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Ann. I. H. Poincaré – AN 35 (2018) 1175–1207

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

On the dual formulation of obstacle problems for the total variation and the area functional

Christoph Scheven [a], Thomas Schmidt [b,*]

[a] *Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann-Str. 9, 45127 Essen, Germany*
[b] *Fachbereich Mathematik, Universität Hamburg, Bundesstr. 55, 20146 Hamburg, Germany*

Received 19 March 2017; received in revised form 9 August 2017; accepted 9 October 2017
Available online 14 November 2017

Abstract

We investigate the Dirichlet minimization problem for the total variation and the area functional with a one-sided obstacle. Relying on techniques of convex analysis, we identify certain dual maximization problems for bounded divergence-measure fields, and we establish duality formulas and pointwise relations between (generalized) BV minimizers and dual maximizers. As a particular case, these considerations yield a full characterization of BV minimizers in terms of Euler equations with a measure datum. Notably, our results apply to very general obstacles such as BV obstacles, thin obstacles, and boundary obstacles, and they include information on exceptional sets and up to the boundary. As a side benefit, in some cases we also obtain assertions on the limit behavior of p-Laplace type obstacle problems for $p \searrow 1$.

On the technical side, the statements and proofs of our results crucially depend on new versions of Anzellotti type pairings which involve general divergence-measure fields and specific representatives of BV functions. In addition, in the proofs we employ several fine results on (BV) capacities and one-sided approximation.

Keywords: (Thin) obstacle problem; Total variation; Convex duality; Optimality conditions; Anzellotti pairing; BV capacity

1. Introduction

Obstacle problems for total variation and area functional. In this paper, on a bounded open set $\Omega \subset \mathbb{R}^n$ of dimension $n \in \mathbb{N}$, we study certain variational problems with unilateral obstacles. More precisely, our primary interest is in the minimization problem for the total variation

$$\int_\Omega |\mathrm{D}u|\,\mathrm{d}x \tag{1.1}$$

* Corresponding author.
E-mail addresses: christoph.scheven@uni-due.de (C. Scheven), thomas.schmidt@math.uni-hamburg.de (T. Schmidt).
URLs: http://www.uni-due.de/mathematik/ag_scheven (C. Scheven), http://www.math.uni-hamburg.de/home/schmidt (T. Schmidt).

https://doi.org/10.1016/j.anihpc.2017.10.003
0294-1449/

among functions $u\colon \Omega \to \mathbb{R}$ which satisfy, for a given obstacle ψ, the zero Dirichlet boundary condition and the obstacle constraint

$$u = 0 \qquad \text{on } \partial\Omega\,, \tag{1.2}$$

$$u \geq \psi \qquad \text{on } \Omega\,. \tag{1.3}$$

We directly remark that instead of (1.2) we will eventually consider the non-homogeneous condition $u = u_o$ on $\partial\Omega$ with a quite general Dirichlet datum u_o, but for the purposes of this introductory exposition we limit ourselves to the homogeneous case in (1.2). Moreover, imposing the same constraints on u, we also study the minimization problem for the area integral

$$\int_\Omega \sqrt{1+|\mathrm{D}u|^2}\,\mathrm{d}x\,, \tag{1.4}$$

which, for sufficiently smooth functions u, gives the n-dimensional area of the graph of u.

It is well known that these problems are naturally set in the space $\mathrm{BV}(\Omega)$ of functions of bounded variation and that, thanks to weak$*$ compactness, general existence results for BV minimizers can be obtained. Indeed, such results involve the BV versions $|\mathrm{D}u|(\overline{\Omega})$ and $\sqrt{1+|\mathrm{D}u|^2}(\overline{\Omega})$ of the functionals in (1.1) and (1.4), where in order to explain[1] $\mathrm{D}u$ also on $\partial\Omega$ one extends u by 0 outside Ω. Moreover, one imposes mild assumptions on the obstacle ψ to ensure compatibility of (1.2) and (1.3) and make sure that the admissible class is non-empty ($\psi \in \mathrm{L}^\infty_{\mathrm{cpt}}(\Omega)$ suffices, but can be weakened). Then, if the obstacle condition (1.3) is understood as an $\mathcal{L}^n$-a.e. inequality, the existence of a minimizer follows by a standard application of the direct method. In particular, the validity of (1.3) is preserved in this reasoning, since BV embeds compactly in L^1 and minimizing sequences possess $\mathcal{L}^n$-a.e. convergent subsequences.

We record that, under specific assumptions on ψ, some finer existence and regularity results for the problems in (1.1)–(1.4) have been obtained in [24–26,37], for instance. However, in the present paper, we are concerned with different issues, which are approached in a general setting with much lighter assumptions on ψ.

Thin obstacles and relaxation. With regard to the case of thin obstacles (i.e. obstacles which are positive only on $(n-1)$-dimensional surfaces), it is also interesting to understand (1.3) as an a.e. condition with respect to the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$. This understanding of the constraint is also natural, since $\mathcal{H}^{n-1}$-a.e. means quasi everywhere with respect to 1-capacity; cf. (2.15). The resulting point of view leads to a more general theory and typically makes an essential difference for obstacles which are — as it occurs in the thin case — upper semicontinuous, but neither continuous nor BV. Precisely, as a substitute for (1.3) we thus employ the condition

$$u^+ \geq \Psi \qquad \mathcal{H}^{n-1}\text{-a.e. on } \Omega\,, \tag{1.5}$$

where the capital letter Ψ is used for the $\mathcal{H}^{n-1}$-a.e. defined obstacle and the approximate upper limit u^+ gives the largest one among the reasonable $\mathcal{H}^{n-1}$-a.e. defined representatives of $u \in \mathrm{BV}(\Omega)$; see Section 2.3 for the precise definition. Under (1.5) the existence issue becomes more subtle, since a (minimizing) sequence of smooth functions $u_k \geq \Psi$ may converge to a limit $u \in \mathrm{BV}(\Omega)$ which does not anymore satisfy (1.5) — even though the usage of u^+ means that (1.5) is understood in the widest possible $\mathcal{H}^{n-1}$-a.e. sense. This difficulty can be overcome by passing to suitable relaxations (or more precisely to L^1-lower semicontinuous envelopes) of the functionals (1.1) and (1.4), and indeed explicit formulas for the relaxations — when starting from competitors $w \in \mathrm{W}^{1,1}_0(\Omega)$ with (1.5) — have been identified by Carriero & Dal Maso & Leaci & Pascali [15]. Their corresponding result [15, Theorem 7.1] states that, if Ω has a Lipschitz boundary and a Borel function Ψ on Ω is compatible with (1.2) in the sense that there exists a competitor $w \in \mathrm{W}^{1,1}_0(\Omega)$ with (1.5), then the relaxed functionals on $\mathrm{BV}(\Omega)$ are given by

$$\mathrm{TV}_{\Psi;\Omega}(u) = |\mathrm{D}u|(\overline{\Omega}) + \int_\Omega (\Psi - u^+)_+\,\mathrm{d}\varsigma\,, \tag{1.6}$$

[1] To be precise, the above quantities are defined for all $u \in \mathrm{BV}(\Omega)$ such that the extension with value 0 on $\mathbb{R}^n \setminus \Omega$ is BV on the whole $\mathbb{R}^n$, and the measure $\mathrm{D}u$ then represents the distributional gradient of the extended function. Correspondingly, the quantities $|\mathrm{D}u|(\overline{\Omega})$ and $\sqrt{1+|\mathrm{D}u|^2}(\overline{\Omega})$ are understood as the total variations over $\overline{\Omega}$ of this $\mathbb{R}^n$-valued measure $\mathrm{D}u$ and the $\mathbb{R}^{1+n}$-valued measure $(\mathcal{L}^n, \mathrm{D}u)$ (with the Lebesgue measure $\mathcal{L}^n$), and they suitably take into account the zero Dirichlet condition in (1.2).

$$\mathcal{A}_{\Psi;\Omega}(u)=\sqrt{1+|\mathrm{D}u|^2}(\overline{\Omega})+\int_\Omega(\Psi-u^+)_+\,\mathrm{d}\varsigma\,. \tag{1.7}$$

Here the subscript $_+$ is used for the positive part of a function, and ς stands for a certain Borel measure on $\mathbb{R}^n$, which is called the De Giorgi measure and coincides with $2\mathcal{H}^{n-1}$ on (countable unions of) regular $(n-1)$-dimensional surfaces, but not on all Borel sets; see [19,20,17,28,15] and Section 2.6. In view of the described relaxation result, the obstacle problems in (1.1)–(1.4) are naturally generalized by the minimization problems for the relaxed functionals $\mathrm{TV}_{\Psi;\Omega}$ and $\mathcal{A}_{\Psi;\Omega}$ *on all of* $\mathrm{BV}(\Omega)$ — with no need to postulate (1.5) anymore, since this constraint is already incorporated, in a weaker fashion, through the ς-terms in $\mathrm{TV}_{\Psi;\Omega}$ and $\mathcal{A}_{\Psi;\Omega}$; compare Remark 3.7 for a more thorough discussion. As the main benefit of the relaxation process, the existence of minimizers of the new functionals $\mathrm{TV}_{\Psi;\Omega}$ and $\mathcal{A}_{\Psi;\Omega}$ is easy to establish. Recently, the relaxed functionals $\mathrm{TV}_{\Psi;\Omega}$ also proved to be useful in the parabolic setting of [14], where solutions to the thin obstacle problem for the total variation flow were constructed.

Main results: duality formulas. In the present paper, we complement the relaxation results from [15] with duality formulas for the above BV obstacle problems, which can (mostly) be obtained as limit cases for $p\searrow 1$ of more standard duality results for p-Laplace type obstacle problems. A first simplified version of our BV duality formulas asserts that, if Ω has a Lipschitz boundary and Ψ is a bounded upper semicontinuous function on $\overline{\Omega}$ with $\Psi\le 0$ on $\partial\Omega$, then we have

$$\min_{u\in\mathrm{BV}(\Omega)}\mathrm{TV}_{\Psi;\Omega}(u)\;=\;\max_{\sigma\in\mathcal{S}^\infty_-(\Omega)}\int_\Omega\Psi\,\mathrm{d}(-\operatorname{div}\sigma)\,, \tag{1.8}$$

$$\min_{u\in\mathrm{BV}(\Omega)}\mathcal{A}_{\Psi;\Omega}(u)\;=\;\max_{\sigma\in\mathcal{S}^\infty_-(\Omega)}\left(\int_\Omega\Psi\,\mathrm{d}(-\operatorname{div}\sigma)+\int_\Omega\sqrt{1-|\sigma|^2}\,\mathrm{d}x\right), \tag{1.9}$$

where $\mathcal{S}^\infty_-(\Omega)$ denotes the collection of all sub-unit vector fields in $\mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ whose distributional divergence exists as a non-positive Radon measure on Ω. Moreover, given any BV obstacle $\psi\in\mathrm{BV}(\Omega)\cap\mathrm{L}^\infty(\Omega)$ with non-positive trace on $\partial\Omega$, the formulas (1.8) and (1.9) remain true for $\Psi=\psi^+$, and likewise they hold for a natural (and essentially optimal) generalization of both upper semicontinuous and BV obstacles, namely for 1-capacity quasi upper semicontinuous obstacles $\Psi\in\mathrm{L}^\infty(\overline{\Omega};\mathcal{H}^{n-1})$ with $\Psi\le 0$ on $\partial\Omega$; see Sections 2.8 and 3.2 for more details. Here, the significance of the quasi semicontinuity requirement is also supported by classical results which guarantee, in a variety of settings, that every convex unilateral set can be represented with the help of a quasi semicontinuous obstacle; see [9, Thm. 3.2], for instance.

We emphasize, however, that, even though our results hold under this very general assumption on Ψ and include the thin case, they seem to be new even in more standard situations with n-dimensional obstacles ψ. Indeed, by taking $\Psi=\psi^+$ we cover continuous obstacles ψ, bounded BV obstacles ψ and more generally all bounded $\mathcal{L}^n$-a.e. obstacles ψ which $\mathcal{L}^n$-a.e. satisfy $\psi^+=\psi$; clearly, in these cases the ς-term in $\mathrm{TV}_{\Psi;\Omega}$ and $\mathcal{A}_{\Psi;\Omega}$ can only take the values 0 and ∞, and thus it is equivalent to drop this term and return to the constraint (1.5).

In the proofs of the duality formulas we rely on fine one-sided approximation results and the theory of variational (1-)capacity, and we crucially involve certain product constructions, which are known as Anzellotti type pairings (see also the next paragraph). Moreover, a part of our reasoning is based on the passage to the limit $p\searrow 1$ in obstacle problems for p-Laplace type operators, and it yields a statement on the convergence of p-energy minimizers to total variation or area minimizers as a side benefit.

Anzellotti type pairings and optimality conditions. The main motivation for our interest in (1.8) and (1.9) stems from the possibility to deduce first-order optimality criteria for BV minimizers. The deduction of such criteria parallels recent work by Beck and the second author, who identified in [10, Theorem 2.2] necessary and sufficient pointwise conditions for BV minimizers of a wide class of functionals, but without presence of an obstacle (compare also [8,13] for previous results). Here, in contrast, we treat the functionals $\mathrm{TV}_{\Psi;\Omega}$ and $\mathcal{A}_{\Psi;\Omega}$, which include the (possibly very general) obstacle Ψ. On a heuristic level with sufficiently smooth minimizers u and the functionals in (1.1) and (1.4), one expects the first-order criteria for the obstacle problems to take the form of variational inequalities or, equivalently, of Euler–Lagrange equations with right-hand side measures

$$-\operatorname{div}\frac{\mathrm{D}u}{|\mathrm{D}u|}=\mu \qquad \text{and} \qquad -\operatorname{div}\frac{\mathrm{D}u}{\sqrt{1+|\mathrm{D}u|^2}}=\mu\,, \tag{1.10}$$

where the non-negative measure μ is supported in the coincidence set of u and Ψ. However, a definition of BV solutions (and, in the first case, even of any solutions u with zeroes of $\mathrm{D}u$) to these equations is not immediate and has only been addressed, as a topic of independent interest, in our predecessor paper [35]. In that paper, extending classical ideas of Kohn & Temam [29] and Anzellotti [6], we indeed introduced a measure $[\![\sigma,\mathrm{D}u^+]\!]_0$ on $\overline{\Omega}$ as a generalized product of a gradient measure $\mathrm{D}u$ and a divergence-measure field $\sigma\in S^\infty_-(\Omega)$. Our measure $[\![\sigma,\mathrm{D}u^+]\!]_0$ is a variant of the one described by Chen & Frid [16, Theorem 3.2] and Mercaldo & Segura de León & Trombetti [32, Appendix A] (their measure corresponds to $[\![\sigma,\mathrm{D}u^*]\!]$ in our terminology), but still our product construction is somewhat refined and is perfectly suited for giving a precise meaning to the equations in (1.10) — in the same general framework in which the functionals $\mathrm{TV}_{\Psi;\Omega}$ and $\mathcal{A}_{\Psi;\Omega}$ and the duality formulas (1.8) and (1.9) have been described. At this stage, we state our corresponding result only for the total variation case, in which it also allows to overcome difficulties due to non-differentiability of $|\mathrm{D}u|$ and singular behavior of $\frac{\mathrm{D}u}{|\mathrm{D}u|}$ at zeroes of $\mathrm{D}u$. Concretely, for every pair of competitors $(u,\sigma)\in\mathrm{BV}(\Omega)\times S^\infty_-(\Omega)$, we obtain: (u,σ) is a minimizer–maximizer pair in (1.8) if and only if the optimality relations

$$[\![\sigma,\mathrm{D}u^+]\!]_0=|\mathrm{D}u| \text{ on } \overline{\Omega}\,, \qquad -\operatorname{div}\sigma=\mu \text{ in } \mathscr{D}'(\Omega)\,, \tag{1.11}$$

$$\mu\equiv 0 \text{ on } \Omega\cap\{u^+>\Psi\}\,, \qquad \mu=\varsigma \text{ on } \Omega\cap\{u^+<\Psi\} \tag{1.12}$$

hold with a non-negative measure μ. The equations in (1.11) are a BV (up-to-the-boundary) analog, as devised in [35], of the first heuristic equation in (1.10), and in particular the first equality in (1.11) encodes that σ takes over the role of the non-well-defined[2] quantity $\frac{\mathrm{D}u}{|\mathrm{D}u|}$. Moreover, the equalities in (1.12) specify that μ is supported only on the coincidence set $\Omega\cap\{u^+=\Psi\}$ and the exceptional set $\Omega\cap\{u^+<\Psi\}$ and that, on the latter set, μ is even fully determined and equal to the De Giorgi measure ς. Here, it is possible that $\mathcal{H}^{n-1}(\Omega\cap\{u^+<\Psi\})>0$ occurs, since — as we recall once more — the strict requirement in (1.5) has been dropped in favor of the weaker penalizing ς-term in (1.6).

Extensions: thin boundary obstacles and mildly regular domains. At this stage we would like to point out that the previously described statements, though quite general, do not yet give a complete picture of our main results in two regards.

Indeed, the more relevant extra feature, which we can handle, are (1-capacity quasi upper semicontinuous) obstacles Ψ which do *not* anymore respect the boundary condition (1.2). In other words, on top of the results described so far, we can dispense with the requirement $\Psi\le 0$ on $\partial\Omega$. Then, since we allow all functions in $\mathrm{BV}(\Omega)$ as competitors, the admissible class is still non-empty and consists of functions which are forced to jump at the boundary. Specifically, we can even cover, for instance, the case of thin boundary obstacles (i.e. obstacles which are positive *only* on $\partial\Omega$). The generalization of the duality formulas (1.8) and (1.9) to these cases actually requires an extension of the ς-terms in $\mathrm{TV}_{\Psi;\Omega}$ and $\mathcal{A}_{\Psi;\Omega}$ to $\overline{\Omega}$ and also — maybe more surprisingly — the addition of an extra term $\int_{\partial\Omega}(1-\sigma_{\mathrm{n}}^*)\Psi_+\,\mathrm{d}\mathcal{H}^{n-1}$ on the right-hand sides of (1.8) and (1.9). Here, σ_{n}^* stands for the inner normal trace of a divergence-measure field $\sigma\in S^\infty_-(\Omega)$ in the sense of [35]. With regard to the optimality criteria, these changes lead to some additional conditions on the boundary, which can partially be understood with the help of a modified up-to-the-boundary pairing $[\![\sigma,\mathrm{D}u^+]\!]_0^*$, which has also been introduced in [35]; see Sections 2.5, 3.2, and 3.3 for more details.

With regard to the regularity of $\partial\Omega$, the above statements are made for the case of Lipschitz domains Ω, but in the sequel we will even cover a large class of bounded open sets Ω of finite perimeter with well-behaved topological boundary. The precise mild regularity condition, on which we rely, has been introduced in [36] and is stated in (2.1) below. This condition ensures, for instance, that Ω cannot locally lie on both sides of $\partial\Omega$ and is relevant in connection with interior approximation of Ω in measure and perimeter. Our adaptation of an up-to-the-boundary Anzellotti type pairing to this kind of mildly regular domains follows the approach of [10, Section 5].

[2] We stress that in our context the Radon–Nikodým derivative $\frac{\mathrm{dD}u}{\mathrm{d}|\mathrm{D}u|}$ is *not* suited as a replacement for $\frac{\mathrm{D}u}{|\mathrm{D}u|}$. Indeed, a first indication for this is that $\frac{\mathrm{dD}u}{\mathrm{d}|\mathrm{D}u|}$ is $|\mathrm{D}u|$-a.e. defined. In contrast, σ is $\mathcal{L}^n$-a.e. defined, thus its distributional divergence can be taken, but still the equality $[\![\sigma,\mathrm{D}u^+]\!]_0=|\mathrm{D}u|$ also contains information on the singular parts of $\mathrm{D}u$.

Comparison with Anzellotti's directional-derivative approach to Euler equations in BV. Finally, we comment on an alternative way to identify a BV version of Euler equations. While the alternative is clearly more straightforward than our duality-based considerations, we believe (and try to explain in the sequel) that its results are conceptionally weaker than the characterization via (1.11)–(1.12) above.

Indeed, in order to characterize BV minimizers u one may simply require, for a given variational functional on BV, the vanishing of all those directional derivatives which exist at the point u. In the obstacle-free case, a computation of the relevant derivatives, which leads to a weak BV formulation of the Euler equation, has been carried out by Anzellotti [7]. Specifically, for a generalized minimizer $u \in \mathrm{BV}(\Omega)$ of the total variation with respect to a boundary datum u_o on a Lipschitz domain Ω, Anzellotti obtained in [7, Theorem 3.9] the equation (with the Radon–Nikodým derivative $\frac{\mathrm{D}u}{\mathrm{d}|\mathrm{D}u|}$)

$$\int_\Omega \frac{\mathrm{dD}u}{\mathrm{d}|\mathrm{D}u|} \cdot \mathrm{D}\varphi + \int_{\partial\Omega} \frac{u-u_o}{|u-u_o|}\varphi \,\mathrm{d}\mathcal{H}^{n-1} = 0 \tag{1.13}$$

for all $\varphi \in \mathrm{BV}(\Omega)$ such that $|\mathrm{D}\varphi|$ is absolutely continuous w.r.t. $|\mathrm{D}u|$ on Ω and such that the trace φ vanishes $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega \cap \{u = u_o\}$. However, in case of the total variation and for similar non-differentiable functionals, it has not been clarified in [7] if this necessary criterion is also sufficient for minimality. For functionals with differentiable integrands such as the area, the statement of Anzellotti's equation requires some more notation and involves BV test functions such that $|\mathrm{D}\varphi|$ is absolutely continuous w.r.t. $\mathcal{L}^n + |\mathrm{D}u|$. In these latter cases, at least, the equation does indeed characterize BV solutions of the Dirichlet minimization problem (which results from the fact that the directions φ with the relevant absolute-continuity property are dense in a suitable topology); see [7, Theorem 3.7] and [7, Theorem 3.10] for the necessity and the sufficiency of the equation, respectively.

In principle, Anzellotti's approach via directional derivatives also applies to *obstacle* problems for BV functionals (possibly with a small enhancement based on *one-sided* directional derivatives). However, already in the obstacle-free case the Anzellotti Euler equation seems less explicit and meaningful than the corresponding duality-based extremality relations of [10, Theorem 2.2]. In fact, in the total variation case (still without obstacle as above), the duality information consists in the simplified version of (1.11) with $\mu \equiv 0$ which asserts $[\![\sigma, \mathrm{D}u]\!]_{u_o} = |\mathrm{D}u|$ on $\overline{\Omega}$ for some sub-unit field σ with vanishing distributional divergence. In contrast, it seems difficult to find a conceptual interpretation of (1.13) based on more standard classes of test functions φ. Actually, whenever some level set of u possesses an interior point, some functions $\varphi \in \mathscr{D}(\Omega)$ are non-admissible in (1.13), and (1.13) does not yield the existence of a divergence-free quantity of type $\frac{\mathrm{D}u}{|\mathrm{D}u|}$ on Ω. In view of such problems, we believe that our approach is preferable over the use of directional derivatives and yields more useful results. Nevertheless, at least for area minimization without obstacle, the Euler equations obtained by the two approaches are, a posteriori, equivalent, even though this is not at all obvious (to us) from the equations themselves.

Last but not least we remark that our duality considerations and specifically equation (1.11) are also consistent with plenty of more recent literature [21,2–5,11,22,12,33,31,32,34,35], in which it has become common to model the 1-Laplace operator and the total variation flow on BV functions with the help of Anzellotti type pairings. All in all, we thus believe that the strategy of this paper via duality and pairings is the state-of-the-art technique in order to extract first-order information on BV minimizers.

Organization of this article. We start with a preliminary section in which we gather the technical tools that are necessary to formulate our results. After these preparations, we give the precise statement of our results in Section 3. Section 4 contains some results on the p-Laplace obstacle problems for $p > 1$, which we use as approximating problems. The proofs of our results on the obstacle problem for the total variation functional are presented in Section 5. In the final Section 6, we comment on the changes that are necessary to derive corresponding results for the area functional.

2. Preliminaries

2.1. General notation

We write $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$ for the n-dimensional Lebesgue measure and the $(n-1)$-dimensional spherical Hausdorff-measure, respectively. The perimeter of a Borel set $A \subset \mathbb{R}^n$ is denoted by $\mathbf{P}(A)$. The $\{0, 1\}$-valued characteristic function of a set A is abbreviated by $\mathbb{1}_A$.

For the positive and the negative part of a function $f : \Omega \to \mathbb{R}$, we use the customary notation

$$f_+(x) := \max\{f(x), 0\} \qquad \text{and} \qquad f_-(x) := \max\{-f(x), 0\}.$$

This is not to be confused with the approximate upper and lower limits f^+ and f^-, which are defined in Section 2.3 below.

As in [1], we use the standard notations L^p, $\mathrm{W}^{k,p}$, C^k, $\mathrm{C}^{k,\alpha}$, BV for spaces of integrable, (weakly) differentiable, Hölder, and bounded-variation functions, respectively. Moreover, $\mathscr{D}(\Omega)$ denotes the space of smooth functions with compact support in the open set Ω, while $\mathscr{D}'(\Omega)$ stands for the corresponding space of distributions.

2.2. Domains

In the sequel, we will always work on an open set Ω in $\mathbb{R}^n$, and mostly we will assume that Ω is bounded and satisfies the mild regularity requirement

$$\mathcal{H}^{n-1}(\partial\Omega) = \mathbf{P}(\Omega) < \infty\,, \tag{2.1}$$

where $\mathbf{P}(\Omega)$ stands for the perimeter of Ω. In view of De Giorgi's structure theorem, condition (2.1) can be equivalently reformulated to $\mathbf{P}(\Omega) < \infty$ and $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^*\Omega) = 0$ with the reduced boundary $\partial^*\Omega$ of Ω, and in particular (2.1) implies that $\partial\Omega$ is $\mathcal{H}^{n-1}$-rectifiable; compare [1, Theorem 3.59].

2.3. Representatives and traces of BV functions

Consider $u \in \mathrm{BV}(\mathbb{R}^n)$. We recall that $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{R}^n$ is either a Lebesgue point (also called an approximate continuity point) of u or an approximate jump point of u; compare [1, Sections 3.6, 3.7]. We write u^+ for the approximate upper limit of u, defined by

$$u^+(x) = \inf\left\{\lambda \in \mathbb{R} : \lim_{r \searrow 0} \frac{\mathcal{L}^n(\{u > \lambda\} \cap \mathrm{B}_r(x))}{r^n} = 0\right\},$$

where $\mathrm{B}_r(x) \subset \mathbb{R}^n$ denotes the open ball with radius $r > 0$ and center $x \in \mathbb{R}^n$. This defines an $\mathcal{H}^{n-1}$-a.e. defined representative u^+ of u which takes the Lebesgue values in the Lebesgue points and the larger of the two jump values in the approximate jump points. Analogously, one defines a representative u^- which takes the lesser jump value in the approximate jump points, and we set $u^* := \frac{1}{2}(u^+ + u^-)$. Observe that $(u+v)^+ \le u^+ + v^+$ and $(u+v)^- \ge u^- + v^-$ hold $\mathcal{H}^{n-1}$-a.e. and that, on the set $\mathrm{J}_u \cap \mathrm{J}_v$ of joint approximative jump points, these inequalities need not be equalities. However, for the mean value representative, one always has the $\mathcal{H}^{n-1}$-a.e. equality $(u+v)^* = u^* + v^*$.

We also recall that $u \in \mathrm{BV}(\mathbb{R}^n)$ possesses traces in the sense of [1, Theorem 3.77] on every oriented $\mathcal{H}^{n-1}$-rectifiable set Γ in $\mathbb{R}^n$. These traces from the two sides of Γ are defined and finite $\mathcal{H}^{n-1}$-a.e. on Γ, but are not necessarily equal to each other or integrable in any sense. In particular, an interior and an exterior trace with respect to Ω exist whenever $\Gamma = \partial^*\Omega$ is the reduced boundary of a set Ω of finite perimeter in $\mathbb{R}^n$, and these traces are then denoted (with slight abuse of terminology) by $u^{\mathrm{int}}_{\partial^*\Omega}$ and $u^{\mathrm{ext}}_{\partial^*\Omega}$. Under the hypothesis (2.1) we can and do regard the traces as $\mathcal{H}^{n-1}$-a.e. defined functions even on the topological boundary $\partial\Omega$, and thus we then use the corresponding notations $u^{\mathrm{int}}_{\partial\Omega}$ and $u^{\mathrm{ext}}_{\partial\Omega}$.

2.4. Admissible function classes

Given a Borel measurable obstacle $\Psi : \overline{\Omega} \to \mathbb{R}$ and a boundary datum $u_o \in \mathrm{W}^{1,1}(\mathbb{R}^n) \cap \mathrm{L}^\infty(\Omega)$, we introduce the admissible classes

$$\begin{aligned} \mathrm{BV}_{u_o}(\Omega) &:= \{u \in \mathrm{BV}(\mathbb{R}^n) \,:\, u = u_o \text{ holds } \mathcal{L}^n\text{-a.e. on } \mathbb{R}^n \setminus \Omega\}, \\ K_\Psi(\Omega) &:= \{u \in \mathrm{BV}_{u_o}(\Omega) \,:\, u^+ \ge \Psi \text{ holds } \mathcal{H}^{n-1}\text{-a.e. on } \overline{\Omega}\}. \end{aligned}$$

Occasionally, we will also use the more precise notation $K_{\Psi,u_o}(\Omega)$ if it is not clear from the context which boundary data are considered.

For the formulation of dual problems, we introduce the notation

$$S^\infty_-(\Omega) := \{\sigma \in \mathrm{L}^\infty(\Omega, \mathbb{R}^n) \,:\, |\sigma| \le 1 \text{ holds } \mathcal{L}^n\text{-a.e. on } \Omega,\ \operatorname{div}\sigma \le 0 \text{ in } \mathscr{D}'(\Omega)\}$$

for the class of sub-unit vector fields on Ω with non-positive distributional divergence.

2.5. Anzellotti type pairings for divergence-measure fields

Following [16], we denote the space of bounded divergence-measure fields on Ω by

$$\mathcal{DM}^\infty(\Omega) := \{\sigma \in \mathrm{L}^\infty(\Omega, \mathbb{R}^n) \,:\, \operatorname{div}\sigma \text{ is a finite signed Borel measure}\}\,.$$

The following result by Chen & Frid [16, Proposition 3.1] is crucial for our purposes.

Lemma 2.1. *Assume that $\sigma \in \mathcal{DM}^\infty(\Omega)$. Then, for every Borel set $A \subset \Omega$ with $\mathcal{H}^{n-1}(A) = 0$, we have $|\operatorname{div}\sigma|(A) = 0$.*

The following lemma enables us to make sense of a pairing $[\![\sigma, \mathrm{D}u^+]\!]_{u_o}$ when u and σ are in the admissible classes $\mathrm{BV}_{u_o}(\Omega)$ and $S^\infty_-(\Omega)$, respectively.

Lemma 2.2 *(Finiteness of divergences with a sign). If Ω is a bounded open set in $\mathbb{R}^n$ with $\mathcal{H}^{n-1}(\partial\Omega) < \infty$, then we have*

$$S^\infty_-(\Omega) \subset \mathcal{DM}^\infty(\Omega)$$

and

$$(-\operatorname{div}\sigma)(\Omega) \le \frac{n\omega_n}{\omega_{n-1}}\mathcal{H}^{n-1}(\partial\Omega) \qquad \text{for every } \sigma \in S^\infty_-(\Omega)\,,$$

where ω_n denotes the volume of the n-dimensional unit ball.

For the proof, we refer to [35, Lemma 2.2]. The constant $\frac{n\omega_n}{\omega_{n-1}}$ can be improved, but this is not relevant for our purposes, and so we have chosen to state the outcome of the simple proof in [35], which still covers the very general domains Ω considered here.

With Lemma 2.1 and Lemma 2.2 at hand, we next define our Anzellotti type pairing.

Definition 2.3 *(Distributional up-to-the-boundary pairing).* Consider a bounded open set $\Omega \subset \mathbb{R}^n$. For every $u_o \in \mathrm{W}^{1,1}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, $\sigma \in S^\infty_-(\Omega)$ and $U \in \mathrm{L}^\infty(\Omega; \mathcal{H}^{n-1})$, we define a distribution $[\![\sigma, \mathrm{D}U]\!]_{u_o} \in \mathscr{D}'(\mathbb{R}^n)$ by setting

$$[\![\sigma, \mathrm{D}U]\!]_{u_o}(\varphi) := \int_{\overline{\Omega}} \varphi(U - u_o^*)\,\mathrm{d}(-\operatorname{div}\sigma) - \int_\Omega (U - u_o)(\sigma \cdot \mathrm{D}\varphi)\,\mathrm{d}x + \int_\Omega \varphi\sigma \cdot \mathrm{D}u_o\,\mathrm{d}x$$

for $\varphi \in \mathscr{D}(\mathbb{R}^n)$. This distribution is well-defined in view of Lemma 2.1 and Lemma 2.2 and, since U and u_o^* are $\mathcal{H}^{n-1}$-a.e. defined and bounded.

Remark 2.4 *(On the up-to-the-boundary pairing for* BV *functions).* Given a function $u \in \mathrm{BV}_{u_o}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, we may choose the representative $U = u^+$ of u in the preceding definition, because u^+ is $\mathcal{H}^{n-1}$-a.e. defined and bounded. In this way, we obtain the distribution $[\![\sigma, \mathrm{D}u^+]\!]_{u_o} \in \mathscr{D}'(\mathbb{R}^n)$, which was already considered in [34,35] and is given by

$$[\![\sigma, \mathrm{D}u^+]\!]_{u_o}(\varphi) := \int_{\overline{\Omega}} \varphi(u^+ - u_o^*)\,\mathrm{d}(-\operatorname{div}\sigma) - \int_\Omega (u - u_o)(\sigma \cdot \mathrm{D}\varphi)\,\mathrm{d}x + \int_\Omega \varphi\sigma \cdot \mathrm{D}u_o\,\mathrm{d}x$$

for $\varphi \in \mathscr{D}(\mathbb{R}^n)$. In the same way, by choosing the representatives u^- or u^* of u instead of u^+, one obtains the distributions $[\![\sigma, \mathrm{D}u^-]\!]_{u_o}$ and $[\![\sigma, \mathrm{D}u^*]\!]_{u_o}$ (which coincide with $[\![\sigma, \mathrm{D}u^+]\!]_{u_o}$ for $u \in \mathrm{W}^{1,1}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, since then $u^+ = u^* = u^-$ holds $\mathcal{H}^{n-1}$-a.e.). However, in connection with obstacle problems, it turns out that u^+ is the suitable choice. This is related to the occurrence of u^+ in the penalization terms in (1.6) and (1.7) and also to the fact that the maximum on the right-hand sides of the duality formulas (1.8) and (1.9) would not necessarily be attained if we used the representative $\Psi = \psi^-$ or $\Psi = \psi^*$ of a BV obstacle ψ, rather than $\Psi = \psi^+$.

From [35, Lemma 3.3] we have the following statement.

Lemma 2.5 *(The pairing on* $\mathrm{W}^{1,1}$ *functions and the generalized normal trace). For an arbitrary bounded open set* $\Omega \subset \mathbb{R}^n$*,* $\sigma \in \mathcal{S}^{\infty}_{-}(\Omega)$ *and* $u \in (u_o|_\Omega + \mathrm{W}^{1,1}_0(\Omega)) \cap \mathrm{L}^\infty(\Omega)$*, it holds*

$$[\![\sigma, \mathrm{D}u^+]\!]_{u_o}(\varphi) = \int_\Omega \varphi(\sigma \cdot \mathrm{D}u)\,\mathrm{d}x \qquad \text{for every } \varphi \in \mathscr{D}(\mathbb{R}^n)\,. \tag{2.2}$$

If Ω *additionally satisfies (2.1), then for every* $\sigma \in \mathcal{S}^{\infty}_{-}(\Omega)$ *there exists a uniquely determined normal trace* $\sigma^*_{\mathrm{n}} \in \mathrm{L}^\infty(\partial\Omega; \mathcal{H}^{n-1})$ *with* $|\sigma^*_{\mathrm{n}}| \leq 1$ *so that for all* $u \in \mathrm{W}^{1,1}(\mathbb{R}^n) \cap \mathrm{L}^\infty(\Omega)$ *and* $\varphi \in \mathscr{D}(\mathbb{R}^n)$*, it holds*

$$[\![\sigma, \mathrm{D}u^+]\!]_{u_o}(\varphi) = \int_\Omega \varphi(\sigma \cdot \mathrm{D}u)\,\mathrm{d}x + \int_{\partial\Omega} \varphi\,(u-u_o)^{\mathrm{int}}_{\partial\Omega}\,\sigma^*_{\mathrm{n}}\,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{2.3}$$

where the trace $(u-u_o)^{\mathrm{int}}_{\partial\Omega}$ *is taken in the sense of Section 2.3.*

Following [34,35] once more, we also define the following modification of the up-to-the-boundary pairing.

Definition 2.6 *(Modified distributional up-to-the-boundary pairing).* Consider a bounded open set $\Omega \subset \mathbb{R}^n$ with (2.1). For every $u_o \in \mathrm{W}^{1,1}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, $\sigma \in \mathcal{S}^{\infty}_{-}(\Omega)$, $U \in \mathrm{L}^\infty(\overline{\Omega}; \mathcal{H}^{n-1})$, and $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we set

$$[\![\sigma, \mathrm{D}U]\!]^*_{u_o}(\varphi) := [\![\sigma, \mathrm{D}U]\!]_{u_o}(\varphi) + \int_{\partial\Omega} \varphi\big[U-(u_o)^{\mathrm{int}}_{\partial\Omega}\big]_+ (1-\sigma^*_{\mathrm{n}})\,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{2.4}$$

where σ^*_{n} denotes the normal trace of $\sigma \in \mathcal{S}^{\infty}_{-}(\Omega)$ determined by the preceding lemma. This defines a distribution $[\![\sigma, \mathrm{D}U]\!]^*_{u_o} \in \mathscr{D}'(\mathbb{R}^n)$ with support in $\overline{\Omega}$. Therefore, even though $[\![\sigma, \mathrm{D}U]\!]^*_{u_o}$ need not be a measure, it makes sense to write $[\![\sigma, \mathrm{D}U]\!]^*_{u_o}(\overline{\Omega})$ in the sense of $[\![\sigma, \mathrm{D}U]\!]^*_{u_o}(\varphi)$ for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\varphi \equiv 1$ on $\overline{\Omega}$, i.e.

$$[\![\sigma, \mathrm{D}U]\!]^*_{u_o}(\overline{\Omega}) = \int_\Omega (U-u^*_o)\,\mathrm{d}(-\operatorname{div}\sigma) + \int_\Omega \sigma \cdot \mathrm{D}u_o\,\mathrm{d}x + \int_{\partial\Omega} \big[U-(u_o)^{\mathrm{int}}_{\partial\Omega}\big]_+ (1-\sigma^*_{\mathrm{n}})\,\mathrm{d}\mathcal{H}^{n-1}\,.$$

Remark 2.7 *(On the modified up-to-the-boundary pairing for* BV *functions).* Once more, for $u_o \in \mathrm{W}^{1,1}(\mathbb{R}^n) \cap \mathrm{L}^\infty(\Omega)$, $u \in \mathrm{BV}_{u_o}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, the preceding definition is applicable to $U = u^+$. Since $u^+ = \max\{u^{\mathrm{int}}_{\partial\Omega}, (u_o)^{\mathrm{int}}_{\partial\Omega}\}$ holds $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, in this case the pairing takes the form

$$[\![\sigma, \mathrm{D}u^+]\!]^*_{u_o}(\varphi) = [\![\sigma, \mathrm{D}u^+]\!]_{u_o}(\varphi) + \int_{\partial\Omega} \varphi\big[(u-u_o)^{\mathrm{int}}_{\partial\Omega}\big]_+ (1-\sigma^*_{\mathrm{n}})\,\mathrm{d}\mathcal{H}^{n-1}\,,$$

and we again use the notation

$$[\![\sigma, \mathrm{D}u^+]\!]^*_{u_o}(\overline{\Omega}) := \int_\Omega (u^+-u^*_o)\,\mathrm{d}(-\operatorname{div}\sigma) + \int_\Omega \sigma \cdot \mathrm{D}u_o\,\mathrm{d}x + \int_{\partial\Omega} \big[(u-u_o)^{\mathrm{int}}_{\partial\Omega}\big]_+ (1-\sigma^*_{\mathrm{n}})\,\mathrm{d}\mathcal{H}^{n-1}\,.$$

While the pairings with general U need only exist as distributions, the next result shows that in the case $U = u^+$ (and similarly for $U = u^*$ and $U = u^-$) with a BV function u they turn out to be measures.

Proposition 2.8 *(The pairings with* BV *functions are bounded measures). Suppose that* Ω *is a bounded open set in* $\mathbb{R}^n$ *with (2.1). For* $u_o \in \mathrm{W}^{1,1}(\mathbb{R}^n) \cap \mathrm{L}^\infty(\Omega)$*,* $\sigma \in \mathcal{S}^{\infty}_{-}(\Omega)$*, and* $u \in \mathrm{BV}_{u_o}(\Omega) \cap \mathrm{L}^\infty(\Omega)$*, the distributions* $[\![\sigma, \mathrm{D}u^+]\!]_{u_o}$ *and* $[\![\sigma, \mathrm{D}u^+]\!]^*_{u_o}$ *both are finite signed Borel measures supported in* $\overline{\Omega}$*, and there hold*

$$\big|[\![\sigma, \mathrm{D}u^+]\!]_{u_o}\big| \leq |\mathrm{D}u| \tag{2.5}$$

and

$$\left|[\![\sigma, \mathrm{D}u^+]\!]^*_{u_o}\right| \le |\mathrm{D}u| \tag{2.6}$$

as measures on $\overline{\Omega}$.

The proofs can be retrieved from [35, Proposition 3.5 and formula (4.11)].

Finally, we note that the measures $[\![\sigma, \mathrm{D}u^+]\!]_{u_o}$ and $[\![\sigma, \mathrm{D}u^+]\!]^*_{u_o}$ differ only on the boundary, and there we have

$$[\![\sigma, \mathrm{D}u^+]\!]_{u_o} \llcorner \partial\Omega = (u-u_o)^{\mathrm{int}}_{\partial\Omega}\sigma^*_{\mathrm{n}}\mathcal{H}^{n-1}\,, \tag{2.7}$$

$$[\![\sigma, \mathrm{D}u^+]\!]^*_{u_o} \llcorner \partial\Omega = \Big([(u-u_o)^{\mathrm{int}}_{\partial\Omega}]_+ - [(u-u_o)^{\mathrm{int}}_{\partial\Omega}]_-\sigma^*_{\mathrm{n}}\Big)\mathcal{H}^{n-1}\,, \tag{2.8}$$

for all $u \in \mathrm{BV}_{u_o}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and $\sigma \in S^\infty_-(\Omega)$, see [35, Eqns. (3.10), (4.10)].

2.6. De Giorgi's measure

According to [17], the following definition of the De Giorgi measure is equivalent to the original one.

Definition 2.9 *(De Giorgi measure).* For any $\varepsilon > 0$, one defines a set function ς_ε on $\mathbb{R}^n$ by

$$\varsigma_\varepsilon(E) := \inf\left\{\mathbf{P}(B) + \tfrac{1}{\varepsilon}\mathcal{L}^n(B) :\ B \subset \mathbb{R}^n \text{ is open with } B \supset E\right\}$$

for every $E \subset \mathbb{R}^n$. Then one lets

$$\varsigma(E) := \sup_{\varepsilon>0}\varsigma_\varepsilon(E) = \lim_{\varepsilon\searrow 0}\varsigma_\varepsilon(E)\,. \tag{2.9}$$

The above definition yields a Borel-regular outer measure ς on $\mathbb{R}^n$ (and thus, in particular, a σ-additive measure on Borel sets in $\mathbb{R}^n$). This measure satisfies

$$\varsigma(E) = 2\mathcal{H}^{n-1}(E) \tag{2.10}$$

for every Borel set E that is contained in a countable union of $(n-1)$-dimensional C^1-surfaces, see [20]. However, [28] provides an example of a Borel set E for which (2.10) does not hold. In general, the De Giorgi measure is only comparable to the Hausdorff measure in the sense

$$c_1\mathcal{H}^{n-1}(E) \le \varsigma(E) \le c_2\mathcal{H}^{n-1}(E) \tag{2.11}$$

for every subset $E \subset \mathbb{R}^n$, with dimensional constants $c_1(n), c_2(n) > 0$, cf. [20] once more.

The following lemma is a slight technical adaptation of [15, Proposition 4.4]. It allows to approximate thin obstacles by $\mathrm{W}^{1,1}$ functions from above.

Lemma 2.10. *Consider an open set Ω in $\mathbb{R}^n$ and an $\mathcal{H}^{n-1}$-a.e. defined, real-valued function Ψ with compact support in Ω such that there holds $\Psi_+ \in \mathrm{L}^1(\Omega;\varsigma)$. Then there exist $w_k \in \mathrm{W}^{1,1}_0(\Omega)$ such that $w_k^* \ge \Psi$ holds $\mathcal{H}^{n-1}$-a.e. on Ω and such that we have*

$$\limsup_{k\to\infty}\int_\Omega |\mathrm{D}w_k|\,\mathrm{d}x \le \int_\Omega \Psi_+\,\mathrm{d}\varsigma\,.$$

Proof. By [15, Proposition 4.4(b)] there exist $u_k \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ such that $u_k^* \ge \Psi$ holds $\mathcal{H}^{n-1}$-a.e. on Ω, such that u_k converges to 0 in $\mathrm{L}^1(\mathbb{R}^n)$, and such that

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}|\mathrm{D}u_k|\,\mathrm{d}x = \int_\Omega \Psi_+\,\mathrm{d}\varsigma\,.$$

Choosing a cut-off function $\eta \in \mathscr{D}(\Omega)$ with $\mathbb{1}_{\mathrm{spt}\Psi} \le \eta \le 1$ on Ω, we take

$$w_k := \eta u_k \in \mathrm{W}^{1,1}_0(\Omega)\,.$$

Then $w_k^* \geq \Psi$ evidently holds $\mathcal{H}^{n-1}$-a.e. on Ω, and via the product rule we moreover get

$$\limsup_{k\to\infty} \int_\Omega |\mathrm{D}w_k|\,\mathrm{d}x \leq \limsup_{k\to\infty} \left[\int_{\mathbb{R}^n} |\mathrm{D}u_k|\,\mathrm{d}x + \|\nabla\eta\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}\|u_k\|_{\mathrm{L}^1(\mathbb{R}^n)}\right] = \int_\Omega \Psi_+\,\mathrm{d}\varsigma\,.$$

This finishes the proof. □

We use the preceding result for a comparison between the De Giorgi measure and the measure $-\operatorname{div}\sigma$ for an arbitrary $\sigma \in \mathcal{S}^\infty_-(\Omega)$.

Proposition 2.11. *For every open set Ω in $\mathbb{R}^n$ and every $\sigma \in \mathcal{S}^\infty_-(\Omega)$, we have*

$$\varsigma \geq -\operatorname{div}\sigma \qquad \textit{as measures on } \Omega\,.$$

Proof. It is sufficient to compare the measures on Borel sets $A \subset \Omega$ with $\varsigma(A) < \infty$ and $A \Subset \Omega$. Then Lemma 2.10, applied with $\Psi = \mathbb{1}_A$, gives $w_k \in \mathrm{W}^{1,1}_0(\Omega)$ such that $w_k^* \geq \mathbb{1}_A$ holds $\mathcal{H}^{n-1}$-a.e. on Ω and such that we have

$$\limsup_{k\to\infty} \int_\Omega |\mathrm{D}w_k|\,\mathrm{d}x \leq \varsigma(A)\,.$$

Using this together with the bound $|\sigma| \leq 1$, the representation (2.2) in Lemma 2.5 and the definition of the pairing, we obtain

$$\begin{aligned}\varsigma(A) &\geq \limsup_{k\to\infty} \int_\Omega |\mathrm{D}w_k|\,\mathrm{d}x \geq \limsup_{k\to\infty} \int_\Omega \sigma\cdot\mathrm{D}w_k\,\mathrm{d}x \\ &= \limsup_{k\to\infty} [\![\sigma,\mathrm{D}w_k]\!]_0\big(\overline{\Omega}\big) = \limsup_{k\to\infty} \int_\Omega w_k^*\,\mathrm{d}(-\operatorname{div}\sigma) \geq (-\operatorname{div}\sigma)(A)\,.\end{aligned}$$

This proves the claim. □

The next proposition can be understood as a boundary version of the preceding result.

Proposition 2.12. *For every bounded open set $\Omega \subset \mathbb{R}^n$ with (2.1) and every $\sigma \in \mathcal{S}^\infty_-(\Omega)$ we have*

$$\varsigma = 2\mathcal{H}^{n-1} \geq (1-\sigma_{\mathrm{n}}^*)\mathcal{H}^{n-1} \qquad \textit{as measures on } \partial\Omega\,.$$

Proof. Since, by De Giorgi's structure theorem, the assumption (2.1) implies that Ω is a set of finite perimeter with $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^*\Omega) = 0$, its boundary $\partial\Omega$ is $\mathcal{H}^{n-1}$-rectifiable. This means that, up to a set of $\mathcal{H}^{n-1}$-measure zero, it is the countable union of $(n-1)$-dimensional C^1-surfaces, and we infer from (2.10) that we have $\varsigma = 2\mathcal{H}^{n-1}$ on $\partial\Omega$. This implies the claimed identity, and the asserted inequality is an immediate consequence since $1-\sigma_{\mathrm{n}}^* \leq 2$ holds $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$. □

2.7. *Total variation and area functionals for relaxed obstacle problems*

For $u \in K_\Psi(\Omega)$, we set

$$\mathrm{TV}_\Omega(u) := |\mathrm{D}u|(\overline{\Omega}) = |\mathrm{D}u|(\Omega) + \int_{\partial\Omega} |(u-u_o)^{\mathrm{int}}_{\partial\Omega}|\,\mathrm{d}\mathcal{H}^{n-1}$$

(where the second equality holds under (2.1) at least) and extend the functional to $\mathrm{L}^1(\Omega)$ by letting $\mathrm{TV}_\Omega(u) := \infty$ for $u \in \mathrm{L}^1(\Omega) \setminus K_\Psi(\Omega)$. Since in case of a thin obstacle Ψ the class $K_\Psi(\Omega)$ is not closed with respect to L^1 convergence (or weak$*$ BV convergence), it is necessary to consider the L^1-lower semicontinuous relaxation of TV_Ω. For Lipschitz domains Ω and for obstacles Ψ that are compatible with the boundary datum, the relaxation has been identified in [15, Theorem 7.1] as the functional

$$\mathrm{TV}_{\Psi;\Omega}(u) := |\mathrm{D}u|(\overline{\Omega}) + \int_{\overline{\Omega}} (\Psi - u^+)_+ \,\mathrm{d}\varsigma\,, \tag{2.12}$$

with the De Giorgi measure ς introduced in Section 2.6 and with the understanding that $\mathrm{TV}_{\Psi;\Omega}(u) = \infty$ whenever $u \notin \mathrm{BV}_{u_o}(\Omega)$. Here we take (2.12) as general definition of $\mathrm{TV}_{\Psi;\Omega}$, and we recall that we use the subscript $_+$ to denote the non-negative part of a function, while the superscript $^+$ stands for an approximate upper limit. Assuming (2.1), we also record the $\mathcal{H}^{n-1}$-a.e. equality $u^+ = \max\{u^{\mathrm{int}}_{\partial\Omega}, (u_o)^{\mathrm{int}}_{\partial\Omega}\}$ *on the boundary* $\partial\Omega$ (which results from our setting with $u_o \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ and $u = u_o$ on $\mathbb{R}^n \setminus \Omega$). In particular, whenever the compatibility condition $\Psi \le (u_o)^{\mathrm{int}}_{\partial\Omega}$ is satisfied $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, it makes no difference if the ς-integral in (2.12) is restricted from $\overline{\Omega}$ to Ω. Therefore, (2.12) is consistent with the definition of $\mathrm{TV}_{\Psi;\Omega}$ in the introduction, and if we think of this compatible situation, we sometimes define Ψ only on Ω and still use the notation $\mathrm{TV}_{\Psi;\Omega}$ (with the background understanding that we may take $\Psi = (u_o)^{\mathrm{int}}_{\partial\Omega}$ on $\partial\Omega$). However, when we allow up-to-the-boundary obstacles Ψ with $\Psi > (u_o)^{\mathrm{int}}_{\partial\Omega}$ on a portion of $\partial\Omega$ (as we will do in Section 3.2 below), then we still rely on (2.12) as a definition, and then it will be essential to stick to $\overline{\Omega}$ as the domain of the ς-integral.

Similarly, we treat the model case of the area functional. For $u_o \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ and $u \in K_\Psi(\Omega) \subset \mathrm{BV}_{u_o}(\Omega)$, this functional is given by

$$\mathcal{A}_\Omega(u) := \sqrt{1+|\mathrm{D}u|^2}(\overline{\Omega}) = \int_\Omega \sqrt{1+|\mathrm{D}^{\mathrm{a}}u|^2}\,\mathrm{d}x + |\mathrm{D}^{\mathrm{s}}u|(\Omega) + \int_{\partial\Omega} |(u-u_o)^{\mathrm{int}}_{\partial\Omega}|\,\mathrm{d}\mathcal{H}^{n-1}$$

(where the second equality holds under (2.1) at least). Here the measure $\sqrt{1+|\mathrm{D}u|^2}$ is defined as the variation measure of the $\mathbb{R}^{1+n}$-valued measure $(\mathcal{L}^n, \mathrm{D}u)$, and $\mathrm{D}u = (\mathrm{D}^{\mathrm{a}}u)\mathcal{L}^n + \mathrm{D}^{\mathrm{s}}u$ is the decomposition into the absolutely continuous and the singular part with respect to the Lebesgue measure $\mathcal{L}^n$. We extend this functional to $\mathrm{L}^1(\Omega)$ by letting $\mathcal{A}_\Omega(u) := \infty$ whenever $u \in \mathrm{L}^1(\Omega) \setminus K_\Psi(\Omega)$. As before, from [15, Theorem 7.1] we infer that under mild assumptions on Ω and Ψ, the L^1-lower semicontinuous relaxation of the resulting functional is given by

$$\mathcal{A}_{\Psi;\Omega}(u) := \sqrt{1+|\mathrm{D}u|^2}(\overline{\Omega}) + \int_{\overline{\Omega}} (\Psi - u^+)_+ \,\mathrm{d}\varsigma \tag{2.13}$$

with the De Giorgi measure ς.

We crucially rely on the following semicontinuity property which results from [15, Theorem 6.1].

Theorem 2.13 *(Carriero–Dal Maso–Leaci–Pascali lower semicontinuity). Consider a bounded open set Ω in $\mathbb{R}^n$, an obstacle $\Psi \in \mathrm{L}^\infty(\overline{\Omega}; \mathcal{H}^{n-1})$, and assume that u_k converges to u in $\mathrm{L}^1(\Omega)$. Then we have*

$$\mathrm{TV}_{\Psi;\Omega}(u) \le \liminf_{k\to\infty} \mathrm{TV}_{\Psi;\Omega}(u_k)\,,$$
$$\mathcal{A}_{\Psi;\Omega}(u) \le \liminf_{k\to\infty} \mathcal{A}_{\Psi;\Omega}(u_k)\,.$$

Proof. We consider a bounded open set $\widetilde{\Omega}$ in $\mathbb{R}^n$ with $\Omega \Subset \widetilde{\Omega}$ and extensions $\widetilde{\Psi}$, $\widetilde{u}_k$, $\widetilde{u}$ of Ψ, u_k, u by the values of u_o on $\widetilde{\Omega} \setminus \Omega$. We can now write

$$\mathrm{TV}_{\Psi;\Omega}(u) + \int_{\widetilde{\Omega}\setminus\overline{\Omega}} |\mathrm{D}u_o|\,\mathrm{d}x = \begin{cases} |\mathrm{D}\widetilde{u}|\big(\widetilde{\Omega}\big) + \int_{\widetilde{\Omega}} (\widetilde{\Psi} - \widetilde{u}^+)_+ \,\mathrm{d}\varsigma & \text{if } \widetilde{u} \in \mathrm{BV}(\widetilde{\Omega})\,, \\ \infty & \text{otherwise}\,, \end{cases}$$

where the right-hand side (as a functional in $\widetilde{u}$) is identified by [15, Theorem 6.1] as a relaxed functional in $\mathrm{L}^1\big(\widetilde{\Omega}\big)$, i.e. as an $\mathrm{L}^1\big(\widetilde{\Omega}\big)$ lower semicontinuous envelope. In particular, this functional is itself lower semicontinuous in $\mathrm{L}^1\big(\widetilde{\Omega}\big)$, and then our claim for the total variation follows immediately. The statement for the area functional can be deduced from [15, Theorem 6.1] in the same way. □

2.8. Capacities and quasi (semi)continuity

For $q \in [1, \infty)$, the (variational or functional) q-capacity of a set $E \subset \mathbb{R}^n$ can be defined by

$$\mathrm{Cap}_q(E) := \inf\left\{ \int_{\mathbb{R}^n} \left(|u|^q + |\mathrm{D}u|^q\right) \mathrm{d}x \,:\, u \in \mathrm{W}^{1,q}(\mathbb{R}^n),\ \begin{array}{c} u \geq 1 \text{ holds } \mathcal{L}^n\text{-a.e. on an} \\ \text{open neighborhood of } E \end{array} \right\}.$$

With this definition at hand, we often say that a property holds Cap_q-q.e. on a set U in $\mathbb{R}^n$ if it holds on $U \setminus E$ with some subset E of U such that $\mathrm{Cap}_q(E) = 0$. It is clear from the definition that a set with zero q-capacity is negligible for the Lebesgue measure. Hence, a Cap_q-q.e. requirement is stronger than the corresponding $\mathcal{L}^n$-a.e. requirement.

A modified q-capacity $\widetilde{\mathrm{Cap}}_q(E)$ can be defined for $E \subset \mathbb{R}^n$ by replacing the integrand $\left(|u|^q + |\mathrm{D}u|^q\right)$ above with the simpler quantity $|\mathrm{D}u|^q$. For $q < n$ or $q = n = 1$, this modification is marginal in the sense that Cap_q and $\widetilde{\mathrm{Cap}}_q$ have the same zero sets and satisfy $c\,\mathrm{Cap}_q(E) \leq \widetilde{\mathrm{Cap}}_q(E) \leq \mathrm{Cap}_q(E)$ for every bounded set $E \subset \mathbb{R}^n$ with a positive constant $c(n, q, \operatorname{diam} E)$; this can be shown via cut-off and the Gagliardo–Nirenberg inequality, compare [30, Lemma 2.6]. For $1 \neq q \geq n$, however, by using the competitors $\left[1 - \varepsilon \log_+(|x|/r)\right]_+$ one sees that $\widetilde{\mathrm{Cap}}_q$ vanishes on all bounded subsets of $\mathbb{R}^n$ and does not exhibit the behavior expected for q-capacity. In contrast, Cap_q with $q > n$ or $q = n = 1$ is positive on all non-empty sets $E \subset \mathbb{R}^n$ by Sobolev's inequality, so that Cap_q behaves reasonably, but trivially in these cases. In particular, a Cap_q-q.e. requirement with $q > n$ or $q = n = 1$ is an everywhere requirement. However, in this paper we are interested in letting $q \searrow 1$, so most of the time both Cap_q and $\widetilde{\mathrm{Cap}}_q$ are non-trivial, reasonable, and only marginally different.

Specifically for the case $q = 1$ of the BV capacities Cap_1 and $\widetilde{\mathrm{Cap}}_1$ we record (see, for instance, [15, Section 2])

$$\widetilde{\mathrm{Cap}}_1(E) = \inf\left\{ \mathrm{TV}_{\mathbb{R}^n}(u) \,:\, u \in \mathrm{BV}(\mathbb{R}^n),\ u^+ \geq 1 \text{ holds } \mathcal{H}^{n-1}\text{-a.e. on } E \right\}, \tag{2.14}$$

$$\mathrm{Cap}_1(E) = 0 \iff \widetilde{\mathrm{Cap}}_1(E) = 0 \iff \mathcal{H}^{n-1}(E) = 0 \tag{2.15}$$

for every $E \subset \mathbb{R}^n$, so that Cap_1-q.e. means nothing but $\mathcal{H}^{n-1}$-a.e., in particular.

With the notion of capacity at hand it now makes sense to introduce the following concept of (semi)continuity up to small sets.

Definition 2.14 (Cap_q*-quasi (semi)continuity).* For any $q \in [1, \infty)$, one says that a Cap_q-q.e. defined, real-valued function Ψ on an open subset Ω of $\mathbb{R}^n$ is Cap_q-quasi upper semicontinuous on Ω if for every $\varepsilon > 0$ there exists an open subset E of Ω with $\mathrm{Cap}_q(E) < \varepsilon$ such that the restriction of Ψ to $\Omega \setminus E$ is everywhere defined and upper semicontinuous. Clearly, there is a corresponding concept of Cap_q-quasi lower semicontinuity, and a function is called Cap_q-quasi continuous if it is both Cap_q-quasi upper semicontinuous and Cap_q-quasi lower semicontinuous.

The next lemma is a restatement of [15, Theorem 2.5], in which, taking into account the above remarks, $\widetilde{\mathrm{Cap}}_1$ has been replaced with Cap_1.

Lemma 2.15 (Cap_1*-quasi semicontinuous representatives of a* BV *function). For every open subset Ω of $\mathbb{R}^n$ and every $u \in \mathrm{BV}(\Omega)$, the representative u^+ is Cap_1-quasi upper semicontinuous on Ω, while the representative u^- is Cap_1-quasi lower semicontinuous on Ω.*

Finally, consider an open set Ω in $\mathbb{R}^n$ and $u \in \mathrm{W}^{1,q}(\Omega)$. Then, the set of non-Lebesgue points of u has zero q-capacity, and the precise representative u^* is Cap_q-q.e. defined. These facts and the following lemmas have originally been obtained in [23] for $\widetilde{\mathrm{Cap}}_q$ with $q \in [1, n)$. The case of Cap_q with $q \in (1, n]$ is covered in [27, Thm. 4.3, Lemma 4.8], compare also [30, Lemma 2.19]. Finally, the cases $q = n = 1$ and $q > n$ are simple consequences of Sobolev's embedding.

Lemma 2.16 (Cap_q*-quasi continuous representative of a* $\mathrm{W}^{1,q}$ *function). For an open subset Ω of $\mathbb{R}^n$ and $u \in \mathrm{W}^{1,q}(\Omega)$ with $q \in [1, \infty)$, the representative u^* is Cap_q-quasi continuous on Ω.*

Lemma 2.17 *(Convergence in* $\mathrm{W}^{1,q}$ *implies convergence* Cap_q*-q.e.). For an open subset* Ω *of* $\mathbb{R}^n$*, if* u_k *converges to* u *strongly in* $\mathrm{W}^{1,q}(\Omega)$ *with* $q \in [1,\infty)$*, then some subsequence* $u^*_{k_\ell}$ *converges* Cap_q*-q.e. on* Ω *to* u.

2.9. Monotone approximation of quasi upper semicontinuous functions

The following lemma is the special case $m = 1$ of a result of Dal Maso [18, Lemma 1.5], for $q = 1$ see also [18, Section 6].

Lemma 2.18. *For* $q \in [1,\infty)$*, consider a* Cap_q*-quasi upper semicontinuous function* $\Psi : \mathbb{R}^n \to \mathbb{R}$ *with*

$$\Psi \le g^* \qquad \mathrm{Cap}_q\text{-q.e. on } \mathbb{R}^n \text{ for some } g \in \mathrm{W}^{1,q}(\mathbb{R}^n)\,.$$

Then there exists a non-increasing sequence $\psi_k \in \mathrm{W}^{1,q}(\mathbb{R}^n)$*,* $k \in \mathbb{N}$ *such that, in the limit* $k \to \infty$*,*

$$\psi^*_k \to \Psi \qquad \mathrm{Cap}_q\text{-q.e. on } \mathbb{R}^n\,.$$

Moreover, we will need the following version of the previous result with boundary values on a domain Ω.

Lemma 2.19. *For* $q \in [1,\infty)$*, suppose that* $\Psi : \Omega \to \mathbb{R}$ *is* Cap_q*-quasi upper semicontinuous with*

$$\Psi \le g^* \qquad \mathrm{Cap}_q\text{-q.e. on } \Omega \text{ for some } g \in u_o|_\Omega + \mathrm{W}^{1,q}_0(\Omega)\,,$$

where $u_o \in \mathrm{W}^{1,q}(\mathbb{R}^n)$ *is given. Then there exists a non-increasing sequence of functions* $\psi_k \in u_o|_\Omega + \mathrm{W}^{1,q}_0(\Omega)$ *such that*

$$\psi^*_k \to \Psi \qquad \mathrm{Cap}_q\text{-q.e. on } \Omega\,.$$

Proof. Extending g by u_o outside of Ω, we find a function $\widehat{g} \in \mathrm{W}^{1,q}(\mathbb{R}^n)$ such that $\widehat{g}^* \ge \Psi$ holds Cap_q-q.e. on Ω (with the representative $\widehat{g}^*$ of Lemma 2.16). Therefore, the extension of Ψ by u^*_o outside of Ω is Cap_q-quasi upper semicontinuous on $\mathbb{R}^n$. Consequently, we can apply Lemma 2.18 to construct a non-increasing sequence $\chi_k \in \mathrm{W}^{1,q}(\mathbb{R}^n)$ such that $\chi^*_k \to \Psi$ converges Cap_q-q.e. on Ω and $\chi^*_k \to u^*_o$ converges Cap_q-q.e. on $\mathbb{R}^n \setminus \Omega$. Since $\Psi \le g^*$ holds Cap_q-q.e. on Ω with $g \in u_o|_\Omega + \mathrm{W}^{1,q}_0(\Omega)$, we obtain the desired sequence by letting $\psi_k := \min\{\chi_k, g\}$. $\square$

Remark 2.20. If, additionally, the function Ψ in the preceding lemmas is essentially bounded with respect to the Cap_q-capacity and if, in case of Lemma 2.19, also u_o is bounded, then the approximations ψ_k can be chosen as bounded functions. This follows by passing to the truncations $\widehat{\psi}_k := \min\{\psi_k, \max\{q\text{-}\sup\Psi, \sup u_o\}\}$ if necessary.

3. Statement of the full results

Next we provide the full statements and a more thorough discussion of our results.

3.1. Duality formulas for obstacle problems in the limit $p \searrow 1$

We consider a Borel measurable obstacle $\Psi : \overline{\Omega} \to \mathbb{R}$ and a boundary datum $u_o \in \mathrm{W}^{1,1}(\mathbb{R}^n) \cap \mathrm{L}^\infty(\Omega)$. From Section 2.4, we recall the definitions of the admissible classes for the minimization problems

$$\mathrm{BV}_{u_o}(\Omega) = \{u \in \mathrm{BV}(\mathbb{R}^n) \,:\, u = u_o \text{ holds } \mathcal{L}^n\text{-a.e. on } \mathbb{R}^n \setminus \Omega\}\,,$$
$$K_\Psi(\Omega) = \{u \in \mathrm{BV}_{u_o}(\Omega) \,:\, u^+ \ge \Psi \text{ holds } \mathcal{H}^{n-1}\text{-a.e. on } \overline{\Omega}\}$$

and of the admissible class for the dual problems

$$S^\infty_-(\Omega) = \{\sigma \in \mathrm{L}^\infty(\Omega,\mathbb{R}^n) \,:\, |\sigma| \le 1 \text{ holds } \mathcal{L}^n\text{-a.e. on } \Omega,\ \operatorname{div}\sigma \le 0 \text{ in } \mathscr{D}'(\Omega)\}.$$

We now consider the obstacle problem for the 1-Laplacian, in a generalized formulation based on functional $\mathrm{TV}_{\Psi;\Omega}$ from (2.12). Our first main result identifies a dual formulation of this problem and shows consistency with the corresponding duality theory for the p-Laplacian with $p > 1$.

Theorem 3.1 *(TV duality as the limit $p \searrow 1$ of p-Laplace obstacle problems). We consider a bounded open set $\Omega \subset \mathbb{R}^n$ with (2.1), $u_o \in \mathrm{W}^{1,q}(\mathbb{R}^n) \cap \mathrm{L}^\infty(\Omega)$, and a bounded Borel function $\Psi : \Omega \to \mathbb{R}$ so that*

$$\Psi \text{ is } \mathrm{Cap}_q\text{-quasi upper semicontinuous on } \Omega \tag{3.1}$$

for some $q > 1$. We also suppose that Ψ is compatible with the boundary values in the sense that there exists a function $g \in u_o|_\Omega + \mathrm{W}_0^{1,q}(\Omega)$ with

$$\Psi \le g^* \qquad \mathrm{Cap}_q\text{-q.e. on } \Omega\,. \tag{3.2}$$

Then we have

$$\min_{u \in \mathrm{BV}_{u_o}(\Omega)} \mathrm{TV}_{\Psi;\Omega}(u) = \max_{\sigma \in S_-^\infty(\Omega)} [\![\sigma, \mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega})\,, \tag{3.3}$$

and every minimizer u of the left-hand side is bounded with

$$\inf_\Omega u_o \le u \le \max\left\{1\text{-}\sup_{\overline{\Omega}} \Psi, \sup_\Omega u_o\right\} \qquad \text{a.e. in } \Omega\,, \tag{3.4}$$

where $1\text{-}\sup_{\overline{\Omega}} \Psi$ stands for the essential supremum of Ψ up to an $\mathcal{H}^{n-1}$-negligible set, while $\inf_\Omega u_o$ and $\sup_\Omega u_o$ denote the usual essential infimum and supremum of u_o (up to a Lebesgue negligible set).

Moreover, there is a minimizer–maximizer pair (u, σ) for (3.3) that can be obtained as a limit of solutions of p-Laplace obstacle problems in the following sense: for some sequence $p_i \searrow 1$ and the solution $u_{p_i} \in u_o|_\Omega + \mathrm{W}_0^{1,p_i}(\Omega)$ of the obstacle problem for the p_i-Laplacian with obstacle Ψ (cf. Definition 4.1), there holds

$$\begin{cases} u_{p_i} \overset{*}{\rightharpoonup} u & \text{weakly}* \text{ in } \mathrm{BV}_{u_o}(\Omega)\,, \\ |\mathrm{D}u_{p_i}|^{p_i-2}\mathrm{D}u_{p_i} \rightharpoonup \sigma & \text{weakly in } \mathrm{L}^{\frac{q}{q-1}}(\Omega, \mathbb{R}^n)\,. \end{cases} \tag{3.5}$$

Remark 3.2 *(On the distributional product $[\![\sigma, \mathrm{D}\Psi]\!]_{u_o}$).* The distributional product $[\![\sigma, \mathrm{D}\Psi]\!]_{u_o}$ and the quantity $[\![\sigma, \mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega})$ on the right-hand side of (3.3) have been explained in Definition 2.6. In particular, we record that, even though $[\![\sigma, \mathrm{D}\Psi]\!]_{u_o}$ is not necessarily a measure, it makes sense to understand

$$[\![\sigma, \mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega}) = \int_\Omega (\Psi - u_o^*)\,\mathrm{d}(-\operatorname{div}\sigma) + \int_\Omega \sigma \cdot \mathrm{D}u_o\,\mathrm{d}x\,.$$

Remark 3.3 *(On thin obstacles).* Our assumptions include the case of a thin obstacle such as the characteristic function of a closed, $(n-1)$-dimensional regular surface contained in Ω. If we consider boundary data $u_o \equiv 0$, the assumptions (3.1) and (3.2) are satisfied because the obstacle function is upper semicontinuous with compact support in Ω. In this case, even for minimizers $u \in \mathrm{BV}_0(\Omega)$, the obstacle constraint $u^+ \ge \Psi$ might not be satisfied $\mathcal{H}^{n-1}$-a.e. on Ω. However, the possible violation of the constraint is penalized by the term

$$\int_{\overline{\Omega}} (\Psi - u^+)_+\,\mathrm{d}\varsigma \tag{3.6}$$

in the functional $\mathrm{TV}_{\Psi;\Omega}$. Since this functional must be finite for any minimizer $u \in \mathrm{BV}_0(\Omega)$ in (3.3) and the De Giorgi measure is comparable to the Hausdorff measure in the sense of estimate (2.11), we can conclude that a minimizer can violate the obstacle constraint at most on a set which is σ-finite with respect to $\mathcal{H}^{n-1}$. In particular, for a minimizer u, the exceptional set $\{x \in \overline{\Omega} : u^+(x) < \Psi(x)\}$ has Hausdorff dimension at most $n-1$.

Remark 3.4 *(On $\mathrm{W}^{1,q}$ obstacles).* Another case that is covered by our assumptions is an obstacle function $\psi \in \mathrm{W}^{1,q}(\Omega)$ that satisfies $(\psi - u_o)_+ \in \mathrm{W}_0^{1,q}(\Omega)$ for some $q > 1$. More precisely, the theorem is then applicable to the Cap_q-quasi continuous representative $\Psi = \psi^*$ of ψ. In this particular case, the penalization term (3.6) is either infinite or zero for all $u \in \mathrm{BV}(\mathbb{R}^n)$, which means that any minimizer $u \in \mathrm{BV}_{u_o}(\Omega)$ in (3.3) must, $\mathcal{H}^{n-1}$-a.e. on $\overline{\Omega}$, satisfy the obstacle constraint $u^+ \ge \Psi = \psi^*$ and thus lie in $K_{\psi^*}(\Omega)$. Consequently, formula (3.3) is then equivalent to

$$\min_{u \in K_{\psi^*}(\Omega)} \mathrm{TV}_\Omega(u) = \max_{\sigma \in S_-^\infty(\Omega)} [\![\sigma, \mathrm{D}\psi^*]\!]_{u_o}(\overline{\Omega})\,. \tag{3.7}$$

If we even have $\psi \in u_o|_\Omega + \mathrm{W}_0^{1,q}(\Omega)$, this can be re-written to

$$\min_{u\in K_{\psi^*}(\Omega)} \mathrm{TV}_\Omega(u) = \max_{\sigma\in S_-^\infty(\Omega)} \int_\Omega \sigma\cdot \mathrm{D}\psi\,\mathrm{d}x\,, \tag{3.8}$$

see Lemma 2.5.

As a second model case, we consider the minimization problem for the non-parametric area functional, again in a generalized formulation based on the functional $\mathcal{A}_{\Psi;\Omega}$ from (2.13). The following theorem is the equivalent of Theorem 3.1.

Theorem 3.5 *(Area duality as the limit $p \searrow 1$ of p-Laplace type obstacle problems). Under the assumptions of Theorem 3.1, it holds*

$$\min_{u\in \mathrm{BV}_{u_o}(\Omega)} \mathcal{A}_{\Psi;\Omega}(u) = \max_{\sigma\in S_-^\infty(\Omega)} \left([\![\sigma, \mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega}) + \int_\Omega \sqrt{1-|\sigma|^2}\,\mathrm{d}x \right), \tag{3.9}$$

and every minimizer of the left-hand side is bounded with (3.4). Moreover, there is a minimizer–maximizer pair (u,σ) for (3.9) that arises as the limit of solutions u_p of obstacle problems for the non-degenerate p-Laplacian with obstacle Ψ in the sense that

$$\begin{cases} u_{p_i} \overset{*}{\rightharpoonup} u & \text{weakly* in } \mathrm{BV}_{u_o}(\Omega)\,, \\ (1+|\mathrm{D}u_{p_i}|^2)^{\frac{p_i-2}{2}}\mathrm{D}u_{p_i} \rightharpoonup \sigma & \text{weakly in } \mathrm{L}^{\frac{q}{q-1}}(\Omega,\mathbb{R}^n)\,, \end{cases}$$

holds for some sequence $p_i \searrow 1$. Here, by the solution of the obstacle problem for the non-degenerate p-Laplacian we mean the uniquely determined map $u_p \in u_o|_\Omega + \mathrm{W}_0^{1,p}(\Omega)$ such that there hold the Cap_p-q.e. inequality $u_p \geq \Psi$ and the minimality property

$$\int_\Omega (1+|\mathrm{D}u_p|^2)^{\frac{p}{2}}\,\mathrm{d}x \leq \int_\Omega (1+|\mathrm{D}v|^2)^{\frac{p}{2}}\,\mathrm{d}x$$

for all $v \in u_o|_\Omega + \mathrm{W}_0^{1,p}(\Omega)$ that Cap_p-q.e. satisfy $v \geq \Psi$.

We finally remark that the area case differs from the total variation case insofar that the functional on the right-hand side of (3.9) is strictly concave and its maximizer σ is uniquely determined.

3.2. Duality correspondence for even more general obstacles

When compared to the preceding theorems, our next main result is concerned with obstacle functions that are more general in two aspects. First, $\Psi : \overline{\Omega} \to \mathbb{R}$ needs only be Cap_1-quasi upper semicontinuous instead of Cap_q-quasi upper semicontinuous, which is the more natural assumption for obstacle problems with linear growth and allows in particular the application to BV obstacles (cf. Remark 3.8). Second, we can treat obstacles that are not compatible with the boundary values in the sense that $\Psi > u_o$ may hold on the boundary (or parts thereof). The corresponding obstacle problems do still admit minimizers because functions $u \in \mathrm{BV}_{u_o}(\Omega)$ can develop jumps along the boundary $\partial\Omega$. However, in the dual formulation on the right-hand side of (3.10) below one has to use the modified distributional product $[\![\sigma, \mathrm{D}\Psi]\!]^*_{u_o}$, whose relevance will be discussed in Remark 3.7 below.

Theorem 3.6 *(Duality for general TV obstacle problems). Consider a bounded open set $\Omega \subset \mathbb{R}^n$ with (2.1), $u_o \in \mathrm{W}^{1,1}(\mathbb{R}^n) \cap \mathrm{L}^\infty(\Omega)$, and an obstacle $\Psi \in \mathrm{L}^\infty(\overline{\Omega}; \mathcal{H}^{n-1})$ that is Cap_1-quasi upper semicontinuous on $\overline{\Omega}$. Then we have*

$$\min_{u\in \mathrm{BV}_{u_o}(\Omega)} \mathrm{TV}_{\Psi;\Omega}(u) = \max_{\sigma\in S_-^\infty(\Omega)} [\![\sigma, \mathrm{D}\Psi]\!]^*_{u_o}(\overline{\Omega})\,. \tag{3.10}$$

Moreover, every minimizer $u \in \mathrm{BV}_{u_o}(\Omega)$ of the left-hand side is bounded with (3.4).

Remark 3.7 *(On up-to-the-boundary obstacles).* The Anzellotti type pairing $[\![\sigma, \mathrm{D}\Psi]\!]^*_{u_o}$ already appeared in [34,35], where it was shown to be the natural pairing for the definition of weak super-1-harmonicity up to the boundary. Here, a corresponding definition has been given in Section 2.5, and the term on the right-hand side of (3.10) can be written out as

$$[\![\sigma, \mathrm{D}\Psi]\!]^*_{u_o}(\overline{\Omega}) = \int_\Omega (\Psi - u_o^*)\,\mathrm{d}(-\operatorname{div}\sigma) + \int_\Omega \sigma \cdot \mathrm{D}u_o\,\mathrm{d}x + \int_{\partial\Omega} (1-\sigma_{\mathrm{n}}^*)[\Psi - (u_o)^{\mathrm{int}}_{\partial\Omega}]_+ \mathrm{d}\mathcal{H}^{n-1}\,, \tag{3.11}$$

where the normal trace $\sigma_{\mathrm{n}}^* \in \mathrm{L}^\infty(\partial\Omega; \mathcal{H}^{n-1})$ of $\sigma \in S^\infty_-(\Omega)$ exists by Lemma 2.5. The main difference to the pairing $[\![\sigma, \mathrm{D}\Psi]\!]_{u_o}$ and the previous duality formulas thus lies in the occurrence of the boundary integral on the right-hand side of (3.11), which can be interpreted in a purely heuristic way as follows. Clearly, the boundary integral vanishes if the obstacle is compatible with the boundary values in the sense $\Psi \le u_o$ on $\partial\Omega$. On portions of $\partial\Omega$ with $\Psi > u_o$, the minimizers $u \in \mathrm{BV}_{u_o}(\Omega)$ are forced to jump up from u_o to Ψ (seen from the outside of Ω), so that one expects the gradient $\mathrm{D}u$ to point in the direction of the inner unit normal to $\partial\Omega$. Since σ can be understood as a generalization of $\frac{\mathrm{D}u}{|\mathrm{D}u|}$, this means that σ should coincide with the inner unit normal and its normal trace (with respect to the inward normal) should equal 1 on boundary portions where $\Psi > u_o$. In this light, one may say that the boundary integral in (3.11) measures on one hand the incompatibility of Ψ and u_o and on the other hand the deviation of the normal trace σ_{n}^* from the expected value 1 on the relevant boundary portions.

Remark 3.8 *(On* BV *obstacles).* For an obstacle $\psi \in \mathrm{BV}_{u_o}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ the preceding theorem is applicable since the representative $\Psi = \psi^+$ of ψ is Cap_1-upper semicontinuous by Lemma 2.15. Similarly to the situation considered in Remark 3.4, in this case we have $\mathrm{TV}_{\Psi;\Omega}(u) = \infty$ whenever $u \notin K_{\psi^+}(\Omega)$. Consequently, for obstacle functions $\psi \in \mathrm{BV}_{u_o}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, the duality formula (3.10) can be written in the equivalent form

$$\min_{u \in K_{\psi^+}(\Omega)} \mathrm{TV}_\Omega(u) = \max_{\sigma \in S^\infty_-(\Omega)} [\![\sigma, \mathrm{D}\psi^+]\!]^*_{u_o}\big(\overline{\Omega}\big)\,. \tag{3.12}$$

Replacing the total variation by the area functional in the situation of Theorem 3.6, we get the following result.

Theorem 3.9 *(Duality for area minimization with general obstacles). Under the assumptions of Theorem 3.6, we have*

$$\min_{u \in \mathrm{BV}_{u_o}(\Omega)} \mathcal{A}_{\Psi;\Omega}(u) = \max_{\sigma \in S^\infty_-(\Omega)} \left([\![\sigma, \mathrm{D}\Psi]\!]^*_{u_o}(\overline{\Omega}) + \int_\Omega \sqrt{1-|\sigma|^2}\,\mathrm{d}x \right). \tag{3.13}$$

Moreover, every minimizer $u \in \mathrm{BV}_{u_o}(\Omega)$ of the left-hand side is bounded with the bounds (3.4).

3.3. Optimality relations with measure data for minimizer–maximizer pairs

As a consequence of the duality formula (3.10), the solutions of general TV obstacle problems can be seen to solve a 1-Laplace equation with a measure datum on the right-hand side in the sense of the following corollary, which involves the pairing $[\![\sigma, \mathrm{D}u^+]\!]^*_{u_o}$ from Section 2.5.

Corollary 3.10 *(Optimality relations for general* TV *obstacle problems). Under the assumptions of Theorem 3.6, consider a pair of competitors $(u, \sigma) \in \mathrm{BV}_{u_o}(\Omega) \times S^\infty_-(\Omega)$. Then, (u, σ) is a minimizer–maximizer pair in (3.10) if and only if it solves the equation*

$$[\![\sigma, \mathrm{D}u^+]\!]^*_{u_o} = |\mathrm{D}u| \text{ on } \overline{\Omega}\,, \qquad -\operatorname{div}\sigma = \mu \text{ in } \mathscr{D}'(\Omega) \tag{3.14}$$

with a non-negative Radon measure μ on Ω such that

$$\mu \equiv 0 \text{ on the non-contact set } \Omega \cap \{u^+ > \Psi\}\,, \tag{3.15}$$

$$\mu = \varsigma \text{ on the exceptional set } \Omega \cap \{u^+ < \Psi\}\,, \tag{3.16}$$

and if (u, σ) also satisfies $\mathcal{H}^{n-1}$-a.e. the boundary coupling conditions

$$\sigma_{\mathrm{n}}^* \equiv 1 \text{ on } \partial\Omega \cap \left\{u^{\mathrm{int}}_{\partial\Omega} > \max\{\Psi, (u_o)^{\mathrm{int}}_{\partial\Omega}\}\right\}, \tag{3.17}$$

$$\sigma_{\mathrm{n}}^* \equiv -1 \text{ on } \partial\Omega \cap \left\{u^{\mathrm{int}}_{\partial\Omega} < \max\{\Psi, (u_o)^{\mathrm{int}}_{\partial\Omega}\}\right\}. \tag{3.18}$$

The proof of this corollary is given in Section 5.3.

Several remarks on interpretations and reformulations of the conditions (3.14)–(3.18) are in order. However, before coming to these points, we briefly clarify that the conditions (3.15) and (3.16) make sense even though $\Omega \cap \{u^+ > \Psi\}$ and $\Omega \cap \{u^+ < \Psi\}$ are only defined up to $\mathcal{H}^{n-1}$-negligible sets, simply because μ and ς vanish on $\mathcal{H}^{n-1}$ null sets, see Lemma 2.1 and formula (2.11) in the preliminaries section. Moreover, we find it worth pointing out that the first equality of measures in (3.14) can be replaced by the equality of numbers $[\![\sigma, \mathrm{D}u^+]\!]^*_{u_o}(\overline{\Omega}) = |\mathrm{D}u|(\overline{\Omega})$, which may seem weaker, but is actually equivalent by estimate (2.6).

Remark 3.11 *(On (3.14) and super-1-harmonicity up to the boundary).* In the terminology introduced in [34,35], the validity of (3.14) for some $\sigma \in \mathcal{S}^\infty_-(\Omega)$ and $\mu \geq 0$ means exactly that $u \in \mathrm{BV}_{u_o}(\Omega)$ is weakly super-1-harmonic on $\overline{\Omega}$ with respect to the Dirichlet datum u_o.

In order to further explicate this point of view we record that the first condition in (3.14) can be split in an interior statement on Ω and a boundary statement on $\partial\Omega$. The interior information is just that $[\![\sigma, \mathrm{D}u^+]\!] = |\mathrm{D}u|$ holds on Ω (for a local pairing $[\![\sigma, \mathrm{D}u^+]\!]$), and together with the second condition in (3.14) this means that u is weakly super-1-harmonic on Ω in the terminology of [34,35]. The boundary information in (3.14) turns out to be that $\sigma^*_{\mathrm{n}} \equiv -1$ holds $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega \cap \{u^{\mathrm{int}}_{\partial\Omega} < (u_o)^{\mathrm{int}}_{\partial\Omega}\}$, cf. (2.8). This last piece of information is also contained in (3.18) and is thus redundant — but this one redundancy in the statement seems acceptable in order to gather all information related to super-1-harmonicity in (3.14) and all boundary information in (3.17) and (3.18).

Remark 3.12 *(On obstacles compatible with the boundary condition).* If the obstacle Ψ respects the Dirichlet boundary datum u_o in the sense of the $\mathcal{H}^{n-1}$-a.e. inequality

$$\Psi \leq (u_o)^{\mathrm{int}}_{\partial\Omega} \qquad \text{on } \partial\Omega\,, \tag{3.19}$$

the statement of Corollary 3.10 simplifies in some regards.

So, we record that the boundary coupling conditions in (3.17) and (3.18) then read as

$$\sigma^*_{\mathrm{n}} \equiv 1 \text{ on } \partial\Omega \cap \{u^{\mathrm{int}}_{\partial\Omega} > (u_o)^{\mathrm{int}}_{\partial\Omega}\}\,, \tag{3.20}$$

$$\sigma^*_{\mathrm{n}} \equiv -1 \text{ on } \partial\Omega \cap \{u^{\mathrm{int}}_{\partial\Omega} < (u_o)^{\mathrm{int}}_{\partial\Omega}\}\,. \tag{3.21}$$

This is precisely the boundary requirement which one imposes in defining 1-harmonicity on $\overline{\Omega}$ with regard to the Dirichlet datum u_o, and this seems indeed very reasonable in the presently considered case where the obstacle is inactive on $\partial\Omega$.

Moreover, (3.21) fully coincides with the boundary information in (3.14) and is thus contained already there. In addition, also (3.20) can be incorporated in a condition of type (3.14), simply by replacing $[\![\sigma, \mathrm{D}u^+]\!]^*_{u_o}$ with $[\![\sigma, \mathrm{D}u^+]\!]_{u_o}$ in the latter, cf. (2.7). All in all, under the compatibility condition (3.19), minimizer–maximizer pairs (u, σ) are thus characterized by the equalities

$$[\![\sigma, \mathrm{D}u^+]\!]_{u_o} = |\mathrm{D}u| \text{ on } \overline{\Omega}\,, \qquad -\operatorname{div}\sigma = \mu \text{ in } \mathscr{D}'(\Omega)$$

for some non-negative Radon measure μ on Ω with (3.15) and (3.16) — without any need for further conditions on $\partial\Omega$. In this case without boundary obstacles we thus recover the criteria specified in (1.11)–(1.12) in the introduction.

Finally, we state the corresponding result for the area functional. For the formulation of the measure data problem, we now need a generalized way of saying that $\sigma = \frac{\mathrm{D}u}{\sqrt{1+|\mathrm{D}u|^2}}$ holds a.e. on Ω. For a function $u \in \mathrm{W}^{1,1}(\Omega)$, this identity turns out to be equivalent to the $\mathcal{L}^n$-a.e. equality

$$\sigma \cdot \mathrm{D}u = \sqrt{1+|\mathrm{D}u|^2} - \sqrt{1-|\sigma|^2}\,,$$

see the discussion in Section 6.1. Therefore, the version of the preceding corollary in the case of the area functional reads as follows.

Corollary 3.13 *(Optimality relations for area minimization with general obstacles). In the situation of Theorem 3.6, consider $(u, \sigma) \in \mathrm{BV}_{u_o}(\Omega) \times \mathcal{S}^\infty_-(\Omega)$. Then, (u, σ) is a minimizer–maximizer pair in (3.13) if and only if it solves the equation*

$$[\![\sigma, \mathrm{D}u^+]\!]^*_{u_o} = \sqrt{1+|\mathrm{D}u|^2} - \sqrt{1-|\sigma|^2}\mathcal{L}^n \text{ on } \overline{\Omega}, \qquad -\operatorname{div}\sigma = \mu \text{ in } \mathscr{D}'(\Omega) \tag{3.22}$$

for some non-negative Radon measure μ on Ω such that (3.15), (3.16), (3.17), (3.18) hold.

If $\Psi \le (u_o)^{\mathrm{int}}_{\partial\Omega}$ holds $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, it is possible to replace $[\![\sigma, \mathrm{D}u^+]\!]^*_{u_o}$ by $[\![\sigma, \mathrm{D}u^+]\!]_{u_o}$ in this characterization. Furthermore, the statement remains generally valid if one requires (3.22) only on Ω instead of $\overline{\Omega}$ and thus without dependence on u_o (since the relevant boundary information is contained in (3.18) anyway).

Finally, we record that the dual solution σ in the area case is fully determined by u via the first relation in (3.22). This has already been pointed out in [35, Section 5], and it is in accordance with the uniqueness remark made after Theorem 3.5.

4. The p-Laplace obstacle problem with $p > 1$

In this section, we consider a bounded open set $\Omega \subset \mathbb{R}^n$, a boundary datum $u_o \in \mathrm{W}^{1,q}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, and a bounded Borel obstacle $\Psi : \Omega \to \mathbb{R}$ that satisfies (3.1) and (3.2) for some $q > 1$. For any $p \in (1, q]$ we introduce the admissible class

$$K^p_\Psi(\Omega) := \{u \in u_o + \mathrm{W}^{1,p}_0(\Omega) : u^* \ge \Psi \text{ holds } \mathrm{Cap}_p\text{-q.e. on } \Omega\}.$$

Definition 4.1. A function $u \in K^p_\Psi(\Omega)$ is called a solution to the obstacle problem for the p-Laplacian with obstacle Ψ if it satisfies

$$\int_\Omega |\mathrm{D}u|^p \,\mathrm{d}x \le \int_\Omega |\mathrm{D}v|^p \,\mathrm{d}x \qquad \text{for all } v \in K^p_\Psi(\Omega). \tag{4.1}$$

The existence of solutions in the above sense follows by classical arguments, which we briefly recall for the convenience of the reader. First, one observes that the set $K^p_\Psi(\Omega)$ is non-empty by (3.2). Moreover, $K^p_\Psi(\Omega)$ is closed with respect to the strong topology of $\mathrm{W}^{1,p}(\Omega)$ because every sequence with $u_i \to u$ in $\mathrm{W}^{1,p}(\Omega)$ in the limit $i \to \infty$ has a subsequence that converges Cap_p-quasi-everywhere, cf. Lemma 2.17. As a consequence of Mazur's lemma and the convexity of $K^p_\Psi(\Omega)$, this set is even (sequentially) closed with respect to the weak topology of $\mathrm{W}^{1,p}(\Omega)$. Therefore, the existence of a solution $u \in K^p_\Psi(\Omega)$ follows by the direct method of the calculus of variations. By the strict convexity of the p-energy integral in (4.1), it is also clear that u is the *unique* solution.

Replacing $v \in K^p_\Psi(\Omega)$ by $u + t(v-u) \in K^p_\Psi(\Omega)$ for $t > 0$, dividing by t and letting $t \searrow 0$, we deduce that the solution u satisfies the variational inequality

$$\int_\Omega |\mathrm{D}u|^{p-2}\mathrm{D}u \cdot \mathrm{D}(v-u)\,\mathrm{d}x \ge 0 \qquad \text{for all } v \in K^p_\Psi(\Omega). \tag{4.2}$$

The following lemma ensures that, for bounded data, the solutions to the obstacle problems are also bounded.

Lemma 4.2. *For boundary data $u_o \in \mathrm{W}^{1,q}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and a bounded Borel function $\Psi : \Omega \to \mathbb{R}$ with (3.1) and (3.2), the solution $u \in K^p_\Psi(\Omega)$ of the obstacle problem (4.1) satisfies*

$$\inf_\Omega u_o \le u \le \max\left\{p\text{-}\sup\nolimits_\Omega \Psi,\ \sup\nolimits_\Omega u_o\right\} \qquad \text{a.e. on } \Omega,$$

where p-$\sup_\Omega \Psi$ is the essential supremum of Ψ up to a set of zero p-capacity.

Proof. If we had $u > M := \max\{p\text{-}\sup_\Omega \Psi, \sup_\Omega u_o\}$ on a set of positive Lebesgue measure, then the truncated function $\hat{u} := \min\{u, M\} \in K^p_\Psi(\Omega)$ would be an admissible competitor with

$$\int_\Omega |\mathrm{D}\hat{u}|^p \,\mathrm{d}x < \int_\Omega |\mathrm{D}u|^p \,\mathrm{d}x,$$

which would contradict the minimality of u. Analogously, if we had $u < m := \inf_\Omega u_o$ on a set of positive measure, then $\check{u} := \max\{u, m\}$ would be an admissible competitor with strictly smaller energy. $\square$

For the dual formulation of the obstacle problem, we introduce the space

$$S_-^{p'}(\Omega) := \{\sigma \in \mathrm{L}^{p'}(\Omega, \mathbb{R}^n) \,:\, \operatorname{div}\sigma \le 0 \text{ in } \mathscr{D}'(\Omega)\}\,,$$

where we used the customary notation $p' := \frac{p}{p-1}$. For $\sigma \in S_-^{p'}(\Omega)$, the Riesz representation theorem implies that the distribution $\operatorname{div}\sigma$ is a non-positive Radon measure, and by a well-known reasoning[3] this measure vanishes on every set of p-capacity zero.

In view of these observations, the distributional product $[\![\sigma, \mathrm{D}U]\!]_{u_o}$ in Definition 2.3 is also meaningful in the slightly different setting of this section, that is for $u_o \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, $\sigma \in S_-^{p'}(\Omega)$ and a bounded Borel function U on Ω which is at least Cap_p-q.e. defined. For $u \in u_o + \mathrm{W}_0^{1,p}(\Omega)$, specifically, the product $[\![\sigma, \mathrm{D}u^*]\!]_{u_o}$ corresponds to the function $\sigma \cdot \mathrm{D}u \in \mathrm{L}^1(\Omega)$, as it follows from the integration-by-parts formula

$$\int_\Omega \varphi^* \,\mathrm{d}(-\operatorname{div}\sigma) = \int_\Omega \sigma \cdot \mathrm{D}\varphi \,\mathrm{d}x\,. \tag{4.3}$$

Indeed, this formula is trivially valid for $\sigma \in S_-^{p'}(\Omega)$ and $\varphi \in \mathscr{D}(\Omega)$, but it carries over to arbitrary $\varphi \in \mathrm{W}_0^{1,p}(\Omega)$ by strong approximation in $\mathrm{W}_0^{1,p}$ and Lemma 2.17.

Now we can state and prove the duality formula for the p-Laplace obstacle problem.

Proposition 4.3. *Consider a bounded open set $\Omega \subset \mathbb{R}^n$, some $u_o \in \mathrm{W}^{1,q}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, and a bounded Borel function $\Psi : \Omega \to \mathbb{R}$ with (3.1) and (3.2) for some $q > 1$. Then, for any $p \in (1, q]$, we have*

$$\min_{u \in K_\Psi^p(\Omega)} \frac{1}{p} \int_\Omega |\mathrm{D}u|^p \,\mathrm{d}x = \max_{\sigma \in S_-^{p'}(\Omega)} \left([\![\sigma, \mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega}) - \frac{1}{p'} \int_\Omega |\sigma|^{p'} \,\mathrm{d}x \right). \tag{4.4}$$

In fact, for the minimizer $u \in K_\Psi^p(\Omega)$ of the left-hand side and $\sigma := |\mathrm{D}u|^{p-2}\mathrm{D}u \in S_-^{p'}(\Omega)$ we have

$$\frac{1}{p} \int_\Omega |\mathrm{D}u|^p \,\mathrm{d}x = [\![\sigma, \mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega}) - \frac{1}{p'} \int_\Omega |\sigma|^{p'} \,\mathrm{d}x\,.$$

Proof. We begin by considering arbitrary functions $\sigma \in S_-^{p'}(\Omega)$ and $u \in K_\Psi^p(\Omega)$. Since $-\operatorname{div}\sigma$ is a non-negative Radon measure that vanishes on sets with p-capacity zero, we know

$$\int_\Omega (u^* - \Psi) \,\mathrm{d}(-\operatorname{div}\sigma) \ge 0\,,$$

from which we infer

$$\begin{aligned}
[\![\sigma, \mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega}) &= \int_\Omega (\Psi - u_o^*) \,\mathrm{d}(-\operatorname{div}\sigma) + \int_\Omega \sigma \cdot \mathrm{D}u_o \,\mathrm{d}x \\
&\le \int_\Omega (u^* - u_o^*) \,\mathrm{d}(-\operatorname{div}\sigma) + \int_\Omega \sigma \cdot \mathrm{D}u_o \,\mathrm{d}x \\
&= \int_\Omega \mathrm{D}u \cdot \sigma \,\mathrm{d}x
\end{aligned}$$

[3] Indeed, for *compact* $E \subset \mathbb{R}^n$ and $\varepsilon > 0$, via the definition of Cap_p one finds some $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\varphi \ge \mathbb{1}_E$ and $\|\mathrm{D}\varphi\|_{\mathrm{L}^p(\mathbb{R}^n,\mathbb{R}^n)}^p \le \mathrm{Cap}_p(E) + \varepsilon$. This gives $(-\operatorname{div}\sigma)(E) \le \int_{\mathbb{R}^n} \varphi \,\mathrm{d}(-\operatorname{div}\sigma) = \int_{\mathbb{R}^n} \sigma \cdot \mathrm{D}\varphi \,\mathrm{d}x \le (\mathrm{Cap}_p(E)+\varepsilon)^{1/p} \|\sigma\|_{\mathrm{L}^{p'}(\Omega,\mathbb{R}^n)}$, and, by inner regularity of $-\operatorname{div}\sigma$, the resulting inequality $(-\operatorname{div}\sigma)(E) \le \mathrm{Cap}_p(E)^{1/p} \|\sigma\|_{\mathrm{L}^{p'}(\Omega,\mathbb{R}^n)}$ holds true even for arbitrary $E \subset \mathbb{R}^n$.

$$\leq \int_\Omega \left(\tfrac{1}{p}|\mathrm{D}u|^p + \tfrac{1}{p'}|\sigma|^{p'}\right) \mathrm{d}x\,.$$

In the last two steps, we used formula (4.3) and Young's inequality. All in all, we deduce

$$\min_{u\in K^p_\Psi(\Omega)} \frac{1}{p}\int_\Omega |\mathrm{D}u|^p\,\mathrm{d}x \geq \sup_{\sigma\in S^{p'}_-(\Omega)} \left([\![\sigma,\mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega}) - \frac{1}{p'}\int_\Omega |\sigma|^{p'}\,\mathrm{d}x \right). \tag{4.5}$$

For the reverse inequality, we choose the solution $u \in K^p_\Psi(\Omega)$ of the obstacle problem for the p-Laplacian with obstacle Ψ and let $\sigma := |\mathrm{D}u|^{p-2}\mathrm{D}u$. Then we have

$$\begin{aligned}\frac{1}{p}\int_\Omega |\mathrm{D}u|^p\,\mathrm{d}x &= \int_\Omega |\mathrm{D}u|^{p-2}\mathrm{D}u\cdot\mathrm{D}(u-u_o)\,\mathrm{d}x + \int_\Omega |\mathrm{D}u|^{p-2}\mathrm{D}u\cdot\mathrm{D}u_o\,\mathrm{d}x - \frac{1}{p'}\int_\Omega |\mathrm{D}u|^p\,\mathrm{d}x\\ &= \int_\Omega (u^*-u_o^*)\,\mathrm{d}(-\operatorname{div}\sigma) + \int_\Omega \sigma\cdot\mathrm{D}u_o\,\mathrm{d}x - \frac{1}{p'}\int_\Omega |\sigma|^{p'}\,\mathrm{d}x\,.\end{aligned}$$

Now we choose a non-increasing sequence $\psi_k \in u_o + \mathrm{W}^{1,q}_0(\Omega)$ such that $\psi_k^* \to \Psi$ converges Cap_q-q.e. on Ω. Such a sequence exists by Lemma 2.19 and thanks to our assumptions (3.1) and (3.2). Via monotone convergence and the variational inequality (4.2) for the solution u, we infer

$$\begin{aligned}\int_\Omega (\Psi-u^*)\,\mathrm{d}(-\operatorname{div}\sigma) &= \lim_{k\to\infty}\int_\Omega (\psi_k^*-u^*)\,\mathrm{d}(-\operatorname{div}\sigma)\\ &= \lim_{k\to\infty}\int_\Omega |\mathrm{D}u|^{p-2}\mathrm{D}u\cdot\mathrm{D}(\psi_k-u)\,\mathrm{d}x \geq 0\,.\end{aligned}$$

Joining the two preceding formulas, we arrive at

$$\begin{aligned}\frac{1}{p}\int_\Omega |\mathrm{D}u|^p\,\mathrm{d}x &\leq \int_\Omega (\Psi-u_o^*)\,\mathrm{d}(-\operatorname{div}\sigma) + \int_\Omega \sigma\cdot\mathrm{D}u_o\,\mathrm{d}x - \frac{1}{p'}\int_\Omega |\sigma|^{p'}\,\mathrm{d}x\\ &= [\![\sigma,\mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega}) - \frac{1}{p'}\int_\Omega |\sigma|^{p'}\,\mathrm{d}x\,.\end{aligned}$$

This proves the reverse inequality in (4.5) and shows moreover that the supremum on the right-hand side is attained for $\sigma = |\mathrm{D}u|^{p-2}\mathrm{D}u$. $\square$

5. The obstacle problem for the total variation

5.1. *The duality formula in the limit $p \searrow 1$*

This section is devoted to the proof of Theorem 3.1. We still consider $u_o \in \mathrm{W}^{1,q}(\mathbb{R}^n) \cap \mathrm{L}^\infty(\Omega)$ and a bounded Borel function $\Psi : \Omega \to \mathbb{R}$ with (3.1) and (3.2) for some $q > 1$. Then we analyze the asymptotic behavior of the solutions to the obstacle problems for the p-Laplacian with data u_o and Ψ in the limit $p \searrow 1$.

Proof of Theorem 3.1. The proof is divided into four steps.

Step 1. The easy inequality in the duality formula. We begin by considering arbitrary $u \in \mathrm{BV}_{u_o}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and $\sigma \in S^\infty_-(\Omega)$. We use Proposition 2.8 and Proposition 2.11 with $U = \Psi - u^+$ in order to estimate

$$\begin{aligned}[\![\sigma,\mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega}) &= [\![\sigma,\mathrm{D}u^+]\!]_{u_o}(\overline{\Omega}) + \int_\Omega (\Psi-u^+)\mathrm{d}(-\operatorname{div}\sigma)\\ &\leq |\mathrm{D}u|(\overline{\Omega}) + \int_\Omega (\Psi-u^+)_+\mathrm{d}\varsigma = \mathrm{TV}_{\Psi;\Omega}(u)\,.\end{aligned} \tag{5.1}$$

Moreover, we observe that every minimizer $u \in \mathrm{BV}_{u_o}(\Omega)$ of $\mathrm{TV}_{\Psi;\Omega}$ satisfies the bounds

$$m := \inf_\Omega u_o \le u \le \max\left\{1\text{-}\sup_{\overline{\Omega}} \Psi, \sup_\Omega u_o\right\} =: M \qquad \text{a.e. in } \Omega\,. \tag{5.2}$$

In fact, if one of these inequalities did not hold on a set of positive measure, the truncation $\widehat{u} := \max\{\min\{u, M\}, m\}$ would satisfy $\mathrm{TV}_{\Psi;\Omega}(\widehat{u}) < \mathrm{TV}_{\Psi;\Omega}(u)$, which contradicts the minimality property of u. This implies the claim (3.4) and also shows

$$\inf_{u \in \mathrm{BV}_{u_o}(\Omega)\cap \mathrm{L}^\infty(\Omega)} \mathrm{TV}_{\Psi;\Omega}(u) \quad = \inf_{u \in \mathrm{BV}_{u_o}(\Omega)} \mathrm{TV}_{\Psi;\Omega}(u)\,. \tag{5.3}$$

Combining (5.3) with (5.1), we infer the estimate

$$\inf_{u \in \mathrm{BV}_{u_o}(\Omega)} \mathrm{TV}_{\Psi;\Omega}(u) \quad \ge \sup_{\sigma \in \mathcal{S}^\infty_{\ge}(\Omega)} [\![\sigma, \mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega})\,, \tag{5.4}$$

which yields one inequality of the claimed identity (3.3). It remains to prove the reverse inequality and to show that the infimum and the supremum are attained. To this end, we analyze the asymptotic behavior of the solutions to the p-Laplacian obstacle problems as $p \searrow 1$.

Step 2. Solutions to the obstacle problems for $p > 1$. For every $p \in (1, q)$, we choose $u_p \in K^p_\Psi(\Omega)$ as the solution of the obstacle problem for the p-Laplacian in the sense of Definition 4.1. From Lemma 4.2 we infer $u_p \in \mathrm{L}^\infty(\Omega)$ with

$$\inf_\Omega u_o \le u_p \le \max\{\sup_\Omega \Psi\,, \sup_\Omega u_o\} \qquad \text{a.e. on } \Omega\,, \tag{5.5}$$

for every $p \in (1, q)$ (where, for simplicity, we estimated p-$\sup_\Omega \Psi$ by the pointwise supremum $\sup_\Omega \Psi$). Moreover, by using $v = g$ as comparison map in the minimality condition (4.1) for u_p, where $g \in u_o|_\Omega + \mathrm{W}^{1,q}_0(\Omega) \subset u_o|_\Omega + \mathrm{W}^{1,p}_0(\Omega)$ is provided by assumption (3.2), we deduce

$$\int_\Omega |\mathrm{D}u_p|\,\mathrm{d}x \le c\Big(\int_\Omega |\mathrm{D}u_p|^p\,\mathrm{d}x\Big)^{\frac{1}{p}} \le c\Big(\int_\Omega |\mathrm{D}g|^p\,\mathrm{d}x\Big)^{\frac{1}{p}} \le c\Big(\int_\Omega |\mathrm{D}g|^q\,\mathrm{d}x\Big)^{\frac{1}{q}} \tag{5.6}$$

for every $p \in (1, q)$. Here the constant c can be chosen only in dependence on q and $|\Omega|$.

Step 3. Letting $p \to 1$. Extending u_p by u_o outside of Ω, we can interpret u_p as element of $\mathrm{BV}_{u_o}(\Omega)$. The bounds (5.5) and (5.6) enable us to extract a subsequence $p_i \to 1$ with

$$\begin{cases} u_{p_i} \overset{*}{\rightharpoondown} u & \text{weakly}{*} \text{ in } \mathrm{BV}_{u_o}(\Omega)\,, \\ u_{p_i} \overset{*}{\rightharpoondown} u & \text{weakly}{*} \text{ in } \mathrm{L}^\infty(\Omega)\,, \end{cases} \tag{5.7}$$

as $i \to \infty$, for some limit map $u \in \mathrm{BV}_{u_o}(\Omega) \cap \mathrm{L}^\infty(\Omega)$. Due to (5.6), the functions $\sigma_i := |\mathrm{D}u_{p_i}|^{p_i-2}\mathrm{D}u_{p_i}$ satisfy

$$\sup_{i\in\mathbb{N}} \int_\Omega |\sigma_i|^{p_i'}\,\mathrm{d}x < \infty\,. \tag{5.8}$$

Possibly passing to another subsequence, we can therefore assume

$$\sigma_i \rightharpoondown \sigma \qquad \text{weakly in } \mathrm{L}^{q'}(\Omega, \mathbb{R}^n) \tag{5.9}$$

as $i \to \infty$ for some $\sigma \in \mathrm{L}^{q'}(\Omega, \mathbb{R}^n)$. We claim that we have $\sigma \in \mathrm{L}^\infty(\Omega, \mathbb{R}^n)$ with $\|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \le 1$. To verify this claim, we consider the truncated sequence

$$T\sigma_i := \begin{cases} \dfrac{\sigma_i}{|\sigma_i|} = \dfrac{\mathrm{D}u_{p_i}}{|\mathrm{D}u_{p_i}|}\,, & \text{if } |\sigma_i| \ge 1\,, \\[2ex] \sigma_i = |\mathrm{D}u_{p_i}|^{p_i-2}\mathrm{D}u_{p_i}\,, & \text{if } |\sigma_i| < 1\,. \end{cases}$$

Clearly, we can also assume $T\sigma_i \xrightarrow{*} \widetilde{\sigma}$ in $\mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ as $i\to\infty$ for some $\widetilde{\sigma}\in\mathrm{L}^\infty(\Omega,\mathbb{R}^n)$, which satisfies $\|\widetilde{\sigma}\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}\le 1$ by lower semicontinuity of the norm with respect to weak$*$ convergence. In order to identify $\sigma=\widetilde{\sigma}$, we calculate for arbitrary $\lambda>1$

$$\begin{aligned}\int_\Omega |\sigma_i - T\sigma_i|\,\mathrm{d}x &= \int_{\Omega\cap\{|\mathrm{D}u_{p_i}|\ge 1\}} \Big|\frac{\mathrm{D}u_{p_i}}{|\mathrm{D}u_{p_i}|}\Big|\big(|\mathrm{D}u_{p_i}|^{p_i-1}-1\big)\,\mathrm{d}x\\ &= \int_{\Omega\cap\{1\le|\mathrm{D}u_{p_i}|\le\lambda\}} \big(|\mathrm{D}u_{p_i}|^{p_i-1}-1\big)\,\mathrm{d}x + \int_{\Omega\cap\{|\mathrm{D}u_{p_i}|>\lambda\}} \big(|\mathrm{D}u_{p_i}|^{p_i-1}-1\big)\,\mathrm{d}x\\ &=: I_i + II_i\,.\end{aligned}$$

The first integral on the right-hand side can be bounded by

$$I_i \le |\Omega|(\lambda^{p_i-1}-1) \qquad \text{for every } i\in\mathbb{N}\,. \tag{5.10}$$

For the estimate of the second integral, we recall (5.6), which yields

$$II_i \le \frac{1}{\lambda}\int_\Omega |\mathrm{D}u_{p_i}|^{p_i}\,\mathrm{d}x \le \frac{1}{\lambda}\int_\Omega |\mathrm{D}g|^{p_i}\,\mathrm{d}x \le \frac{c}{\lambda}\big(\|\mathrm{D}g\|^q_{\mathrm{L}^q(\Omega,\mathbb{R}^n)}+1\big)\,. \tag{5.11}$$

Collecting the estimates, we arrive at

$$\int_\Omega |\sigma_i - T\sigma_i|\,\mathrm{d}x \le |\Omega|(\lambda^{p_i-1}-1) + \frac{c}{\lambda}\big(\|\mathrm{D}g\|^q_{\mathrm{L}^q(\Omega,\mathbb{R}^n)}+1\big)\,.$$

The right-hand side can be made arbitrarily small by first choosing $\lambda>1$ large enough and then p_i sufficiently close to 1. Hence, we have shown

$$\sigma_i - T\sigma_i \to 0 \qquad \text{in } \mathrm{L}^1(\Omega,\mathbb{R}^n), \text{ as } i\to\infty\,.$$

But this implies $\sigma=\widetilde{\sigma}\in\mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ and consequently,

$$\|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} = \|\widetilde{\sigma}\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \le 1\,, \tag{5.12}$$

as claimed. Next, we recall that the approximating solutions u_{p_i} satisfy

$$\int_\Omega \sigma_i\cdot\mathrm{D}\varphi\,\mathrm{d}x = \int_\Omega |\mathrm{D}u_{p_i}|^{p_i-2}\mathrm{D}u_{p_i}\cdot\mathrm{D}\varphi\,\mathrm{d}x \ge 0$$

for all $\varphi\in\mathscr{D}(\Omega)$ with $\varphi\ge 0$ on Ω. Passing to the limit $i\to\infty$, this implies

$$\int_\Omega \sigma\cdot\mathrm{D}\varphi \ge 0 \qquad \text{for all } \varphi\in\mathscr{D}(\Omega) \text{ with } \varphi\ge 0\,.$$

Keeping in mind (5.12), we thereby deduce $\sigma\in S^\infty_-(\Omega)$.

Step 4. The limit maps are extremal points. Since the solutions $u_{p_i}\in\mathrm{W}^{1,p_i}(\Omega)$ satisfy Cap_1-q.e. the obstacle constraint $u^*_{p_i}\ge\Psi$, we have $\mathrm{TV}_{\Psi;\Omega}(u_{p_i})=\int_\Omega|\mathrm{D}u_{p_i}|\,\mathrm{d}x$. Using the lower semicontinuity of $\mathrm{TV}_{\Psi;\Omega}$ with respect to weak$*$ convergence in BV, which is guaranteed by Theorem 2.13, we hence deduce

$$\begin{aligned}\mathrm{TV}_{\Psi;\Omega}(u) &\le \liminf_{i\to\infty}\int_\Omega |\mathrm{D}u_{p_i}|\,\mathrm{d}x \le \liminf_{i\to\infty}|\Omega|^{1-\frac{1}{p_i}}\Big(\int_\Omega |\mathrm{D}u_{p_i}|^{p_i}\,\mathrm{d}x\Big)^{\frac{1}{p_i}}\\ &= \liminf_{i\to\infty}\Big(\frac{1}{p_i}\int_\Omega |\mathrm{D}u_{p_i}|^{p_i}\,\mathrm{d}x\Big)^{\frac{1}{p_i}}\,.\end{aligned} \tag{5.13}$$

From Proposition 4.3 we know

$$\frac{1}{p_i}\int_\Omega |\mathrm{D}u_{p_i}|^{p_i}\,\mathrm{d}x = [\![\sigma_i,\mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega}) - \frac{1}{p_i'}\int_\Omega |\sigma_i|^{p_i'}\,\mathrm{d}x \tag{5.14}$$

for every $i\in\mathbb{N}$. Now, we join (5.13) with (5.14) and make use of the bound (5.8). This leads us to

$$\begin{aligned}\mathrm{TV}_{\Psi;\Omega}(u) &\le \liminf_{i\to\infty}\Big([\![\sigma_i,\mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega}) - \frac{1}{p_i'}\int_\Omega |\sigma_i|^{p_i'}\,\mathrm{d}x\Big)^{\frac{1}{p_i}}\\ &= \liminf_{i\to\infty}[\![\sigma_i,\mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega})\,. \end{aligned}\tag{5.15}$$

Next, we apply Lemma 2.19 to obtain a non-increasing sequence of functions $\psi_k\in u_o|_\Omega+\mathrm{W}^{1,q}_0(\Omega)$ such that ψ_k^* converges Cap_q-q.e. on Ω to Ψ. We subsequently use the Cap_q-q.e. inequality $\Psi\le\psi_k^*$ on Ω, the triviality of the pairing on $\mathrm{W}^{1,q}$-functions (see Lemma 2.5 or (4.3)), the convergence (5.9), and again the triviality of the pairing on $\mathrm{W}^{1,q}$ in order to estimate

$$\begin{aligned}\limsup_{i\to\infty}[\![\sigma_i,\mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega}) \le \limsup_{i\to\infty}[\![\sigma_i,\mathrm{D}\psi_k^*]\!]_{u_o}(\overline{\Omega}) &= \limsup_{i\to\infty}\int_\Omega \sigma_i\cdot\mathrm{D}\psi_k\,\mathrm{d}x\\ &= \int_\Omega \sigma\cdot\mathrm{D}\psi_k\,\mathrm{d}x = [\![\sigma,\mathrm{D}\psi_k^*]\!]_{u_o}(\overline{\Omega})\end{aligned}\tag{5.16}$$

for every $k\in\mathbb{N}$. Combining (5.15) and (5.16) and letting $k\to\infty$, we deduce

$$\mathrm{TV}_{\Psi;\Omega}(u)\le\lim_{k\to\infty}[\![\sigma,\mathrm{D}\psi_k^*]\!]_{u_o}(\overline{\Omega}) = [\![\sigma,\mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega})\,,\tag{5.17}$$

where the last equality follows from the definition of the pairing and the monotone Cap_q-q.e. (hence also $(-\operatorname{div}\sigma)$-a.e.) convergence $\psi_k^*\to\Psi$ on Ω. Combining this with (5.4), we deduce the claim (3.3). The asymptotic behavior (3.5) in the limit $p\searrow 1$ is satisfied by (5.7) and (5.9). This completes the proof of the theorem. □

5.2. *The duality formula for general obstacles*

In this section, we establish the general duality statement in Theorem 3.6.

Proof of Theorem 3.6. We consider $u_o\in\mathrm{W}^{1,1}(\mathbb{R}^n)\cap\mathrm{L}^\infty(\Omega)$ and, for a start, an arbitrary $\Psi\in\mathrm{L}^\infty(\overline{\Omega};\mathcal{H}^{n-1})$. From the Propositions 2.8, 2.11, 2.12, the definitions of the pairings, and the sublinearity of the function $x\mapsto x_+$, it follows that we have, for all $u\in\mathrm{BV}_{u_o}(\Omega)\cap\mathrm{L}^\infty(\Omega)$ and $\sigma\in\mathcal{S}^\infty_-(\Omega)$,

$$\begin{aligned}\mathrm{TV}_{\Psi;\Omega}(u) &= |\mathrm{D}u|(\overline{\Omega}) + \int_{\overline{\Omega}}(\Psi-u^+)_+\,\mathrm{d}\varsigma\\ &\ge [\![\sigma,\mathrm{D}u^+]\!]^*_{u_o}(\overline{\Omega}) + \int_\Omega(\Psi-u^+)\,\mathrm{d}(-\operatorname{div}\sigma) + \int_{\partial\Omega}(\Psi-u^+)_+(1-\sigma_{\mathrm{n}}^*)\,\mathrm{d}\mathcal{H}^{n-1}\\ &= [\![\sigma,\mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega}) + \int_{\partial\Omega}\big([u^+-(u_o)^{\mathrm{int}}_{\partial\Omega}]_+ + (\Psi-u^+)_+\big)(1-\sigma_{\mathrm{n}}^*)\,\mathrm{d}\mathcal{H}^{n-1}\\ &\ge [\![\sigma,\mathrm{D}\Psi]\!]^*_{u_o}(\overline{\Omega})\,.\end{aligned}$$

Moreover, by the truncation argument already used for the derivation of (5.2) we deduce

$$\inf_\Omega u_o\le u\le\max\big\{1\text{-}\sup_{\overline{\Omega}}\Psi,\sup_\Omega u_o\big\}\qquad\text{a.e. in }\Omega\tag{5.18}$$

for every minimizer $u\in\mathrm{BV}_{u_o}(\Omega)$ of $\mathrm{TV}_{\Psi;\Omega}$ and

$$\inf_{u\in\mathrm{BV}_{u_o}(\Omega)\cap\mathrm{L}^\infty(\Omega)}\mathrm{TV}_{\Psi;\Omega}(u) = \inf_{u\in\mathrm{BV}_{u_o}(\Omega)}\mathrm{TV}_{\Psi;\Omega}(u)\,.$$

Thus, we have verified (3.4) and

$$\inf_{u\in \mathrm{BV}_{u_o}(\Omega)} \mathrm{TV}_{\Psi;\Omega}(u) \ge \sup_{\sigma\in \mathcal{S}^\infty_-(\Omega)} [\![\sigma, \mathrm{D}\Psi]\!]^*_{u_o}\big(\overline{\Omega}\big)\,. \tag{5.19}$$

The reverse inequality and the fact that the infimum and supremum are attained will be verified in several steps for more and more general obstacles.

Step 1. $\mathrm{W}^{1,1}$ *obstacles* ψ *with* $\psi = u_o$ *on* $\partial\Omega$. Possibly replacing $u_o \in \mathrm{W}^{1,1}(\mathbb{R}^n)\cap \mathrm{L}^\infty(\Omega)$ with the truncation $\max\{\min\{u_o, \|u_o\|_{\mathrm{L}^\infty(\Omega)}\}, -\|u_o\|_{\mathrm{L}^\infty(\Omega)}\}$, we can assume that u_o is indeed bounded on the whole $\mathbb{R}^n$. Moreover, in this first step, we consider $\psi \in (u_o|_\Omega + \mathrm{W}^{1,1}_0(\Omega))\cap \mathrm{L}^\infty(\Omega)$. Extending ψ by u_o outside of Ω, we can understand $\psi\in \mathrm{W}^{1,1}(\mathbb{R}^n)\cap \mathrm{L}^\infty(\mathbb{R}^n)$. Standard approximation of this extension yields functions $\psi_k \in \mathrm{W}^{1,2}(\mathbb{R}^n)$ with $\psi_k \to \psi$ in $\mathrm{W}^{1,1}(\mathbb{R}^n)$, as $k\to\infty$. We can assume $\|\psi_k\|_{\mathrm{L}^\infty(\mathbb{R}^n)} \le \|\psi\|_{\mathrm{L}^\infty(\mathbb{R}^n)}$ by passing to the truncations $\max\{\min\{\psi_k, \sup_{\mathbb{R}^n}\psi\}, \inf_{\mathbb{R}^n}\psi\}$ if necessary. As new boundary data, we consider the same functions $u_{o,k} := \psi_k \in \mathrm{W}^{1,2}(\Omega)\cap \mathrm{L}^\infty(\Omega)$. In this way, we find approximating obstacles $\psi_k \in u_{o,k} + \mathrm{W}^{1,2}_0(\Omega)$, and we have the convergences

$$\begin{aligned} \psi_k &\to \psi && \text{in } \mathrm{W}^{1,1}(\Omega)\\ u_{o,k}\big|_{\mathbb{R}^n\setminus\overline{\Omega}} &\to u_o\big|_{\mathbb{R}^n\setminus\overline{\Omega}} && \text{in } \mathrm{W}^{1,1}(\mathbb{R}^n\setminus\overline{\Omega}) \end{aligned} \tag{5.20}$$

in the limit $k\to\infty$. From Theorem 3.1 and Remark 3.4 thereafter (compare (3.8), in particular), we obtain minimizer–maximizer pairs $(u_k,\sigma_k)\in K_{\psi_k^*,u_{o,k}}(\Omega)\times \mathcal{S}^\infty_-(\Omega)$ for the obstacle problems with obstacles ψ_k such that we have

$$\mathrm{TV}_\Omega(u_k) = \int_\Omega \sigma_k\cdot \mathrm{D}\psi_k\,\mathrm{d}x\,.$$

Moreover, since (5.18) applies to each of the u_k, we have

$$\|u_k\|_{\mathrm{L}^\infty(\Omega)} \le \max\left\{\sup\nolimits_\Omega \psi_k, \|u_{o,k}\|_{\mathrm{L}^\infty(\Omega)}\right\} \le \|\psi\|_{\mathrm{L}^\infty(\mathbb{R}^n)} \tag{5.21}$$

for every $k\in\mathbb{N}$. By minimality we have $|\mathrm{D}u_k|\big(\overline{\Omega}\big) \le \int_\Omega |\mathrm{D}\psi_k|\,\mathrm{d}x$, and hence the convergence (5.20) implies that the functions $u_k\in \mathrm{BV}_{u_{o,k}}(\Omega)$ are bounded in $\mathrm{BV}(\mathbb{R}^n)$. Moreover, by definition of $\mathcal{S}^\infty_-(\Omega)$, the σ_k are bounded in $\mathrm{L}^\infty(\Omega,\mathbb{R}^n)$. Possibly passing to subsequences, we can thus assume that u_k and σ_k weakly$*$ converge in $\mathrm{BV}(\mathbb{R}^n)\cap \mathrm{L}^\infty(\Omega)$ and $\mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ to limits $u_\infty\in \mathrm{BV}_{u_o}(\Omega)\cap \mathrm{L}^\infty(\Omega)$ and $\sigma_\infty\in \mathcal{S}^\infty_-(\Omega)$, respectively, and it follows that $u\ge\psi$ holds a.e. on Ω. Furthermore, relying on the above convergences, the lower semicontinuity of the total variation, and the bounds $|\sigma_k|\le 1$, we obtain

$$\begin{aligned} \mathrm{TV}_\Omega(u_\infty) &\le \liminf_{k\to\infty}\mathrm{TV}_\Omega(u_k) = \liminf_{k\to\infty}\int_\Omega \sigma_k\cdot\mathrm{D}\psi_k\,\mathrm{d}x\\ &\le \lim_{k\to\infty}\int_\Omega \sigma_k\cdot \mathrm{D}\psi\,\mathrm{d}x + \lim_{k\to\infty}\int_\Omega |\mathrm{D}(\psi_k-\psi)|\,\mathrm{d}x\\ &= \int_\Omega \sigma_\infty\cdot\mathrm{D}\psi\,\mathrm{d}x = [\![\sigma_\infty, \mathrm{D}\psi^+]\!]^*_{u_o}(\overline{\Omega})\,, \end{aligned}$$

where the last identity follows from Lemma 2.5 and the fact that $[\![\sigma_\infty,\mathrm{D}\psi^+]\!]_{u_o}$ equals $[\![\sigma_\infty,\mathrm{D}\psi^+]\!]^*_{u_o}$ under the present assumption at the boundary. Joining this with (5.19) for $\Psi=\psi^+$ and observing that the a.e. inequality $u_\infty\ge\psi$ suffices to guarantee $u_\infty\in K_{\psi^+,u_o}(\Omega)$, we deduce that $(u_\infty,\sigma_\infty)\in K_{\psi^+,u_o}(\Omega)\times\mathcal{S}^\infty_-(\Omega)$ is a minimizer–maximizer pair with the claimed L^∞-bound for u_∞ and that (3.10) holds for obstacles in $u_o|_\Omega+\mathrm{W}^{1,1}_0(\Omega)$.

Step 2. $\mathrm{W}^{1,1}$ *obstacles* ψ*, possibly with* $\psi > u_o$ *on* $\partial\Omega$. Next, we consider an obstacle $\psi\in \mathrm{W}^{1,1}(\Omega)\cap\mathrm{L}^\infty(\Omega)$ that need not agree with the boundary datum u_o on $\partial\Omega$, but satisfies only $(\psi-u_o)_-\in \mathrm{W}^{1,1}_0(\Omega)$. Since $\mathbb{1}_\Omega$ is lower semicontinuous, we can find an increasing sequence of Lipschitz functions $\eta_k\in \mathrm{C}^{0,1}_0(\Omega)$ with $\eta_k\to\mathbb{1}_\Omega$ pointwisely on Ω. We use these to define an increasing sequence of approximating obstacles by

$$\psi_k := u_o|_\Omega + \eta_k(\psi-u_o)_+ - (\psi-u_o)_- \in u_o|_\Omega + \mathrm{W}^{1,1}_0(\Omega)\,.$$

From Step 1 we infer the existence of solutions $(u_k,\sigma_k)\in K_{\psi_k^+,u_o}(\Omega)\times S^\infty_-(\Omega)$ for the obstacle problems with obstacles ψ_k, in particular

$$\mathrm{TV}_\Omega(u_k)=\int_\Omega \sigma_k\cdot \mathrm{D}\psi_k\,\mathrm{d}x=[\![\sigma_k,\mathrm{D}\psi_k^+]\!]_{u_o}(\overline{\Omega})\,. \tag{5.22}$$

Using $u_o+\mathbb{1}_\Omega(\psi-u_o)$ as a competitor, we see that the u_k are bounded in $\mathrm{BV}(\mathbb{R}^n)$, and the property (5.18) for the u_k implies

$$\|u_k\|_{\mathrm{L}^\infty(\Omega)}\le \max\{\sup_\Omega\psi_k,\,\|u_o\|_{\mathrm{L}^\infty(\Omega)}\}\le \max\{\sup_\Omega\psi,\,\|u_o\|_{\mathrm{L}^\infty(\Omega)}\} \tag{5.23}$$

for every $k\in\mathbb{N}$. Consequently, after passage to a subsequence we can assume that u_k weakly$*$ converges to a limit $u_\infty\in\mathrm{BV}_{u_o}(\Omega)\cap\mathrm{L}^\infty(\Omega)$, which in fact satisfies $u_\infty\in K_{\psi^+,u_o}(\Omega)$. Moreover, we can assume that $\sigma_k\overset{*}{\rightharpoonup}\sigma_\infty$ weakly$*$ in $\mathrm{L}^\infty(\Omega,\mathbb{R}^n)$. Taking into account the definition of the pairing, the Cap_1-q.e. inequality $\psi_k^+\le\psi^+$ on Ω for any $k\in\mathbb{N}$ and the identity (2.3) with $\varphi\equiv1$ on $\overline{\Omega}$, we find

$$\begin{aligned}[\![\sigma_k,\mathrm{D}\psi_k^+]\!]_{u_o}(\overline{\Omega})&\le[\![\sigma_k,\mathrm{D}\psi^+]\!]_{u_o}(\overline{\Omega})\\&=\int_\Omega\sigma_k\cdot\mathrm{D}\psi\,\mathrm{d}x+\int_{\partial\Omega}(\psi-u_o)^{\mathrm{int}}_{\partial\Omega}\,(\sigma_k)^*_{\mathrm{n}}\,\mathrm{d}\mathcal{H}^{n-1}\\&\le\int_\Omega\sigma_k\cdot\mathrm{D}\psi\,\mathrm{d}x+\int_{\partial\Omega}(\psi-u_o)^{\mathrm{int}}_{\partial\Omega}\,\mathrm{d}\mathcal{H}^{n-1}\,,\end{aligned}$$

where we used the $\mathcal{H}^{n-1}$-a.e. inequalities $(\psi-u_o)^{\mathrm{int}}_{\partial\Omega}\ge0$ and $(\sigma_k)^*_{\mathrm{n}}\le1$ on $\partial\Omega$ for the last step. Plugging the last estimate into (5.22), letting $k\to\infty$, and again applying (2.3), we arrive at

$$\begin{aligned}\liminf_{k\to\infty}\mathrm{TV}_\Omega(u_k)&\le\int_\Omega\sigma_\infty\cdot\mathrm{D}\psi\,\mathrm{d}x+\int_{\partial\Omega}(\psi-u_o)^{\mathrm{int}}_{\partial\Omega}\,\mathrm{d}\mathcal{H}^{n-1}\\&=[\![\sigma_\infty,\mathrm{D}\psi^+]\!]_{u_o}(\overline{\Omega})+\int_{\partial\Omega}(1-(\sigma_\infty)^*_{\mathrm{n}})(\psi-u_o)^{\mathrm{int}}_{\partial\Omega}\,\mathrm{d}\mathcal{H}^{n-1}\\&=[\![\sigma_\infty,\mathrm{D}\psi^+]\!]^*_{u_o}(\overline{\Omega})\,,\end{aligned}$$

where the last identity is valid since $(\psi)^{\mathrm{int}}_{\partial\Omega}\ge(u_o)^{\mathrm{int}}_{\partial\Omega}$ holds $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$ by assumption. By using the lower semicontinuity of the total variation, we thereby deduce

$$\mathrm{TV}_\Omega(u_\infty)\le[\![\sigma_\infty,\mathrm{D}\psi^+]\!]^*_{u_o}(\overline{\Omega})\,.$$

Comparison with (5.19) shows that $u_\infty\in K_{\psi^+,u_o}(\Omega)$ is a minimizer with the claimed L^∞-bound and that (3.10) holds also in the case $\psi\in\mathrm{W}^{1,1}(\Omega)$ with $(\psi-u_o)_-\in\mathrm{W}^{1,1}_0(\Omega)$ if we choose the representative $\Psi=\psi^+$ of ψ.

Step 3. Cap_1*-quasi upper semicontinuous obstacles.* Finally, we consider an arbitrary bounded Borel function $\Psi:\overline{\Omega}\to\mathbb{R}$ that is Cap_1-quasi upper semicontinuous. We decompose the obstacle function according to

$$\Psi=\max\{\Psi,u_o^*\}-(\Psi-u_o^*)_-\qquad\text{on }\overline{\Omega}\,.$$

Since $-(\Psi-u_o^*)_-$ is majorized by the constant function $g\equiv0$, we can apply Lemma 2.19 and the remark thereafter to find a non-increasing sequence of functions $\eta_{k,1}\in\mathrm{W}^{1,1}_0(\Omega)\cap\mathrm{L}^\infty(\Omega)$ with

$$\eta^*_{k,1}\to-(\Psi-u_o^*)_-\qquad\mathrm{Cap}_1\text{-q.e. on }\Omega,\text{ as }k\to\infty\,. \tag{5.24}$$

By passing to $\min\{\eta_{k,1},0\}$ if necessary, we can moreover assume that $\eta_{k,1}\le0$ holds a.e. on Ω. Since $\max\{\Psi,u_o^*\}$ is a bounded Cap_1-quasi upper semicontinuous function on $\overline{\Omega}$, its extension by u_o^* outside of $\overline{\Omega}$ is Cap_1-quasi upper semicontinuous on $\mathbb{R}^n$ and can be majorized by a $\mathrm{W}^{1,1}$ function. We are thus in a position to apply Lemma 2.18, which, together with Remark 2.20, yields a non-increasing sequence of functions $\eta_{k,2}\in\mathrm{W}^{1,1}(\Omega)\cap\mathrm{L}^\infty(\Omega)$ with $\eta_{k,2}\le\max\{1\text{-}\sup_{\overline{\Omega}}\Psi,\,\|u_o\|_{\mathrm{L}^\infty(\Omega)}\}$ a.e. on Ω and

$$\begin{aligned} \eta^*_{k,2} &\to \max\{\Psi, u^*_o\} && \mathrm{Cap}_1\text{-q.e. on } \Omega\,, \\ (\eta_{k,2})^{\mathrm{int}}_{\partial\Omega} &\to \max\{\Psi, (u_o)^{\mathrm{int}}_{\partial\Omega}\} && \mathrm{Cap}_1\text{-q.e. on } \partial\Omega\,, \end{aligned} \tag{5.25}$$

in the limit $k \to \infty$. Consequently, the functions

$$\psi_k := \eta_{k,1} + \eta_{k,2} \in W^{1,1}(\Omega) \cap \mathrm{L}^\infty(\Omega)$$

form a non-increasing sequence with

$$\begin{aligned} \|\psi_k\|_{\mathrm{L}^\infty(\Omega)} \le \max\{1\text{-}\sup_{\overline{\Omega}}|\Psi|, \|u_o\|_{\mathrm{L}^\infty(\Omega)}\} && \text{for every } k \in \mathbb{N}\,, \\ \psi^*_k \to \Psi && \mathrm{Cap}_1\text{-q.e. on } \Omega\,, \\ (\psi_k - u_o)^{\mathrm{int}}_{\partial\Omega} \to [\Psi - (u_o)^{\mathrm{int}}_{\partial\Omega}]_+ && \mathrm{Cap}_1\text{-q.e. on } \partial\Omega\,, \end{aligned}$$

in the limit $k \to \infty$. Since $\eta_{k,2} \ge u_o$ as a consequence of our construction, for each $k \in \mathbb{N}$ we have $0 \ge -(\psi_k - u_o)_- \ge \eta_{k,1} \in \mathrm{W}^{1,1}_0(\Omega)$. Therefore, it holds $(\psi_k - u_o)_- \in \mathrm{W}^{1,1}_0(\Omega)$, so that we are in a situation covered by Step 2 above. The earlier reasoning thus ensures the existence of minimizer–maximizer pairs $(u_k, \sigma_k) \in K_{\psi_k^+, u_o}(\Omega) \times S^\infty_-(\Omega)$ for the obstacle problems with obstacles ψ_k, which, by (5.18), satisfy

$$\|u_k\|_{\mathrm{L}^\infty(\Omega)} \le \max\left\{\sup_\Omega \psi_k, \|u_o\|_{\mathrm{L}^\infty(\Omega)}\right\} \le \max\left\{\sup_\Omega \psi_1, \|u_o\|_{\mathrm{L}^\infty(\Omega)}\right\}$$

and

$$\mathrm{TV}_\Omega(u_k) = [\![\sigma_k, \mathrm{D}\psi^+_k]\!]^*_{u_o}(\overline{\Omega}) \tag{5.26}$$

for each $k \in \mathbb{N}$. Since $u_o + \mathbb{1}_\Omega(\psi_1 - u_o)$ is an admissible competitor for each of the u_k, we infer that the sequence u_k is bounded in $\mathrm{BV}_{u_o}(\Omega)$. We may thus assume convergence $u_k \overset{*}{\rightharpoonup} u_\infty$ weakly$*$ in $\mathrm{BV}_{u_o}(\Omega)$ and $\sigma_k \overset{*}{\rightharpoonup} \sigma_\infty$ weakly$*$ in $\mathrm{L}^\infty(\Omega, \mathbb{R}^n)$ in the limit $k \to \infty$, for some $u_\infty \in \mathrm{BV}_{u_o}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and some $\sigma_\infty \in S^\infty_-(\Omega)$. We emphasize that, in contrast to the previous steps, at the present stage the exceptional set $\overline{\Omega} \cap \{u_\infty < \Psi\}$ may have positive $\mathcal{H}^{n-1}$-measure. In order to analyze the convergence of the right-hand side in (5.26), we observe that for every $\ell \in \mathbb{N}$, the monotonicity of the sequence ψ_k implies

$$\begin{aligned} &\limsup_{k\to\infty} [\![\sigma_k, \mathrm{D}\psi^+_k]\!]^*_{u_o}(\overline{\Omega}) \\ &\quad= \limsup_{k\to\infty}\left([\![\sigma_k, \mathrm{D}\psi^+_k]\!]_{u_o}(\overline{\Omega}) + \int_{\partial\Omega}(1 - (\sigma_k)^*_{\mathrm{n}})(\psi_k - u_o)^{\mathrm{int}}_{\partial\Omega}\,\mathrm{d}\mathcal{H}^{n-1}\right) \\ &\quad\le \limsup_{k\to\infty}\left([\![\sigma_k, \mathrm{D}\psi^+_\ell]\!]_{u_o}(\overline{\Omega}) + \int_{\partial\Omega}(1 - (\sigma_k)^*_{\mathrm{n}})(\psi_\ell - u_o)^{\mathrm{int}}_{\partial\Omega}\,\mathrm{d}\mathcal{H}^{n-1}\right) \\ &\quad= \limsup_{k\to\infty}\int_\Omega \sigma_k \cdot \mathrm{D}\psi_\ell\,\mathrm{d}x + \int_{\partial\Omega}(\psi_\ell - u_o)^{\mathrm{int}}_{\partial\Omega}\,\mathrm{d}\mathcal{H}^{n-1} \\ &\quad= \int_\Omega \sigma_\infty \cdot \mathrm{D}\psi_\ell\,\mathrm{d}x + \int_{\partial\Omega}(\psi_\ell - u_o)^{\mathrm{int}}_{\partial\Omega}\,\mathrm{d}\mathcal{H}^{n-1} \\ &\quad= [\![\sigma_\infty, \mathrm{D}\psi^+_\ell]\!]_{u_o}(\overline{\Omega}) + \int_{\partial\Omega}(1 - (\sigma_\infty)^*_{\mathrm{n}})(\psi_\ell - u_o)^{\mathrm{int}}_{\partial\Omega}\,\mathrm{d}\mathcal{H}^{n-1}\,. \end{aligned}$$

In the last three steps, we employed the weak$*$ convergence $\sigma_k \overset{*}{\rightharpoonup} \sigma_\infty$ in $\mathrm{L}^\infty(\Omega, \mathbb{R}^n)$ as $k \to \infty$ and twice the identity (2.3). Passing $\ell \to \infty$ and making use of the monotone Cap_1-q.e. convergences $\psi^+_\ell = \psi^*_\ell \to \Psi$ on Ω and $(\psi_\ell - u_o)^{\mathrm{int}}_{\partial\Omega} \to [\Psi - (u_o)^{\mathrm{int}}_{\partial\Omega}]_+$ on $\partial\Omega$, we deduce

$$\begin{aligned} \limsup_{k\to\infty} [\![\sigma_k, \mathrm{D}\psi^+_k]\!]^*_{u_o}(\overline{\Omega}) &\le [\![\sigma_\infty, \mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega}) + \int_{\partial\Omega}(1 - (\sigma_\infty)^*_{\mathrm{n}})[\Psi - (u_o)^{\mathrm{int}}_{\partial\Omega}]_+\,\mathrm{d}\mathcal{H}^{n-1} \\ &= [\![\sigma_\infty, \mathrm{D}\Psi]\!]^*_{u_o}(\overline{\Omega})\,. \end{aligned} \tag{5.27}$$

Since we have $u_k \in K_{\psi_k^+,u_o}(\Omega) \subset K_{\Psi,u_o}(\Omega)$, the lower semicontinuity result from Theorem 2.13 ensures

$$\mathrm{TV}_{\Psi;\Omega}(u_\infty) \le \liminf_{k\to\infty} \mathrm{TV}_{\Psi;\Omega}(u_k) = \liminf_{k\to\infty} \mathrm{TV}_{\Omega}(u_k)\,. \tag{5.28}$$

Combining (5.28), (5.26), and (5.27), we conclude

$$\mathrm{TV}_{\Psi;\Omega}(u_\infty) \le [\![\sigma_\infty, \mathrm{D}\Psi]\!]^*_{u_o}\big(\overline{\Omega}\big)\,.$$

Comparing the last inequality with (5.19), it turns out that u_∞ is a minimizer and σ_∞ a maximizer. Therefore, we have verified (3.10) in the general case, and the proof of Theorem 3.6 is complete. □

5.3. *Derivation of the optimality relations*

In this short subsection, we give the

Proof of Corollary 3.10. The corollary is a consequence of the following chain of inequalities that holds for any pair of competitors $(u,\sigma) \in \mathrm{BV}_{u_o}(\Omega) \times S^\infty_-(\Omega)$ in (3.10):

$$\begin{aligned}
[\![\sigma, \mathrm{D}\Psi]\!]^*_{u_o}(\overline{\Omega}) &= [\![\sigma, \mathrm{D}u^+]\!]^*_{u_o}(\overline{\Omega}) + \int_\Omega (\Psi - u^+)\,\mathrm{d}(-\operatorname{div}\sigma) \\
&\qquad + \int_{\partial\Omega} (1-\sigma_{\mathrm{n}}^*)\big([\Psi-(u_o)^{\mathrm{int}}_{\partial\Omega}]_+ - [u^+-(u_o)^{\mathrm{int}}_{\partial\Omega}]_+\big)\,\mathrm{d}\mathcal{H}^{n-1} \\
&\le [\![\sigma, \mathrm{D}u^+]\!]^*_{u_o}(\overline{\Omega}) + \int_\Omega (\Psi - u^+)\,\mathrm{d}(-\operatorname{div}\sigma) + \int_{\partial\Omega} (1-\sigma_{\mathrm{n}}^*)(\Psi-u^+)_+\,\mathrm{d}\mathcal{H}^{n-1} \\
&\le |\mathrm{D}u|(\overline{\Omega}) + \int_\Omega (\Psi - u^+)_+\,\mathrm{d}(-\operatorname{div}\sigma) + 2\int_{\partial\Omega} (\Psi-u^+)_+\,\mathrm{d}\mathcal{H}^{n-1} \\
&\le |\mathrm{D}u|(\overline{\Omega}) + \int_\Omega (\Psi - u^+)_+\,\mathrm{d}\varsigma + \int_{\partial\Omega} (\Psi-u^+)_+\,\mathrm{d}\varsigma \\
&= \mathrm{TV}_{\Psi;\Omega}(u)\,.
\end{aligned} \tag{5.29}$$

Here, the first line follows from the definition of $[\![\sigma, \mathrm{D}u^+]\!]^*_{u_o}(\overline{\Omega})$, and the second one from the $\mathcal{H}^{n-1}$-a.e. bound $\sigma_{\mathrm{n}}^* \le 1$ on $\partial\Omega$ and the elementary inequality $a_+ - b_+ \le (a-b)_+$ for $a, b \in \mathbb{R}$, which specifically gives

$$[\Psi-(u_o)^{\mathrm{int}}_{\partial\Omega}]_+ - [u^+-(u_o)^{\mathrm{int}}_{\partial\Omega}]_+ \le (\Psi-u^+)_+\,. \tag{5.30}$$

Furthermore, in the third line we used Proposition 2.8, the non-negativity of $(-\operatorname{div}\sigma)$, and the $\mathcal{H}^{n-1}$-a.e. bound $\sigma_{\mathrm{n}}^* \ge -1$ on $\partial\Omega$, while the fourth line is a consequence of Propositions 2.11 and 2.12.

It is clear from Theorem 3.6 that (u,σ) is a minimizer–maximizer pair if and only if all the inequalities in (5.29) are in fact equalities, and going through the above argument this occurs precisely if we have the following five identities, which we will show to be equivalent with (3.14)–(3.18) in the statement of the corollary:

$$\begin{aligned}
&[\![\sigma, \mathrm{D}u^+]\!]^*_{u_o}(\overline{\Omega}) = |\mathrm{D}u|(\overline{\Omega})\,, \\
&\qquad \Psi - u^+ = (\Psi-u^+)_+ && (-\operatorname{div}\sigma)\text{-a.e. on } \Omega\,, \\
&\qquad\quad \sigma_{\mathrm{n}}^* \equiv 1 && \mathcal{H}^{n-1}\text{-a.e. where inequality (5.30) is strict on } \partial\Omega\,, \\
&\qquad\quad \sigma_{\mathrm{n}}^* \equiv -1 && \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega \cap \{u^+ < \Psi\}\,,
\end{aligned}$$

and

$$\int_\Omega (\Psi-u^+)_+\,\mathrm{d}(-\operatorname{div}\sigma) = \int_\Omega (\Psi-u^+)_+\,\mathrm{d}\varsigma\,. \tag{5.31}$$

Now, in view of (2.6) the first identity is equivalent with the equality $[\![\sigma, \mathrm{D}u^+]\!]^*_{u_o} = |\mathrm{D}u|$ of measures on $\overline{\Omega}$ and thus with the validity of (3.14) for $\mu := -\operatorname{div}\sigma$. The second line then clearly corresponds to $\mu(\Omega \cap \{u^+ > \Psi\}) = 0$, that is to (3.15). The boundary portion in the third identity is quickly identified by observing that the strict inequality $a_+ - b_+ < (a-b)_+$ holds if and only if one has $b > a_+$ or $b < -a_-$. In the case of (5.30), the second alternative can not occur because $u^+ = \max\{u^{\mathrm{int}}_{\partial\Omega}, (u_o)^{\mathrm{int}}_{\partial\Omega}\} \geq (u_o)^{\mathrm{int}}_{\partial\Omega}$ holds $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$. So, the boundary portion in the third identity turns out to be the one where $u^+ - (u_o)^{\mathrm{int}}_{\partial\Omega} > [\Psi - (u_o)^{\mathrm{int}}_{\partial\Omega}]_+$ or, equivalently, $u^{\mathrm{int}}_{\partial\Omega} > \max\{\Psi, (u_o)^{\mathrm{int}}_{\partial\Omega}\}$ holds. In view of this observation, the third identity coincides with (3.17). The fourth identity is (only) a part of (3.18), since indeed it requires $\sigma^*_{\mathrm{n}} \equiv -1$ only on the portion $\{u^+ < \Psi\}$ of $\{u^{\mathrm{int}}_{\partial\Omega} < \max\{\Psi, (u_o)^{\mathrm{int}}_{\partial\Omega}\} = \{u^+ < \Psi\} \cup \{u^{\mathrm{int}}_{\partial\Omega} < (u_o)^{\mathrm{int}}_{\partial\Omega}\}$ (all sets taken in $\partial\Omega$). However, we have already obtained (3.14), and by (2.8) the validity of (3.14) on $\partial\Omega$ turns out to mean $\big[(u-u_o)^{\mathrm{int}}_{\partial\Omega}\big]_+ - \big[(u-u_o)^{\mathrm{int}}_{\partial\Omega}\big]_- \sigma^*_{\mathrm{n}} = |(u-u_o)^{\mathrm{int}}_{\partial\Omega}|$ on $\partial\Omega$. Thus the equality $\sigma^*_{\mathrm{n}} \equiv -1$ on the other portion $\{u^{\mathrm{int}}_{\partial\Omega} < (u_o)^{\mathrm{int}}_{\partial\Omega}\}$ is already contained in (3.14) (compare Remark 3.11), and hence (3.18) is fully justified. Finally, since Proposition 2.11 generally asserts $-\operatorname{div}\sigma \leq \varsigma$ as measures on Ω, the remaining identity (5.31) means nothing but $-\operatorname{div}\sigma = \varsigma$ on $\Omega \cap \{u^+ < \Psi\}$, that is (3.16). □

6. The obstacle problem for the area functional

In this final section, we briefly explain how to adapt the proofs in the preceding sections to the case of the area functional.

6.1. Convex conjugates and duality for non-degenerate p-energies

In order to specify the approximating problems, we introduce the non-degenerate p-energies

$$\mathcal{A}^p_\Omega(u) := \int_\Omega f_p(\mathrm{D}u)\,\mathrm{d}x \qquad \text{for } u \in \mathrm{W}^{1,p}(\Omega)\,,$$

where we defined

$$f_p(z) := \big(1+|z|^2\big)^{\frac{p}{2}} \qquad \text{for } z \in \mathbb{R}^n$$

for $p \in [1,\infty)$. In particular, with this convention, the functional $\mathcal{A}^1_\Omega$ coincides with the area $\mathcal{A}_\Omega$ on $\mathrm{W}^{1,1}$ functions. The convex conjugate

$$f^*_p(z^*) := \sup_{z\in\mathbb{R}^n} [z^* \cdot z - f_p(z)] \tag{6.1}$$

can be computed in terms of the inverse $\Phi_p(z^*) := (\nabla f_p)^{-1}(z^*)$ of the gradient $\nabla f_p(z) = (1+|z|^2)^{\frac{p-2}{2}} z$ as

$$f^*_p(z^*) = z^* \cdot \Phi_p(z^*) - \big(1+|\Phi_p(z^*)|^2\big)^{\frac{p}{2}} \qquad \text{for } z^* \in \mathbb{R}^n\,.$$

In the case $p = 1$, the above formula is only valid for $|z^*| < 1$ since, otherwise, Φ_1 is undefined. However, in this case we have the explicit formula

$$f^*_1(z^*) = \begin{cases} -\sqrt{1-|z^*|^2} & \text{if } |z^*| \leq 1 \\ \infty & \text{if } |z^*| > 1 \end{cases}$$

for all $z^* \in \mathbb{R}^n$. As a consequence of $f_p \leq f_q$ for $p \leq q$ we have

$$f^*_p \geq f^*_q \qquad \text{whenever } 1 \leq p \leq q\,. \tag{6.2}$$

In addition, we observe:

Lemma 6.1. *In the limit $p \searrow 1$ we have*

$$f^*_p(z^*) \to f^*_1(z^*) \qquad \textit{monotonically, for all } z^* \in \mathbb{R}^n\,. \tag{6.3}$$

Proof. In the case $|z_*| \le 1$ the claim can be checked, for example, by choosing $z = \frac{z^*}{\sqrt{\lambda^2 - |z^*|^2}}$ in the supremum of (6.1), with arbitrary $\lambda > 1$. Letting first $p \searrow 1$ and then $\lambda \searrow 1$, we infer the estimate

$$\lim_{p \searrow 1} f_p^*(z^*) \ge -\sqrt{1-|z^*|^2} = f_1^*(z^*)\,.$$

Since we have $f_p^*(z^*) \le f_1^*(z^*)$ as a consequence of (6.2), this implies the claimed convergence (6.3) for $|z^*| \le 1$. In the case $|z^*| > 1$, however, we choose $z = \lambda z^*$ in (6.1). After letting $p \searrow 1$, this implies

$$\lim_{p \searrow 1} f_p^*(z^*) \ge \lambda\Big(|z^*|^2 - \sqrt{\tfrac{1}{\lambda^2} + |z^*|^2}\Big)\,,$$

and the right-hand side tends to $\infty = f_1^*(z^*)$ in the limit $\lambda \to \infty$. This completes the proof of (6.3). $\square$

In the case $p = 1$, the Fenchel inequality gives

$$z^* \cdot z \le f_1(z) + f_1^*(z^*) = \sqrt{1+|z|^2} - \sqrt{1-|z^*|^2} \quad \text{for } z, z^* \in \mathbb{R}^n \text{ with } |z^*| \le 1\,. \tag{6.4}$$

This inequality induces a corresponding estimate for the Anzellotti pairings. Indeed, from [35, Lemma 5.5], we obtain the following proposition, which serves as a replacement for Proposition 2.8.

Proposition 6.2. *Consider a bounded open set $\Omega \subset \mathbb{R}^n$ with (2.1), $u_o \in W^{1,1}(\mathbb{R}^n) \cap L^\infty(\Omega)$, $u \in BV_{u_o}(\Omega) \cap L^\infty(\Omega)$, and $\sigma \in S_-^\infty(\Omega)$. Then we have the inequalities of measures*

$$\big|[\![\sigma, Du^+]\!]_{u_o}\big| \le \sqrt{1+|Du|^2} - \sqrt{1-|\sigma|^2}\,\mathcal{L}^n \qquad \text{on } \overline{\Omega} \tag{6.5}$$

and

$$\big|[\![\sigma, Du^+]\!]_{u_o}^*\big| \le \sqrt{1+|Du|^2} - \sqrt{1-|\sigma|^2}\,\mathcal{L}^n \qquad \text{on } \overline{\Omega}\,. \tag{6.6}$$

We recall that equality in the Fenchel inequality (6.4) holds if and only if z and z^* are coupled by $z^* = \nabla f_1(z) = \frac{z}{\sqrt{1+|z|^2}}$. As a consequence, for any $u \in W^{1,1}(\Omega)$ and $\sigma \in S_-^\infty(\Omega)$ we have the equivalence

$$\sigma \cdot Du = \sqrt{1+|Du|^2} - \sqrt{1-|\sigma|^2} \iff \sigma = \frac{Du}{\sqrt{1+|Du|^2}}\,.$$

In this sense, the case of equality in (6.5) and (6.6), respectively, can be understood as a generalization of the identity $\sigma = \frac{Du}{\sqrt{1+|Du|^2}}$ a.e. on Ω to BV functions u.

The classical arguments of Section 4 apply also to the non-degenerate p-energies $\mathcal{A}_\Omega^p$ with $p > 1$. In particular, the unique minimizer u of $\mathcal{A}_\Omega^p$ in $K_\Psi^p(\Omega)$ satisfies the variational inequality

$$\int_\Omega \nabla f_p(Du) \cdot D(v-u)\,dx \ge 0 \qquad \text{for all } v \in K_\Psi^p(\Omega)\,,$$

and the bounds in Lemma 4.2 hold analogously. Hence, we have the following analog of Proposition 4.3.

Proposition 6.3. *Consider a bounded domain $\Omega \subset \mathbb{R}^n$, boundary values $u_o \in W^{1,q}(\Omega) \cap L^\infty(\Omega)$ and a bounded Borel function $\Psi : \Omega \to \mathbb{R}$ with (3.1) and (3.2) for $q > 1$. Then, for every $p \in (1, q]$, it holds*

$$\min_{u \in K_\Psi^p(\Omega)} \int_\Omega f_p(Du)\,dx = \max_{\sigma \in S_-^{p'}(\Omega)} \Big([\![\sigma, D\Psi]\!]_{u_o}(\overline{\Omega}) - \int_\Omega f_p^*(\sigma)\,dx\Big)\,. \tag{6.7}$$

Indeed, for the minimizer $u \in K_\Psi^p(\Omega)$ of the left-hand side, the right-hand side is maximized by

$$\sigma := \nabla f_p(Du) = p(1+|Du|^2)^{\frac{p-2}{2}} Du \in S_-^{p'}(\Omega)\,,$$

i.e. for this choice of σ we have

$$\int_\Omega f_p(\mathrm{D}u)\,\mathrm{d}x = [\![\sigma, \mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega}) - \int_\Omega f_p^*(\sigma)\,\mathrm{d}x\,. \tag{6.8}$$

6.2. *The duality formula in the limit $p \searrow 1$*

Sketch of proof for Theorem 3.5. In large parts, the proof is analogous to the arguments that have been carried out in detail in Section 5.1. One difference, however, is that the applications of Proposition 2.8 now are replaced by the corresponding estimate from Proposition 6.2. In particular, the easy inequality in the claimed duality identity (3.9) follows from (6.5) and Proposition 2.11 by estimating

$$\begin{aligned}
[\![\sigma, \mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega}) &= [\![\sigma, \mathrm{D}u^+]\!]_{u_o}(\overline{\Omega}) + \int_\Omega (\Psi - u^+)\mathrm{d}(-\operatorname{div}\sigma)\\
&\le \mathcal{A}_\Omega(u) - \int_\Omega \sqrt{1-|\sigma|^2}\,\mathrm{d}x + \int_\Omega (\Psi-u^+)_+\mathrm{d}\varsigma\\
&= \mathcal{A}_{\Psi;\Omega}(u) - \int_\Omega \sqrt{1-|\sigma|^2}\,\mathrm{d}x
\end{aligned}$$

for all $u \in \mathrm{BV}_{u_o}(\Omega)\cap \mathrm{L}^\infty(\Omega)$ and $\sigma \in S_-^\infty(\Omega)$. For a sequence $p_i \searrow 1$, we next choose minimizers $u_{p_i} \in K_\Psi^{p_i}(\Omega)\cap \mathrm{L}^\infty(\Omega)$ of the left-hand side in (6.7) and the corresponding maximizers $\sigma_i := p_i(1+|\mathrm{D}u_{p_i}|^2)^{\frac{p-2}{2}}\mathrm{D}u_{p_i} \in S_-^{p_i'}(\Omega)$ of the right-hand side in (6.7). Analogously to Section 5.1, we can achieve convergence $u_{p_i} \overset{*}{\rightharpoonup} u$ weakly$*$ in $\mathrm{BV}_{u_o}(\Omega)$ and in $\mathrm{L}^\infty(\Omega)$ as well as $\sigma_i \rightharpoonup \sigma$ weakly in $\mathrm{L}^{q'}(\Omega,\mathbb{R}^n)$, where σ has non-positive distributional divergence. Using the lower semicontinuity of $\mathcal{A}_{\Psi;\Omega}$ according to Theorem 2.13, Hölder's inequality and the identity (6.8) for the solutions u_{p_i}, we deduce

$$\begin{aligned}
\mathcal{A}_{\Psi;\Omega}(u) &\le \liminf_{i\to\infty}\left(\int_\Omega (1+|\mathrm{D}u_{p_i}|^2)^{\frac{p_i}{2}}\,\mathrm{d}x\right)^{\frac{1}{p_i}}\\
&= \liminf_{i\to\infty}\left([\![\sigma_i, \mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega}) - \int_\Omega f_{p_i}^*(\sigma_i)\,\mathrm{d}x\right).
\end{aligned} \tag{6.9}$$

Repeating the arguments from (5.16) and (5.17), we have

$$\limsup_{i\to\infty}[\![\sigma_i, \mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega}) \le [\![\sigma, \mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega})\,. \tag{6.10}$$

The two preceding estimates imply, in particular,

$$\limsup_{i\to\infty}\left(-\int_\Omega f_{p_i}^*(\sigma_i)\,\mathrm{d}x\right) > -\infty\,. \tag{6.11}$$

From the monotonic dependence (6.2) of f_p^* on p and the weak upper semicontinuity of the concave functional $-\int_\Omega f_p^*(\cdot)\,\mathrm{d}x$ we infer

$$\limsup_{i\to\infty}\left(-\int_\Omega f_{p_i}^*(\sigma_i)\,\mathrm{d}x\right) \le \limsup_{i\to\infty}\left(-\int_\Omega f_{1+\varepsilon}^*(\sigma_i)\,\mathrm{d}x\right) \le -\int_\Omega f_{1+\varepsilon}^*(\sigma)\,\mathrm{d}x$$

for every $\varepsilon > 0$. Next, we exploit that $f_{1+\varepsilon}^*$ converges monotonically to f_1^* as $\varepsilon \searrow 0$ according to (6.3). Therefore, we can pass to the limit $\varepsilon \searrow 0$ in the last integral to infer

$$\limsup_{i\to\infty}\left(-\int_\Omega f_{p_i}^*(\sigma_i)\,\mathrm{d}x\right) \le -\int_\Omega f_1^*(\sigma)\,\mathrm{d}x\,.$$

In view of (6.11) and the fact $f_1^*(z^*)=\infty$ for $|z^*|>1$, this implies $|\sigma|\le 1$ a.e. on Ω, i.e. $\sigma\in S^\infty_-(\Omega)$. Consequently, we can rewrite the preceding formula to

$$\limsup_{i\to\infty}\Big(-\int_\Omega f_{p_i}^*(\sigma_i)\,\mathrm{d}x\Big)\le\int_\Omega\sqrt{1-|\sigma|^2}\,\mathrm{d}x\,. \tag{6.12}$$

Plugging (6.10) and (6.12) into (6.9), we deduce the final estimate

$$\mathcal{A}_{\Psi;\Omega}(u)\le[\![\sigma,\mathrm{D}\Psi]\!]_{u_o}(\overline{\Omega})+\int_\Omega\sqrt{1-|\sigma|^2}\,\mathrm{d}x\,.$$

This yields the remaining inequality needed for the claimed identity (3.9). □

Remark 6.4. Compared to the proof of the TV-case in Section 5.1, in the preceding proof we obtained the estimate $|\sigma|\le 1$ a.e. on Ω in a slightly different way. In fact, in Section 5.1 we avoided the use of convex conjugate functions and employed a truncation argument instead. However, the approach of this section would also be applicable in the TV-case.

6.3. The duality formula for general obstacles

Sketch of proof for Theorem 3.9. Here, we follow, up to minor modifications, the line of argument from Section 5.2. Analogously to (5.19), now replacing the application of Proposition 2.8 by the estimate (6.6) from Proposition 6.2, we deduce

$$\mathcal{A}_{\Psi;\Omega}(u)\ge[\![\sigma,\mathrm{D}\Psi]\!]^*_{u_o}(\overline{\Omega})+\int_\Omega\sqrt{1-|\sigma|^2}\,\mathrm{d}x$$

for all $u\in\mathrm{BV}_{u_o}(\Omega)\cap\mathrm{L}^\infty(\Omega)$ and $\sigma\in S^\infty_-(\Omega)$. This readily implies one inequality in the claimed identity (3.13). The other inequality follows by the three-step approximation procedure of Section 5.2. In each of the steps, the behavior of the pairings $[\![\sigma_k,\mathrm{D}\psi_k]\!]^*_{u_o}$ in the limit $k\to\infty$ can be controlled by exactly the same arguments as in Section 5.2. The only difference in the case of the area functional is that we additionally have to deal with the integrals $\int_\Omega\sqrt{1-|\sigma_k|^2}\,\mathrm{d}x$ that now occur in the dual formulation. However, these integrals can be handled by using the weak$*$ upper semicontinuity of the concave functional $\int_\Omega\sqrt{1-|\cdot|^2}\,\mathrm{d}x$ on sub-unit L^∞ vector fields. Also taking into account the lower semicontinuity of $\mathcal{A}_{\Psi;\Omega}$ due to Theorem 2.13, it is now straightforward to modify the proof of Theorem 3.6 and to establish Theorem 3.9. □

6.4. Derivation of the optimality relations

Sketch of proof for Corollary 3.13. We proceed exactly as in (5.29), now applying the estimate (6.6) instead of Proposition 2.8. This yields, for every pair of competitors $(u,\sigma)\in\mathrm{BV}_{u_o}(\Omega)\times S^\infty_-(\Omega)$ in (3.13), the chain of inequalities

$$\begin{aligned}
&[\![\sigma,\mathrm{D}\Psi]\!]^*_{u_o}(\overline{\Omega})+\int_\Omega\sqrt{1-|\sigma|^2}\,\mathrm{d}x\\
&\quad\le[\![\sigma,\mathrm{D}u^+]\!]^*_{u_o}(\overline{\Omega})+\int_\Omega\sqrt{1-|\sigma|^2}\,\mathrm{d}x\\
&\qquad\quad+\int_\Omega(\Psi-u^+)\,\mathrm{d}(-\operatorname{div}\sigma)+\int_{\partial\Omega}(1-\sigma_{\mathrm{n}}^*)(\Psi-u^+)_+\,\mathrm{d}\mathcal{H}^{n-1}\\
&\quad\le\mathcal{A}_\Omega(u)+\int_\Omega(\Psi-u^+)_+\,\mathrm{d}(-\operatorname{div}\sigma)+2\int_{\partial\Omega}(\Psi-u^+)_+\,\mathrm{d}\mathcal{H}^{n-1}
\end{aligned}$$

$$\leq \mathcal{A}_\Omega(u) + \int_\Omega (\Psi - u^+)_+ \, d\varsigma + \int_{\partial\Omega} (\Psi - u^+)_+ \, d\varsigma$$
$$= \mathcal{A}_{\Psi;\Omega}(u) .$$

Again, we deduce that (u, σ) is a minimizer–maximizer pair if and only if all inequalities are equalities, and a discussion of the cases of equality gives the claims. Indeed, (3.15), (3.16), (3.17), (3.18) come out in the same fashion as in the proof of Corollary 3.10, and the only slightly different discussion concerns the case of equality in the estimate

$$[\![\sigma, Du^+]\!]^*_{u_o}(\overline{\Omega}) \leq \mathcal{A}_\Omega(u) - \int_\Omega \sqrt{1-|\sigma|^2} \, dx . \tag{6.13}$$

However, in view of (6.6) equality in (6.13) is equivalent to the first identity in (3.22), and we obtain the claimed characterization.

In the case that $\Psi \leq (u_o)^{\mathrm{int}}_{\partial\Omega}$ holds $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, the preceding chain of inequalities holds without the boundary integrals, and as a consequence, we arrive at (6.13) with $[\![\sigma, Du^+]\!]^*_{u_o}(\overline{\Omega})$ replaced by $[\![\sigma, Du^+]\!]_{u_o}(\overline{\Omega})$. In view of (6.5) it is then possible to replace $[\![\sigma, Du^+]\!]^*_{u_o}$ by $[\![\sigma, Du^+]\!]_{u_o}$ also in (3.22) and the claimed characterization (compare with Remark 3.12 in the TV case).

Finally, the redundancy of the boundary information in (6.13) is a consequence of the discussion in Remark 3.11, and the proof of Corollary 3.13 is complete. □

Conflict of interest statement

There are no conflicts of interest.

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