0\rangle\eta)}{\langle\zeta_0\rangle^{m-|\beta|}} d\eta.$$

Since a is a symbol of order m, we have $|\partial_x^{\alpha} \partial_{\xi_0}^{\beta} a(x, \xi_0)| \le C\langle\xi_0\rangle^{m-|\beta|}$. Hereafter, C denotes a generic constant. Thus, we estimate

$$|\partial_x^{\alpha} \partial_{\zeta_0}^{\beta} b(x, \zeta_0)| \le C\langle\zeta_0\rangle^{2m+1-|\beta|} \int_{\mathbb{R}} \frac{\langle\langle\zeta_0\rangle\eta\rangle^m}{\langle\zeta_0\rangle^m} \cdot \frac{\langle\zeta_0 - \langle\zeta_0\rangle\eta\rangle^{m-|\beta|}}{\langle\zeta_0\rangle^{m-|\beta|}} d\eta.$$

For $m < -\frac{1}{2}$, the integrand is bounded by $C\langle\eta\rangle^{2m}$ (uniformly for ζ_0) hence the integral is finite. Thus, we showed that $b(x, \zeta_0) \in S^{2m+1}(\mathbb{R}^4 \times \mathbb{R})$ which implies $v^2 \in I^{2\mu+\frac{3}{2}}(N^*S)$ for $\mu < -1$. □

In our setup, we shall take $\mu_i < -1$ and obtain $v_i^2 \in I^{2\mu_i+\frac{3}{2}}(N^*S_i)$, $i = 1, 2$ using the lemma. From standard wave front analysis, e.g. [6, Theorem 1.3.6], we obtain that av_i^2 is a well-defined distribution and

$$\text{WF}(av_i^2) \subset (N^*S_i + N^*S_0) \cup N^*S_i \cup N^*S_0 = N^*S_{0i} \cup N^*S_i \cup N^*S_0.$$

Here, we used the transversal intersection assumption to get $N^*S_i + N^*S_0 = N^*S_{0i}$. More precisely, we can apply [12, Lemma 1.1] to get

$$av_i^2 \in I^{2\mu_i+\frac{3}{2},\mu_0+1}(N^*S_{0i}, N^*S_i) + I^{\mu_0,2\mu_i+\frac{5}{2}}(N^*S_{0i}, N^*S_0). \tag{4.4}$$

We note that the orders here have different meanings to those in [12, Lemma 1.1].

Now consider X_i, $i = 1, 2$ and recall that $\mathrm{WF}(Q) \subset N^*\mathrm{Diag} \cup \Lambda_P$. Away from the intersections S_{0i}, we have

$$\mathrm{WF}(X_i) \subset (\Lambda_P \circ N^*S_{0i}) \cup (\Lambda_P \circ N^*S_0) \cup N^*S_i.$$

Here, we used the fact that S_i are characteristic for P to get $\Lambda_P \circ N^*S_i = N^*S_i$. Observe that this part of $\mathrm{WF}(X_i)$ corresponds to the transmitted wave. Next, we know that $N^*S_0 \cap \Sigma_P = \emptyset$ because S_0 is not characteristic for P. So it suffices to consider $\Lambda_i \doteq \Lambda_P \circ N^*S_{0i}$, $i = 1, 2$ and describe these Lagrangians.

For some $p \in S_{0i}$, consider the normal vectors $(1, \alpha) \in N_p^*S_i$ and $(s, \beta) \in N_p^*S_0$, where $g^*(\alpha, \alpha) = 1$ and $s^2 = g^*(\beta, \beta) \neq 1$. Consider their linear combination

$$\zeta = a(1, \alpha) + b(s, \beta) = a(1 + bs/a, \alpha + b/a\beta) \in N_p^*S_{0i}, \quad a, b \in \mathbb{R}\backslash 0.$$

Without loss of generality, we can take $a = 1$ and find b so that $\zeta \in \Sigma_P$ from solving a quadratic equation. Now for the Lorentzian metric $\tilde{g}$, we have

$$\begin{aligned}\tilde{g}^*(\zeta, (s, \beta)) &= -s(1 + bs) + g^*(\alpha + b\beta, \beta) \\ &= -s - bs^2 + g^*(\alpha, \beta) + bg^*(\beta, \beta) = \tilde{g}^*((1, \alpha), (s, \beta))\end{aligned}$$

Thus the vector ζ corresponds to the reflected directions after the interaction at S_0. Finally, we conclude that $\mathrm{WF}(X_i) \subset \Lambda_i \cup N^*S_i$, $i = 1, 2$, with the transmitted waves on N^*S_i and reflected waves on Λ_i.

Away from N^*S_0 and N^*S_i, we obtain from (4.4) that $av_i^2 \in I^{\mu_0 + 2\mu_i + \frac{5}{2}}(N^*S_{012})$. Therefore, using [12, Prop. 2.1] and wave front analysis, we know that away from $N^*S_0 \cup N^*S_i$,

$$X_i = Q(av_i^2) \in I^{\mu_0 + 2\mu_i + 1, -\frac{1}{2}}(N^*S_{012}, \Lambda_i).$$

Thus $X_i \in I^{\mu_0 + 2\mu_i + 1}(\Lambda_i)$ away from $N^*S_0 \cup N^*S_i \cup N^*S_{0i}$ and this is the reflected wave in the nonlinear responses.

4.2. Singularities in X_{12}

The singularities in X_{12} are analyzed in [17] and [20] when S_0 is also characteristic for P. In particular, a conic type singularity is generated. We adapt the analysis to S_0 not characteristic for P. We start with a wave front analysis to locate the singularities of X_{12}.

For $v_i \in I^{\mu_i}(N^*S_i)$, $i = 1, 2$, we can apply [12, Lemma 1.1] to get

$$v_1 v_2 \in I^{\mu_1, \mu_2 + 1}(N^*S_{12}, N^*S_1) + I^{\mu_2, \mu_1 + 1}(N^*S_{12}, N^*S_2).$$

By standard wave front analysis, we know that

$$\mathrm{WF}(av_1 v_2) \subset N^*S_1 \cup N^*S_2 \cup N^*S_{12} \cup N^*S_0 \cup N^*S_{01} \cup N^*S_{02} \cup N^*S_{012},$$

where we used $N^*S_{12} + N^*S_0 = N^*S_{012}$ as a consequence of the transversal intersection assumptions. Now consider $\mathrm{WF}(X_{12})$. We already know that $\Lambda_P \circ N^*S_{0i} = \Lambda_i \cup N^*S_i$. Since S_i are characteristic for P, the normal vectors in N^*S_i are light-like vectors for $\tilde{g}$. As S_1, S_2 intersect transversally, it is a fact that the linear combination of two light-like vectors do not give new light like vectors that is, $N^*S_{12} \cap \Sigma_P = N^*S_1 \cup N^*S_2$. Thus it remains to consider $\Lambda \doteq \Lambda_P \circ N^*S_{012}$.

We claim that S_{012} must be a space-like curve, namely the tangent vectors to S_{012} are space-like for $\tilde{g}$. Consider tangent vectors (a, θ), $a \in \mathbb{R}$, $\theta \in \mathbb{R}^3$ to S_{012} at p. If $a = 0$, the vector is space-like. Otherwise, one can rescale the vector so it suffices to consider $(1, \theta)$, $\theta \in \mathbb{R}^3$. Observe that light-like vectors $(1, \alpha) \in N_p^*S_1$, $(1, \beta) \in N_p^*S_2$, $g^*(\alpha, \alpha) = g^*(\beta, \beta) = 1$ should be normal to S_{012}. So we get

$$-1 + g(\alpha, \theta) = 0$$

where α becomes the corresponding tangent vector in $T_p\mathbb{R}^4$. Since $g(\alpha, \alpha) = 1$, we conclude that $g(\theta, \theta) \geq 1$ so that either $(1, \theta)$ is space-like or $\theta = \alpha$. The latter is impossible because the same argument tells $\theta = \beta$ but α, β are linearly independent. So we conclude that $g(\theta, \theta) > 1$ so $(1, \theta)$ is space-like. Now let's consider all light-like vectors $(1, \eta)$, $g(\eta, \eta) = 1$ at p normal to $(1, \theta)$. They should satisfy $-1 + g(\eta, \theta) = 0$. Because $g(\theta, \theta) > 1$, we conclude that the solution set of η is a one-dimensional subset of $T_p\mathbb{R}^4$. Hence $N^*S_{012} \cap \Sigma_P$ at p is a two dimensional subset and $\Lambda\backslash(N^*S_1 \cup N^*S_2)$ is non-empty. Away from the intersections S_{01}, S_{02}, S_{12} and S_{012}, we have

$$\mathrm{WF}(X_{12}) \subset N^*S_1 \cup N^*S_2 \cup \Lambda_1 \cup \Lambda_2 \cup \Lambda.$$

We summarize the results above and prove the main result of the paper.

Theorem 4.3. *Consider the problem described in Setup 4.1. We have the following conclusions for the nonlinear response w away from the intersection sets S_{01}, S_{02}, S_{12} and S_{012}*

(1) *$WF(w) \subset \Lambda_1 \cup \Lambda_2 \cup N^*S_1 \cup N^*S_2 \cup \Lambda$.*
(2) *Away from $\Lambda_1 \cup \Lambda_2 \cup N^*S_1 \cup N^*S_2$, $w \in I^\mu(\Lambda)$ with $\mu = \sum_{i=0}^2 \mu_i + \frac{1}{2}$.*
(3) *$\Lambda \cap WF(w) \neq \emptyset$ if the principal symbols $\sigma(v_i)$ and $\sigma(a)$ are non-vanishing at S_{012}.*

Proof. (1). The statement summarizes the results we obtained above.

(2) and (3). It remains to show $w \in I^\mu(\Lambda)$, in particular, to show that $Q(av_1v_2) \in I^\mu(\Lambda)$ because X_1, X_2 terms are smooth near Λ.

By our assumptions on the intersections of $S_i, i = 0, 1, 2$, for any $p \in S_{012}$, we can find local coordinates $x = (x^i)_{i=0}^3$ such that $S_i = \{x^i = 0\}$ and $S_{012} = \{x^0 = x^1 = x^2 = 0\}$. We use $\zeta = (\zeta_i)_{i=0}^3$ as the dual variables to x. Then we can express for example $N^*S_0 = \{x^0 = 0, \zeta_1 = \zeta_2 = \zeta_3 = 0\}$ and $N^*S_{012} = \{x^0 = x^1 = x^2 = 0, \zeta_3 = 0\}$. In this local coordinates, we can write down the conormal distributions as

$$v_1(x) = \int_{\mathbb{R}} e^{ix^1\zeta_1} b_1(x, \zeta_1) d\zeta_1, \quad v_2(x) = \int_{\mathbb{R}} e^{ix^2\zeta_2} b_2(x, \zeta_2) d\zeta_2,$$

$$a(x) = \int_{\mathbb{R}} e^{ix^0\zeta_0} b_0(x, \zeta_0) d\zeta_0,$$

where $b_i \in S^{\mu_i + \frac{1}{2}}(\mathbb{R}^4 \times \mathbb{R}), i = 0, 1, 2$ are standard symbols. Then we have the multiplication

$$a(x)v_1(x)v_2(x) = \int_{\mathbb{R}^3} e^{i(x^0\zeta_0 + x^1\zeta_1 + x^2\zeta_2)} b_0(x, \zeta_0) b_1(x, \zeta_1) b_2(x, \zeta_2) d\zeta_0 d\zeta_1 d\zeta_2.$$

We denote $c(x, \tilde{\zeta}) = b_0(x, \zeta_0) b_1(x, \zeta_1) b_2(x, \zeta_2)$ with $\tilde{\zeta} = (\zeta_0, \zeta_1, \zeta_2) \in \mathbb{R}^3$.

Now we let $\phi(t), t \geq 0$ be a smooth cut-off function such that $\phi(t) = 1$ for $t \geq 1$ and $\phi(t) = 0$ for $t < \frac{1}{2}$. For $\delta > 0$, we define

$$\chi_\delta(\tilde{\zeta}) = \prod_{i=0}^{2} \phi\Big(\frac{|\zeta_i|}{\delta|\tilde{\zeta}|}\Big).$$

Then χ_δ is supported on $\{\tilde{\zeta} \in \mathbb{R}^3 : \delta|\tilde{\zeta}| \leq 2|\zeta_i|, i = 0, 1, 2\}$. We conclude that $\chi_\delta c$ is a symbol because

$$\begin{aligned} |\partial_x^\alpha \partial_{\zeta_0}^{\beta_0} \partial_{\zeta_1}^{\beta_1} \partial_{\zeta_2}^{\beta_2} (\chi_\delta(\tilde{\zeta}) c(x, \tilde{\zeta}))| &\leq C_{\chi,\delta} (1 + |\zeta_0|)^{\mu_0 + \frac{1}{2} - \beta_0} (1 + |\zeta_1|)^{\mu_1 + \frac{1}{2} - \beta_1} (1 + |\zeta_2|)^{\mu_2 + \frac{1}{2} - \beta_2} \\ &\leq C_{\chi,\delta} (1 + |\tilde{\zeta}|)^{\mu_0 + \mu_1 + \mu_2 + \frac{3}{2} - |\beta|} \end{aligned}$$

where we used $\mu_i < -1, i = 1, 2$ and also $\mu_0 < -1$ because a in particular belongs to $L^p_{\mathrm{loc}}(\mathbb{R}^4)$ for all $p > 0$. We split av_1v_2 as

$$\begin{aligned} a(x)v_1(x)v_2(x) &= \int_{\mathbb{R}^3} e^{i(x^0\zeta_0 + x^1\zeta_1 + x^2\zeta_2)} \chi_\delta(\tilde{\zeta}) c(x, \tilde{\zeta}) d\zeta_0 d\zeta_1 d\zeta_2 \\ &+ \int_{\mathbb{R}^3} e^{i(x^0\zeta_0 + x^1\zeta_1 + x^2\zeta_2)} (1 - \chi_\delta(\tilde{\zeta})) c(x, \tilde{\zeta}) d\zeta_0 d\zeta_1 d\zeta_2 \doteq U_1 + U_2. \end{aligned} \tag{4.5}$$

Thus near S_{012} and for any $\delta > 0$, $U_1 \in I^\mu(N^*S_{012})$ with $\mu = \sum_{i=0}^2 \mu_i + 2$ and U_2 is a distribution with $\mathrm{WF}(U_2)$ contained in a δ neighborhood of $N^*S_1 \cup N^*S_2 \cup N^*S_0 \cup N^*S_{12} \cup N^*S_{01} \cup N^*S_{02}$. It is clear from the expression that the symbol of U_1 is non-vanishing if $b_i, i = 0, 1, 2$ are non-vanishing. Finally, $w = Q(av_1v_2) = Q(U_1) + Q(U_2)$. By Proposition 2.1, we know that $Q(U_1) \in I^{\mu-\frac{3}{2}}(\Lambda)$ away from N^*S_{012} and the symbol is non-vanishing on Λ. For the other piece, we know that $\mathrm{WF}(Q(U_2))$ is contained in a small neighborhood of $\Lambda_1 \cup \Lambda_2 \cup N^*S_1 \cup N^*S_2$ and $N^*S_{01} \cup N^*S_{02} \cup N^*S_{12} \cup N^*S_{012}$. This finishes the proof. □

From the two subsections, we know that the nonlinear responses consist of reflected waves $X_i \in I^{2\mu_i+\mu_0+1}(\Lambda_i), i = 1, 2$ and the new wave $X_{12} = I^{\mu_1+\mu_2+\mu_0+\frac{1}{2}}(\Lambda)$. These can be distinguished in terms of the order of Lagrangian distributions when $\mu_1 - \mu_2 \neq \pm\frac{1}{2}$.

5. Linear responses versus nonlinear responses

For equation (4.1), we have analyzed the singularities in the asymptotic expansion terms in (4.2). Comparing the wave front sets of the linear response v and the nonlinear response w, we find that the differences are the reflected waves on $\Lambda_i, i = 1, 2$ and the conic wave on Λ. In this section, we demonstrate that if the linear properties of the materials are also different across S_0, the linear response may also contain reflected waves, hence the nonlinear responses on Λ_i are potentially indistinguishable. For this reason, it is reasonable to think of the new conic wave at Λ as the observable nonlinear effect.

We continue using the notations in Section 4. We consider a perturbation problem of (4.1)

$$\begin{aligned} Pu(t,x) + \delta q(t,x)u(t,x) + a(t,x)u^2(t,x) = 0, &\quad \text{in } (0,T)\times\mathbb{R}^3, \\ u(t,x) = \epsilon(u_1(t,x) + u_2(t,x)), &\quad t < 0, \end{aligned} \tag{5.1}$$

where $\epsilon, \delta > 0$ are two small parameters. For ease of elaboration, we lower the regularity requirements as follows. We assume that $q, a \in I^{\mu_0}(N^*S_0)$ are compactly supported in $t > 0$ with $\mu_0 < -3$ so that $q, a \in H^s(\mathbb{R}^4), s = -\mu_0 - 1 > 2$ which is an algebra. We also assume that the incoming waves $u_i \in I^{\mu_i}(N^*S_i), \mu_i < -3$ and $Pu_i = 0$. Thus $u_i \in H^s(\mathbb{R}^4)$ as well.

We remark that the potential q depending on another small parameter simplifies our argument because it allows us to analyze the singularities in the leading term instead of the full solution. In the linear setting when the metric g has a conormal singularity across a submanifold so that the coefficient of Δ_g has conormal singularities, de Hoop, Uhlmann and Vasy studied the transmitted and reflected waves carefully in [5]. Also, in the backscattering setting when the potential has a conormal singularity, a similar problem is studied by Greenleaf and Uhlmann [12]. However, both papers require quite complicated analysis to clarify the singularities in the full solution.

Under our regularity assumptions, the local well-posedness of equation (5.1) is essentially known, see e.g. [18, Appendix B]. In particular, for ϵ sufficiently small, there is a unique solution $u \in H^s_{\mathrm{loc}}((0,T)\times\mathbb{R}^3)$. We also have $u = \epsilon v + \epsilon^2 w + o(\epsilon^2)$ where the $o(\epsilon^2)$ term is small in H^s. Moreover, since the potential depends on δ, v and w actually have expansions in δ as well. Our goal is to analyze the wave front sets of the asymptotic terms of v, w.

Proposition 5.1. *Consider equation (5.1) with the above assumptions. For $\delta, \epsilon > 0$ sufficiently small, there is a unique solution $u \in H^s_{\mathrm{loc}}((0,T)\times\mathbb{R}^3)$ which can be written as*

$$u = \epsilon(u_1 + u_2 + \delta V) + \epsilon^2\delta W + O(\epsilon\delta^2) + O(\epsilon^3),$$

where the remainder terms are in $H^s_{\mathrm{loc}}((0,T)\times\mathbb{R}^3)$. Moreover, away from the sets S_0, S_{12}, we have

(1) *$WF(u_1 + u_2 + \delta V) \subset N^*S_1 \cup \Lambda_1 \cup N^*S_2 \cup \Lambda_2$.*
(2) *$WF(W) \subset N^*S_1 \cup \Lambda_1 \cup N^*S_2 \cup \Lambda_2 \cup \Lambda$.*

Proof. Since v satisfies the linearized equation, we can write $v = v_1 + v_2$ so that

$$\begin{aligned} Pv_i(t,x) + \delta q(t,x)v_i(t,x) = 0, &\quad \text{in } (0,T)\times\mathbb{R}^3, \\ v_i(t,x) = u_i(t,x), &\quad t < 0. \end{aligned}$$

It suffices to analyze the singularities of v_1. Let $\bar{v} = v_1 - u_1$. We get

$$P\bar{v}(t,x) + \delta q(t,x)\bar{v}(t,x) = -q(t,x)u_1(t,x), \quad \text{in } (0,T)\times\mathbb{R}^3,$$
$$\bar{v}(t,x) = 0, \quad t < 0.$$

Using the causal inverse $Q = P^{-1}$, we get $\bar{v} + Q(\delta q\bar{v}) = -Q(\delta q u_1)$, from which we derive

$$\bar{v} = \sum_{n=0}^{\infty}(-1)^n\delta^n(QM_q)^n Q(-\delta q u_1), \tag{5.2}$$

where M_q denotes the operator of multiplication by q. Since $q \in H^s(\mathbb{R}^4), s > 2$, we know that $M_q : H^k(\mathbb{R}^4) \to H^k(\mathbb{R}^4)$ is continuous for $0 \le k \le s$, see e.g. [9, Section 3.2]. We recall that $u_1 \in H^s(\mathbb{R}^4)$ and $Q : H^s_{\text{comp}}(\mathbb{R}^4) \to H^{s+1}_{\text{loc}}(\mathbb{R}^4)$ is continuous. From the finite speed of propagation (for the linearized equation), we know that each term of (5.2) is supported in a compact set of $(0,T)\times\mathbb{R}^3$. For δ sufficiently small, we obtain that the series (5.2) converges in $H^{s+1}(\mathbb{R}^4)$, and

$$v_1 = u_1 - \delta Q(qu_1) + O(\delta^2),$$

where the remainder term is in $H^{s+1}(\mathbb{R}^4)$.

Now we find the singularities in $Q(qu_1)$. Since $q \in I^{\mu_0}(N^*S_0)$, $u_1 \in I^{\mu_1}(N^*S_1)$ and S_0 intersects S_1 transversally, we use [12, Lemma 1.1] to get

$$qu_1 \in I^{\mu_1,\mu_0+1}(N^*S_{01}, N^*S_0) + I^{\mu_0,\mu_1+1}(N^*S_{01}, N^*S_1).$$

More precisely, we can write $qu_1 = \Phi_1 + \Phi_2$ so that $\Phi_1 \in I^{\mu_1,\mu_0+1}(N^*S_{01}, N^*S_0)$ microlocally supported away from N^*S_1 and $\Phi_2 \in I^{\mu_0,\mu_1+1}(N^*S_{01}, N^*S_1)$ microlocally supported away from N^*S_0. Now consider the action of Q on qu_1. Using [12, Proposition 2.1, 2.2] we obtain that

$$Q(\Phi_2) \in I^{\mu_0-1,\mu_1}(N^*S_{01}, N^*S_1) + I^{\mu_0+\mu_1-\frac{1}{2},-\frac{1}{2}}(N^*S_{01}, \Lambda_1).$$

On the other hand, Q acts on Φ_1 as a pseudo-differential operator of order -2 so that $Q(\Phi_1) \in I^{\mu_1-2,\mu_0+1}(N^*S_{01}, N^*S_0)$. We conclude that the wave front set of $Q(qu_1)$ is contained in $N^*S_{01}\cup N^*S_0\cup N^*S_1\cup\Lambda_1$. The analysis for v_2 is the same. So we conclude that

$$v = u_1 + u_2 + \delta V + O(\delta^2)$$

where the wave front set $\mathrm{WF}(V) \subset N^*S_{01}\cup N^*S_0\cup N^*S_1\cup\Lambda_1\cup N^*S_{02}\cup N^*S_2\cup\Lambda_2$. Therefore, the linear responses contain reflected and transmitted waves.

Next, we follow the same lines to analyze the nonlinear response w which satisfies the equation

$$Pw(t,x) + \delta q(t,x)w(t,x) = -a(t,x)v^2(t,x), \quad \text{in } (0,T)\times\mathbb{R}^3,$$
$$w(t,x) = 0, \quad t < 0.$$

Since $v \in H^s(\mathbb{R}^4)$ and $a \in H^s(\mathbb{R}^4)$, we know that $av^2 \in H^s(\mathbb{R}^4)$ is well-defined. Similarly, we obtain that

$$w = \sum_{n=0}^{\infty}(-1)^n\delta^n(QM_q)^n Q(-\delta a v^2)$$

which converges in $H^{s+1}(\mathbb{R}^4)$ for δ sufficiently small. So we have

$$w = \delta W + O(\delta^2), \quad W = -Q(a(u_1+u_2)^2).$$

From wave front analysis as in Section 4, we know that $\mathrm{WF}(W)$ is contained in

$$N^*S_{012}\cup N^*S_{01}\cup N^*S_{02}\cup N^*S_{12}\cup\Lambda_1\cup\Lambda_2\cup N^*S_1\cup N^*S_2\cup\Lambda.$$

This completes the proof of the proposition. □

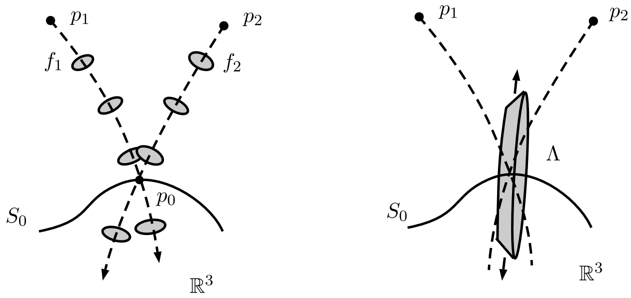

Fig. 3. Illustration of Theorem 6.1. The left picture shows the setup of the theorem. The two top ovals represent the singular supports of the sources f_1, f_2 and the rest represent the singular supports of distorted plane waves v_1, v_2. Similar to Fig. 2, they propagate and concentrate along the geodesics from p_1, p_2 (dashed curves). The right picture shows the nonlinear response on Λ after the nonlinear interactions at p_0, ignoring the transmitted and reflected waves.

6. The inverse problem

As an application of our main results, we address the inverse problem of determining the location of S_0 and the principal symbol of $a(t,x)$ using the nonlinear response. We consider a source problem using the construction in Section 2.

We take two points $(p_i, \zeta_i) \in \Sigma_P, i = 1, 2$ such that the corresponding geodesics γ_{p_i,ζ_i} for the Lorentzian metric $\tilde{g} = -dt^2 + g$ intersect at $p_0 \in \mathbb{R}^4$. See the left picture of Fig. 3. For $s_0, t_0 > 0$, let $f_i \in I^{\mu_i+\frac{3}{2}}(N^*Y_i(p_i, \zeta_i, s_0, t_0))$ and $v_i \in I^{\mu_i}(N^*S_i(p_i, \zeta_i, s_0)), i = 1, 2$ be constructed as in Section 2. Let S_0 be a co-dimension one submanifold of $\mathbb{R}^4$ not characteristic for P, and $a \in I^{\mu_0}(N^*S_0) \cap L^\infty(\mathbb{R}^4)$. As in Section 4, we suppose that S_0, S_1, S_2 intersect in a transversal way when they intersect. We use the notations Λ_1, Λ_2 in Section 4 to denote the Lagrangian submanifolds carrying the reflected waves. Their projections to $\mathbb{R}^4$ are denoted by $\widehat{S}_1, \widehat{S}_2$ respectively. We denote $\mathscr{S} \doteq (\bigcup_{i=0}^2 S_i) \cup \widehat{S}_1 \cup \widehat{S}_2$. In particular, we know that this set contains the singular supports of the reflected and transmitted waves in the nonlinear response.

For fixed $T > 0$ and $\epsilon_1, \epsilon_2 \in (0, \epsilon_0)$, we consider the following source problem

$$\begin{aligned} Pu(t,x) + a(t,x)u^2(t,x) &= \epsilon_1 f_1 + \epsilon_2 f_2, \quad \text{in } (-\infty, T) \times \mathbb{R}^3, \\ u(t,x) &= 0, \quad \text{in } (-\infty, 0) \times \mathbb{R}^3. \end{aligned} \tag{6.1}$$

We assume that the exponents $\mu_i, i = 0, 1, 2$ and ϵ_0 are chosen such that the well-posedness result Theorem 3.1 holds for (6.1). The data set we use for the inverse problem is

$$\mathscr{D}_a(f_1, f_2) \doteq \{u(\epsilon_1, \epsilon_2) : u(\epsilon_1, \epsilon_2) \in \mathscr{X} \text{ is the unique solution to (6.1) for } \epsilon_1, \epsilon_2 \in (0, \epsilon_0)\}.$$

We remark that the data set depends on the choice of (p_i, ζ_i) and $f_i, i = 1, 2$. However, once they are chosen, the data set is a two parameter family of solutions to (6.1).

Theorem 6.1. *Suppose that the principal symbols $\sigma(f_i) \neq 0, i = 1, 2$ on γ_{p_i,ζ_i}, respectively. Under the above assumptions, we have*

(1) *$p_0 \in S_0$ if and only if $\partial_{\epsilon_1}\partial_{\epsilon_2} u(\epsilon_1, \epsilon_2)|_{\epsilon_1=\epsilon_2=0}$ is not smooth away from $\mathscr{S}$ for all s_0 small.*
(2) *If $p_0 \in S_0$, the principal symbol $\sigma(a)$ at p_0 is uniquely determined by $\mathscr{D}_a(f_1, f_2)$. More precisely, suppose $u^{(i)}$ are solutions to (6.1) with $a^{(i)} \in I^{\mu_0^{(i)}}(N^*S_0), i = 1, 2$. If $u^{(1)}(\epsilon_1, \epsilon_2) = u^{(2)}(\epsilon_1, \epsilon_2)$ on Λ, then the orders $\mu_0^{(1)} = \mu_0^{(2)}$ and the principal symbols $\sigma(a^{(1)}) = \sigma(a^{(2)})$ at $(p_0, \xi_0) \in N^*S_0$.*

Proof. (1). We observed from the remark after Assumptions 4.1 that the source problem (6.1) can be reduced to the setup of Theorem 4.3. Following the successive approximation in Section 4, we obtain that

$$\partial_{\epsilon_1}\partial_{\epsilon_2} u(\epsilon_1, \epsilon_2)|_{\epsilon_1=\epsilon_2=0} = -2Q(a(x)v_1(x)v_2(x)).$$

So the conclusion follows from Theorem 4.3 when S_0, S_1, S_2 intersect at p_0. If they do not intersect, the wave front analysis in Section 4 shows that $\mathrm{WF}(Q(a(x)v_1(x)v_2(x)))$ is contained in $(\bigcup_{i=0}^2 N^*S_i) \cup N^*S_{12} \cup \Lambda_1 \cup \Lambda_2$ hence the term is smooth away from the set $\mathscr{S}$.

(2). If $\sigma(f_j) \neq 0, j = 1, 2$ on γ_{p_j,ζ_j}, we know from Proposition 2.1 that $\sigma(v_j) \neq 0$ at $(p_0, \xi_j) \in \Sigma_P$. Also, if $u^{(1)}(\epsilon_1, \epsilon_2) = u^{(2)}(\epsilon_1, \epsilon_2)$ on Λ, we know from Theorem 4.3 that $\mathscr{U}^{(i)} \doteq \partial_{\epsilon_1}\partial_{\epsilon_2} u^{(i)}(\epsilon_1, \epsilon_2)|_{\epsilon_1=\epsilon_2=0}, i = 1, 2$ are Lagrangian distributions of the same order on Λ away from $\Lambda_1 \cup \Lambda_2 \cup N^*S_{12} \cup (\bigcup_{i=0}^2 N^*S_i)$ with the same principal symbols at $(x, \zeta) \in \Lambda$. By Proposition 2.1, we know that the principal symbols of $\mathscr{U}^{(i)}$ at $(p_0, \xi) \in \Sigma_P$ are the same because the matrix $\sigma(Q)(x, \zeta, p_0, \xi)$ is invertible. In the proof of Theorem 4.3, we can read the order and the principal symbols of $\mathscr{U}^{(i)}$ at (p_0, ξ) in terms of the principal symbols of a, v_1, v_2 at $(p_0, \xi_0), (p_0, \xi_1), (p_0, \xi_2)$ respectively with $\xi = \sum_{i=0}^2 \xi_i$, see equation (4.5). This implies that the order $\mu_0^{(1)} = \mu_0^{(2)}$ and the principal symbols $\sigma(a^{(1)})(p_0, \xi_0) = \sigma(a^{(2)})(p_0, \xi_0)$. □

The nonlinear term can be determined in a special case of piecewise constant functions. The corollary below follows from Theorem 6.1 directly.

Corollary 6.2. *In addition to the assumptions in Theorem 6.1, we assume that Ω is a simply connected, bounded open subset of $\mathbb{R}^3$ such that $\partial\Omega$ is a co-dimension one submanifold of $\mathbb{R}^3$. Let $S_0 = \mathbb{R} \times \partial\Omega$ and $a(t, x) \doteq \alpha\chi_\Omega(x), \alpha \in \mathbb{R}$, which is conormal to S_0 and in $L^\infty(\mathbb{R}^4)$. If $p_0 \in S_0$, then α is uniquely determined by $\mathscr{D}_a(f_1, f_2)$.*

Conflict of interest statement

There is no conflict of interest.

References

[1] C. Bär, N. Ginoux, F. Pfäffle, Wave Equations on Lorentzian Manifolds and Quantization, ESI Lect. Math. Phys., European Mathematical Society (EMS), Zürich, 2007.
[2] M. Beals, Propagation and Interaction of Singularities in Nonlinear Hyperbolic Problems, Prog. Nonlinear Differ. Equ. Appl., vol. 3, Birkhäuser, Boston, 1989.
[3] J.M. Bony, Propagation et interaction des singularités pour les solutions des équations au dérivées paritelles non-linéaires, in: Proc. of the Internations Congress of Mathematicians, 1983.
[4] M. Christ, A. Kiselev, Maximal functions associated to filtrations, J. Funct. Anal. 179 (2) (2001) 409–425.
[5] M. de Hoop, G. Uhlmann, A. Vasy, Diffraction from conormal singularities, Ann. Sci. Écol. Norm. Supér., 4e ser. 48 (2015) 351–408.
[6] J.J. Duistermaat, Fourier Integral Operators, Prog. Math., vol. 130, Birkhäuser, Boston, 1996.
[7] M. Fatemi, J.F. Greenleaf, Ultrasound-stimulated vibro-acoustic spectrography, Science 280 (5360) (1998) 82–85.
[8] M. Fatemi, J.F. Greenleaf, Vibro-acoustography: an imaging modality based on ultrasound-stimulated acoustic emission, Proc. Natl. Acad. Sci. 96 (12) (1999) 6603–6608.
[9] T. Hughes, T. Kato, J. Marsden, Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity, Arch. Ration. Mech. Anal. 63 (3) (1977) 273–294.
[10] L. Hörmander, The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators, Class. Math., Springer, Berlin, 2007.
[11] L. Hörmander, The Analysis of Linear Partial Differential Operators IV: Fourier Integral Operators, Class. Math., Springer-Verlag, Berlin, 2009.
[12] A. Greenleaf, G. Uhlmann, Recovering singularities of a potential from singularities of scattering data, Commun. Math. Phys. 157 (3) (1993) 549–572.
[13] P.A. Johnson, K.R. McCall, Observation and implications of nonlinear elastic wave response in rock, Geophys. Res. Lett. (1994), https://doi.org/10.1029/93GL03162.
[14] P.A. Johnson, T.J. Shankland, Nonlinear generation of elastic waves in granite and sandstone: continuous wave and travel time observations, J. Geophys. Res. 94B (12) (1989) 17729–17733.
[15] P.A. Johnson, T.J. Shankland, R.J. O'Connell, J.N. Albricht, Nonlinear generation of elastic waves in crystalline rock, J. Geophys. Res. 92B (5) (1987) 3597–3602.
[16] G. Jones, D.R. Kobett, Interaction of elastic waves in an isotropic solid, J. Acoust. Soc. Am. 35 (1) (1963) 5–10.
[17] Y. Kurylev, M. Lassas, G. Uhlmann, Inverse problems for Lorentzian manifolds and non-linear hyperbolic equations, Invent. Math. 212 (3) (2018) 781–857.
[18] Y. Kurylev, M. Lassas, G. Uhlmann, Inverse problems in spacetime I: Inverse problems for Einstein equations—extended preprint version, arXiv:1405.4503, 2014.
[19] B.N. Kuvshinov, T.J.H. Smit, X.H. Campman, Non-linear interaction of elastic waves in rocks, Geophys. J. Int. 194 (3) (2013) 1920–1940.

[20] M. Lassas, G. Uhlmann, Y. Wang, Inverse problems for semilinear wave equations on Lorentzian manifolds, Commun. Math. Phys. 360 (2) (2018) 555–609, https://doi.org/10.1007/s00220-018-3135-7, arXiv:1606.0626.

[21] R. Melrose, N. Ritter, Interaction of nonlinear progressing waves for semilinear wave equations, Ann. Math. 121 (1) (1985) 187–213.

[22] R. Melrose, G. Uhlmann, Lagrangian intersection and the Cauchy problem, Commun. Pure Appl. Math. 32 (4) (1979) 483–519.

[23] G. Mockenhaupt, A. Seeger, C. Sogge, Local smoothing of Fourier integral operators and Carleson–Sjölin estimates, J. Am. Math. Soc. 6 (1) (1993) 65–130.

[24] J. Rauch, M. Reed, Singularities produced by the nonlinear interaction of three progressing waves, examples, Commun. Partial Differ. Equ. 7 (9) (1982) 1117–1133.

[25] F.R. Rollins, L.H. Taylor, P.H. Todd, Ultrasonic study of three phonon interactions. II. Experimental results, Phys. Rev. 136 (3A) (1964) A597–A601.

[26] H. Smith, C. Sogge, Global Strichartz estimates for nontrapping perturbations of the Laplacian, Commun. Partial Differ. Equ. 25 (11–12) (2000) 2171–2183.

[27] C. Sogge, Lectures on Non-Linear Wave Equations, Monogr. Anal., vol. II, International Press, Boston, 1995.