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Weak solutions of semilinear elliptic equation involving Dirac mass

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Abstract

In this paper, we study the elliptic problem with Dirac mass

$$\begin{cases} -\Delta u = Vu^p + k\delta_0 \quad \text{in} \quad \mathbb{R}^N, \\ \lim\limits_{|x|\to+\infty} u(x) = 0, \end{cases} \tag{1}$$

where $N > 2$, $p > 0$, $k > 0$, δ_0 is the Dirac mass at the origin and the potential V is locally Lipchitz continuous in $\mathbb{R}^N \setminus \{0\}$, with non-empty support and satisfying

$$0 \le V(x) \le \frac{\sigma_1}{|x|^{a_0}(1+|x|^{a_\infty - a_0})},$$

with $a_0 < N$, $a_0 < a_\infty$ and $\sigma_1 > 0$. We obtain two positive solutions of (1) with additional conditions for parameters on a_∞, a_0, p and k. The first solution is a minimal positive solution and the second solution is constructed via Mountain Pass Theorem.

MSC: 35J60; 35J20

Keywords: Weak solution; Mountain Pass theorem; Dirac mass

1. Introduction

The goal of this paper is to study the existence of multiple weak solutions to the nonlinear elliptic problem with Dirac mass

$$\begin{cases} -\Delta u = Vu^p + k\delta_0 \quad \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to+\infty} u(x) = 0 \end{cases} \tag{P_k}$$

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where $N > 2$, $p > 0$, $k > 0$, δ_0 is the Dirac mass at the origin, and the potential V is locally Lipchitz continuous in $\mathbb{R}^N \setminus \{0\}$. Problem (P_k) concerns with source term in contrast with the problem with absorption, namely the semi-linear elliptic equation

$$\begin{cases} -\Delta u + g(u) = \nu & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where ν is a bounded Radon measure, Ω is a bounded C^2 domain in $\mathbb{R}^N$ and $g : \mathbb{R} \to \mathbb{R}$ is nondecreasing and $g(0) \geq 0$. This absorption problem has been extensively studied for the last several decades. A fundamental contribution to the problem is due to Brezis [8], Benilan and Brezis [5], where they showed the existence and uniqueness of weak solution for problem (1.1) if the function $g : \mathbb{R} \to \mathbb{R}$ satisfies the subcritical assumption:

$$\int_1^{+\infty} (g(s) - g(-s)) s^{-1-\frac{N}{N-2}} ds < +\infty.$$

The method is to approximate the measure ν by a sequence of regular functions, and find classical solutions which converges to a weak solution of (1.1). For this approach to work, uniform bounds for the sequence of classical solutions are necessary to be established. The uniqueness is then derived by Kato's inequality. Such a method has been applied to solve equations with boundary measure data in [12,15–18] and other extensions in [3,4,6,7,22].

In the source term case, it is hard to find uniform bounds when using the approximation method from [5,8], so different approaches has to be considered. Moreover, uniqueness is no longer valid in general. Actually, for the problem

$$\begin{cases} -\Delta u = u^q + \lambda\delta_0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where $q \in (1, \frac{N}{N-2})$, $\lambda > 0$ and Ω is a bounded domain containing the origin, it was shown in [14] that there exists $\lambda^* > 0$ such that (1.2) has two non-negative solutions for $\lambda \in (0, \lambda^*)$. For a general Radon measure ν replacing $\lambda\delta_0$ in (1.2), one weak solution was found in [4]. When $q \in (1, \frac{N}{N-2})$, the non-negative solutions of (1.2) are isolated singular solutions of

$$-\Delta u = u^q \quad \text{in} \quad \Omega \setminus \{0\}, \tag{1.3}$$

behaving asymptotically at the origin like $|x|^{2-N}$. The non-negative solutions to (1.3) with isolated singularities have been classified in [1] for $q = \frac{N}{N-2}$, in [11] for $\frac{N}{N-2} < q < \frac{N+2}{N-2}$ and in [9] for $q = \frac{N+2}{N-2}$. Using this classification, solutions of equations like (1.3) with many singular points were constructed in [19,21].

For the problem in the whole space, it was proved in [20] that the equation

$$\begin{cases} -\Delta u + u = u^q + \kappa \sum_{i=1}^{m} \delta_{x_i} & \text{in} \quad \mathcal{D}'(\mathbb{R}^N), \\ \lim_{|x|\to+\infty} u(x) = 0 \end{cases} \tag{1.4}$$

possesses at least two weak solutions for $\kappa > 0$ small and $q \in (1, \frac{N}{N-2})$. The exponential decay of the fundamental solution of the operator $-\Delta + id$ plays an essential role in finding solutions of (1.4), in particular the fact that it belongs to $L^1(\mathbb{R}^N)$. In our problem (P_k) however, the fundamental solution of $-\Delta$ does not belong to $L^1(\mathbb{R}^N)$, fact that brings difficulties in finding solutions for the equation.

In this paper we find two weak solutions for problem (P_k). By a *weak solution* of (P_k) we mean a non-negative function $u \in L^1_{loc}(\mathbb{R}^N)$ such that $Vu^p \in L^1(\mathbb{R}^N)$,

$$\lim_{r\to+\infty} \operatorname{essup}_{x\in\mathbb{R}^N\setminus B_r(0)} u(x) = 0$$

and u satisfies

$$\int_{\mathbb{R}^N} u(-\Delta)\xi dx = \int_{\mathbb{R}^N} Vu^p \xi dx + k\xi(0), \quad \forall \xi \in C_c^{1,1}(\mathbb{R}^N).$$

On the potential function V, we assume throughout the paper that V is has non-empty support and there exist $a_0 < N$, $a_0 < a_\infty$, $\sigma_1 > 0$ such that, for all $x \in \mathbb{R}^N \setminus \{0\}$

$$0 \le V(x) \le V_0(x) := \frac{\sigma_1}{|x|^{a_0}(1+|x|^{a_\infty - a_0})}. \tag{1.5}$$

Condition (1.5) implies that the limiting behavior of V at the origin is controlled by $|x|^{-a_0}$ and that of $V(x)$ at infinity is controlled by $|x|^{-a_\infty}$.

Now we are in a position of stating our first result on the minimal solution of (P_k).

Theorem 1.1. *Suppose condition (1.5) holds and $p > 0$ satisfies*

$$p \in (\frac{N-a_\infty}{N-2}, \frac{N-a_0}{N-2}). \tag{1.6}$$

Then,

(i) *There exists $k^* = k^*(p, V) \in (0, +\infty]$ such that for $k \in (0, k^*)$, there exists a minimal positive solution $u_{k,V}$ of problem (P_k). If $k > k^*$ and $p > 1$, there is no solution for (P_k). Moreover, $k^* < \infty$ if $p > 1$; $k^* = +\infty$ if $0 < p < 1$ or if $p = 1$ and $\sigma_1 > 0$ is small.*

(ii) *For p fixed, $k^*(p, V)$ is decreasing in V and the mapping $V \mapsto u_{k,V}$ is increasing.*

(iii) *If V is radially symmetric, the minimal solution $u_{k,V}$ is also radially symmetric.*

In the sequel, we denote $u_{k,V}$ the minimal solution obtained in Theorem 1.1 corresponding to k and V. We remark that the minimal solution of (P_k) is derived by iterating an increasing sequence $\{v_n\}_n$ defined by

$$v_0 = k\mathbb{G}[\delta_0], \qquad v_n = \mathbb{G}[V v_{n-1}^p] + k\mathbb{G}[\delta_0], \tag{1.7}$$

where $\mathbb{G}[\cdot]$ is the Green operator defined as

$$\mathbb{G}[f](x) = \int_{\mathbb{R}^N} G(x, y) f(y) dy$$

and G is the Green kernel of $-\Delta$ in $\mathbb{R}^N \times \mathbb{R}^N$, where $\mathbb{G}[\delta_0]$ is the fundamental solution of $-\Delta$. To insure the convergence of the sequence $\{v_n\}_n$, we construct a suitable barrier function by using the estimate

$$\mathbb{G}[V\mathbb{G}^p[\delta_0]] \le \sigma_2 \mathbb{G}[\delta_0] \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}, \tag{1.8}$$

where $\sigma_2 > 0$. The optimal range of k, for which the estimate (1.8) is achieved, is

$$k_p = (\sigma_2 p)^{-\frac{1}{p-1}} \frac{p-1}{p}, \tag{1.9}$$

giving the range for constructing the barrier function for (P_k). Thus we have $k^* \ge k_p$.

Once the minimal solution is found, we further explore its properties. Precisely, we show that such a solution is regular except at the origin, and we study its decays at infinity. These properties allow us to establish the stability of the minimal solution, whereas this stability plays a crucial role in finding the second solution.

Denote by $\mathcal{D}^{1,2}(\mathbb{R}^N)$ the Sobolev space which is the closure of $C_c^\infty(\mathbb{R}^N)$ under the norm

$$\|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\nabla v|^2 dx \right)^{\frac{1}{2}}.$$

We say a solution u of (P_k) is *stable (resp. semi-stable)* if

$$\int_{\mathbb{R}^N} |\nabla \xi|^2 dx > p \int_{\mathbb{R}^N} V u^{p-1} \xi^2 dx, \quad (\text{resp. } \ge) \quad \forall \xi \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}.$$

The following theorem establishes the main properties of the minimal solution.

Theorem 1.2. *Suppose that the function V satisfies (1.5) with $a_\infty > a_0$ and $a_0 \in \mathbb{R}$, and p satisfies (1.6).*

(i) *If $a_0 < 2$, $1 < p$ and $k \in (0, k_p)$, then $u_{k,V}$ is a classical solution of the equation*

$$-\Delta u = V u^p \quad in \quad \mathbb{R}^N \setminus \{0\}, \qquad \lim_{|x|\to+\infty} u(x) = 0 \tag{1.10}$$

and it satisfies

$$\sup_{x\in\mathbb{R}^N\setminus\{0\}} u(x)|x|^{N-2} < +\infty. \tag{1.11}$$

Moreover, $u_{k,V}$ is stable and there exists $c_1 > 0$ independent of k such that

$$\int_{\mathbb{R}^N} |\nabla\xi|^2 dx - p\int_{\mathbb{R}^N} V u_{k,V}^{p-1}\xi^2 dx \geq c_1\left((k^*)^{\frac{p-1}{p}} - k^{\frac{p-1}{p}}\right)\int_{\mathbb{R}^N} |\nabla\xi|^2 dx, \tag{1.12}$$

for all $\xi \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}$.

(ii) *If*

$$p \in \left(0, \frac{N}{N-2}\right) \tag{1.13}$$

and $k \in (0, k^*)$, then the minimal solution $u_{k,V}$ is stable and it satisfies (1.12). Moreover, any non-negative weak solution u of (P_k) is a classical solution of problem (1.10) and it satisfies (1.11).

We notice that in (i) and (ii) of Theorem 1.2, the parameter k is bounded by k_p and k^*, respectively. We do not know if $k_p < k^*$. We also remark that (1.11) implies that the singularity and the decays of u at the origin and infinity respectively are the same as the fundamental solution.

The second solution of (P_k) will be constructed using the Mountain Pass Theorem. Indeed, we will look for critical points of the functional

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\, dx - \int_{\mathbb{R}^N} V F(u_{k,V}, v_+)\, dx \tag{1.14}$$

in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, where $t_+ = \max\{0, t\}$ and

$$F(s,t) = \frac{1}{p+1}\left[(s+t_+)^{p+1} - s^{p+1} - (p+1)s^p t_+\right].$$

To assure that the functional E is well defined, we establish the embeddings

$$\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N, V_0 u_{k,V}^{p-1} dx) \tag{1.15}$$

and

$$\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N, V_0 dx), \tag{1.16}$$

which are compact if

$$p+1 \in (2^*(a_\infty), 2^*(a_0)) \cap [1, 2^*), \tag{1.17}$$

where $2^*(t) = \frac{2N-2t}{N-2}$ with $t \in \mathbb{R}$ and $2^* = 2^*(0)$. Using the compact embeddings we may verify that the functional E satisfies the $(PS)_c$ condition. Furthermore, we can prove the mountain pass structure using the stability of the minimal solution.

Taking into account the range of p for the existence of the minimal solution, we suppose

$$p \in \left(\frac{N-a_\infty}{N-2}, \frac{N-a_0}{N-2}\right) \cap (\max\{2^*(a_\infty) - 1, 0\}, \min\{2^*(a_0) - 1, 2^* - 1\}). \tag{1.18}$$

The intersection of intervals in (1.18) is not empty if we further assume that

$$a_0 < 2,\ a_\infty > \max\{0, 1+\frac{a_0}{2}\}. \tag{1.19}$$

Our result on the existence of the second solution can be stated as follows.

Theorem 1.3. *Suppose that the function V satisfies (1.5) with a_0 and a_∞ given in (1.19), $p > 1$ satisfies (1.18) and k_p is given by (1.9). Then, for $k \in (0, k_p)$, problem (P_k) admits a weak solution $u > u_{k,V}$. Moreover, both u and $u_{k,V}$ are classical solutions of (1.10).*

Although we are not able to show $k_p < k^*$, we may prove that if p satisfies (1.13), then problem (P_k) admits a solution u such that $u > u_{k,V}$ for all $k \in (0, k^*)$.

If V is radially symmetric, the range of p can be improved to

$$p \in (\frac{N-a_\infty}{N-2}, \frac{N-a_0}{N-2}) \cap (\max\{2^*(a_\infty)-1, 0\}, 2^*(a_0)-1), \tag{1.20}$$

as we prove in our last theorem, which states as follows.

Theorem 1.4. *Suppose that the function V is radially symmetric satisfying (1.5) with a_0 and a_∞ given in (1.19), $p > 1$ satisfies (1.20) and k_p is given by (1.9). Then, for $k \in (0, k_p)$, problem (P_k) admits a radially symmetric solution $u > u_{k,V}$, and both u and $u_{k,V}$ are classical solutions of (1.10).*

The paper is organized as follows. In Section §2, we show the existence of the minimal solution of (P_k). Section §3 is devoted to prove regularity and stability of the minimal solution. Finally, in Section §4 we find the second solution of (P_k) using the Mountain Pass theorem.

Along the paper we denote by c_i a positive constant, whose value is not important.

2. Minimal solution

In this section we show the existence of the minimal solution for (P_k). To this end, we construct a monotone sequence of approximating solutions by the iterating technique mentioned in the introduction. In order to get an upper bound of such a sequence we construct a suitable super-solution, based on the following result.

Lemma 2.1. *Assume that the function V satisfies (1.5) with $a_0 < N$, $a_\infty > 0$, and that $p > 0$ satisfies (1.6), the we have*

$$\mathbb{G}[V\mathbb{G}^p[\delta_0]] \le \sigma_2 \mathbb{G}[\delta_0] \quad in \quad \mathbb{R}^N \setminus \{0\}, \tag{2.1}$$

where σ_2 linearly depends on σ_1.

Proof. Since

$$\mathbb{G}[\delta_0](x) = \frac{c_N}{|x|^{N-2}}, \tag{2.2}$$

by the assumption on p, we have

$$V(x)\mathbb{G}^p[\delta_0](x) \le \frac{c_N^p \sigma_1}{(1+|x|^{a_\infty - a_0})|x|^{(N-2)p+a_0}}, \quad \forall x \in \mathbb{R}^N \setminus \{0\}, \tag{2.3}$$

where $c_N > 0$ is the normalized constant depending only on N. We notice that this implies $V\mathbb{G}^p[\delta_0] \in L^1(\mathbb{R}^N)$. Continuing with the proof, we deduce by (2.2) and (2.3) that

$$\begin{aligned} &\mathbb{G}[V\mathbb{G}^p[\delta_0]](x) \\ &\le c_N^{p+1}\sigma_1 \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2}} \frac{1}{(1+|y|^{a_\infty-a_0})|y|^{(N-2)p+a_0}}\, dy \end{aligned}$$

$$= c_N^{p+1}\sigma_1|x|^{2-(N-2)p-a_0}\int_{\mathbb{R}^N}\frac{1}{|e_x-y|^{N-2}}\frac{1}{(1+(|x||y|)^{a_\infty-a_0})|y|^{(N-2)p+a_0}}\,dy$$
$$:= c_N^{p+1}\sigma_1|x|^{2-(N-2)p-a_0}\int_{\mathbb{R}^N}\Phi(x,y)\,dy,$$

where $e_x=\frac{x}{|x|}$. To continue we need to estimate the integral above, for which we consider two cases (i) $|x|\ge 1$ and (ii) $|x|\le 1$.

Case (i) $|x|\ge 1$. Recall that by (1.6) we have $(N-2)p+a_0<N$. We decompose the integral in three parts, first:

$$\int_{B_{\frac12}(0)}\Phi(x,y)\,dy$$
$$\le c_2\int_{B_{\frac12}(0)}\frac{1}{1+(|x||y|)^{a_\infty-a_0}}\frac{1}{|y|^{(N-2)p+a_0}}\,dy$$
$$= c_2|x|^{(N-2)p+a_0-N}\int_{B_{\frac{|x|}{2}}(0)}\frac{1}{1+|z|^{a_\infty-a_0}}\frac{1}{|z|^{(N-2)p+a_0}}\,dz$$
$$\le c_3|x|^{(N-2)p+a_0-N}\left(\int_{B_{\frac12}(0)}\frac{dz}{|z|^{(N-2)p+a_0}}+\int_{B_{\frac{|x|}{2}}(0)\setminus B_{\frac12}(0)}\frac{dz}{|z|^{(N-2)p+a_\infty}}\right)$$
$$\le c_4\left(|x|^{(N-2)p+a_0-N}+|x|^{a_0-a_\infty}\right). \tag{2.4}$$

Next we consider $y\in B_{\frac12}(e_x)$ and we find

$$\frac{1}{1+(|x||y|)^{a_\infty-a_0}}\frac{1}{|y|^{(N-2)p+a_0}}\le c_5|x|^{a_0-a_\infty},$$

from where

$$\int_{B_{\frac12}(e_x)}\Phi(x,y)\,dy\le c_6|x|^{a_0-a_\infty}\int_{B_{\frac12}(e_x)}\frac{1}{|e_x-y|^{N-2}}dy\le c_7|x|^{a_0-a_\infty}. \tag{2.5}$$

Recalling that from (1.6) we have $(N-2)(p+1)+a_\infty>N$, we also obtain that

$$\int_{\mathbb{R}^N\setminus(B_{\frac12}(0)\cap B_{\frac12}(e_x))}\Phi(x,y)\,dy\le c_8|x|^{a_0-a_\infty}\int_{\mathbb{R}^N\setminus B_1(0)}\frac{dy}{|y|^{(N-2)(p+1)+a_\infty}}\le c_9|x|^{a_0-a_\infty}.$$

From here, (2.4), (2.5) and using that, by (1.6), $(N-2)p+a_\infty\ge N$, we conclude that

$$\mathbb{G}[V\mathbb{G}^p[\delta_0]](x)\le c_{10}\max\{|x|^{2-N},|x|^{2-(N-2)p-a_\infty}\}\le c_{11}|x|^{2-N}\quad\text{for } |x|\ge 1. \tag{2.6}$$

Case (ii) $|x|\le 1$. It is not difficult to see that, for appropriate constants,

$$\int_{B_{\frac12}(0)}\Phi(x,y)\,dy\le c_{12}\int_{B_{\frac12}(0)}\frac{1}{|y|^{(N-2)p+a_0}}\,dy\le c_{13}\quad\text{and} \tag{2.7}$$
$$\int_{B_{\frac12}(e_x)}\Phi(x,y)\,dy\le c_{14}\int_{B_{\frac12}(e_x)}\frac{1}{|e_x-y|^{N-2}}\,dy\le c_{15}. \tag{2.8}$$

And we also have that

$$\int\limits_{\mathbb{R}^N\setminus(B_{\frac12}(0)\cap B_{\frac12}(e_x))} \Phi(x,y)\,dy$$

$$\le c_{16}\int\limits_{\mathbb{R}^N\setminus B_1(0)} \frac{1}{1+|x|^{a_\infty-a_0}|y|^{a_\infty-a_0}}\,\frac{1}{|y|^{(N-2)p+a_0+N-2}}\,dy$$

$$\le c_{16}|x|^{(N-2)p+a_0-2}\int\limits_{\mathbb{R}^N\setminus B_1(0)} \frac{dz}{|z|^{(N-2)p+a_0+N-2}(1+|z|^{a_\infty-a_0})}$$

$$\le c_{17}|x|^{(N-2)p+a_0-2}.$$

From here, (2.7) and (2.8) we obtain

$$\mathbb{G}[V\mathbb{G}^p[\delta_0]](x)\le c_{18}|x|^{2-N},\qquad \text{for } |x|\le 1. \tag{2.9}$$

Therefore, the assertion follows by (2.6) and (2.9). □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. First, we prove (i). We consider the iteration scheme defined in (1.7). Observing that

$$v_1=\mathbb{G}[Vv_0^p]+k\mathbb{G}[\delta_0]>v_0,$$

and assuming that

$$v_{n-1}(x)\ge v_{n-2}(x),\qquad x\in\mathbb{R}^N\setminus\{0\},$$

we deduce that

$$v_n=\mathbb{G}[Vv_{n-1}^p]+k\mathbb{G}[\delta_0]\ge \mathbb{G}[Vv_{n-2}^p]+k\mathbb{G}[\delta_0]=v_{n-1}. \tag{2.10}$$

Thus $\{v_n\}_n$ is an increasing sequence. Move-over, we have that

$$\int\limits_{\mathbb{R}^N} v_n(-\Delta)\xi\,dx=\int\limits_{\mathbb{R}^N} Vv_{n-1}^p\xi\,dx+k\xi(0),\qquad \forall\xi\in C_c^{1,1}(\mathbb{R}^N). \tag{2.11}$$

Next we build an upper bound for $\{v_n\}_n$. For $t>0$ we have

$$w_t:=tk^p\mathbb{G}[V\mathbb{G}[\delta_0]^p]+k\mathbb{G}[\delta_0]\le(\sigma_2tk^p+k)\mathbb{G}[\delta_0],$$

where $\sigma_2>0$ is from Lemma 2.1. Then

$$\mathbb{G}[Vw_t^p]+k\mathbb{G}[\delta_0]\le(\sigma_2tk^p+k)^p\mathbb{G}[V\mathbb{G}[\delta_0]^p]+k\mathbb{G}[\delta_0]\le w_t$$

if

$$(\sigma_2tk^{p-1}+1)^p\le t. \tag{2.12}$$

Now we choose t such that (2.12) holds. If $p>1$, since the function $f(t)=(\frac1p(\frac{p-1}{p})^{p-1}t+1)^p$ intersects the line $g(t)=t$ at the unique point t_p, we may choose k and t_p such that

$$\sigma_2k^{p-1}\le\frac1p\left(\frac{p-1}{p}\right)^{p-1}\quad\text{and}\quad t_p=\left(\frac{p}{p-1}\right)^p. \tag{2.13}$$

If $p=1$, we choose $\sigma_1>0$ small so that $\sigma_2<1$ and

$$t_p=\frac{1}{1-\sigma_2}.$$

Finally, if $p < 1$, for $t > 1$, we have

$$(\sigma_2 t k^{p-1} + 1)^p \le (\sigma_2 k^{p-1} + 1)^p t^p,$$

so we may choose

$$t_p = (\sigma_2 k^{p-1} + 1)^{\frac{p}{1-p}}.$$

Hence, by the definition of w_{t_p} and for the chosen t_p, we have $w_{t_p} > v_0$ and

$$v_1 = \mathbb{G}[V v_0^p] + k\mathbb{G}[\delta_0] < \mathbb{G}[V w_{t_p}^p] + k\mathbb{G}[\delta_0] = w_{t_p}.$$

Inductively, we obtain

$$v_n \le w_{t_p}, \quad \text{for all } n \in \mathbb{N}. \tag{2.14}$$

Therefore, the sequence $\{v_n\}$ converges to a function $u_{k,V}$. By (2.11), $u_{k,V}$ is a weak solution of (P_k). We claim that $u_{k,V}$ is the minimal solution of (P_k), that is, for any positive solution u of (P_k), we always have $u_{k,V} \le u$. Indeed, there holds

$$u = \mathbb{G}[V u^p] + k\mathbb{G}[\delta_0] \ge v_0,$$

and then

$$u = \mathbb{G}[V u^p] + k\mathbb{G}[\delta_0] \ge \mathbb{G}[V v_0^p] + k\mathbb{G}[\delta_0] = v_1.$$

We may show inductively that $u \ge v_n$ for all $n \in \mathbb{N}$. The claim follows.

Similarly, if problem (P_k) has a non-negative solution u for $k_1 > 0$, then (P_k) admits a minimal solution $u_{k,V}$ for all $k \in (0, k_1]$. As a result, the mapping $k \mapsto u_{k,V}$ is increasing and we may define

$$k^* = \sup\{k > 0 : \ (P_k) \text{ has a minimal solution for } k\}.$$

We clearly have that $k^* > 0$. We remark that when $0 < p < 1$ or $p = 1$, and $\sigma_1 > 0$ small, we may always find a super-solution w_{t_p}. Hence, there exists a minimal solution for all $k > 0$ and $k^* = \infty$.

Now, we prove that for $p > 1$ we have $k^* < +\infty$. Suppose on the contrary that, problem (P_k) admits a minimal solution $u_{k,V}$ for $k > 0$ large. Let x_0 be a point such that $x_0 \ne 0$, $V(x_0) > 0$ and let $r > 0$ be such that

$$V(x) \ge \frac{V(x_0)}{2}, \quad \forall x \in B_r(x_0).$$

Denote by η_0 a C^2 function such that $\eta_0(x) = 1$ for $x \in B_1(0)$ and $\eta_0(x) = 0$, for $x \in \mathbb{R}^N \setminus B_2(0)$. Let $\eta_0^R(x) = \eta_0(\frac{x-x_0}{R})$ and

$$\xi_R(x) = \mathbb{G}[\chi_{B_r(x_0)}]\eta_0^R(x) \in C_c^{1,1}(\mathbb{R}^N)$$

for $R > r$, where χ_Ω is the characteristic function of Ω. We observe that

$$\lim_{R\to+\infty} \xi_R = \mathbb{G}[\chi_{B_r(x_0)}].$$

Taking ξ_R as a test function with $R > 4r$, we obtain

$$\int_{B_r(x_0)} u_{k,V}\,dx + \int_{B_{2R}(x_0)\setminus B_R(x_0)} u_{k,V}(-\Delta)\xi_R\,dx = \int_{\mathbb{R}^N} V u_{k,V}^p \xi_R\,dx + k\xi_R(0). \tag{2.15}$$

For $x \in B_{2R}(x_0) \setminus B_R(x_0)$, we have

$$|(-\Delta)\xi_R(x)| \le |\nabla\mathbb{G}[\chi_{B_r(x_0)}] \cdot \nabla\eta_0^R(x)| + |\mathbb{G}[\chi_{B_r(x_0)}](-\Delta)\eta_0^R(x)|.$$

Since

$$|\nabla\eta_0^R(x)| \le \frac{c}{R}, \ \ |(-\Delta)\eta_0^R(x)| \le \frac{c}{R^2}, \ \ |\nabla\mathbb{G}[\chi_{B_r(x_0)}]| \le cR^{1-N} \text{ and } |\mathbb{G}[\chi_{B_r(x_0)}]| \le cR^{2-N},$$

we have

$$|(-\Delta)\xi_R(x)| \le c_{19}R^{-N},$$

for $x \in B_{2R}(x_0) \setminus B_R(x_0)$. Since $u_{k,V}$ is a weak solution, we have

$$\lim_{R\to+\infty} \sup_{x\in\mathbb{R}^N\setminus B_R(0)} u_{k,V}(x) = 0,$$

which yields

$$\lim_{R\to+\infty} \int_{B_{2R}(x_0)\setminus B_R(x_0)} u_{k,V}(-\Delta)\xi_R dx = 0.$$

Letting $R \to +\infty$ in (2.15), we see that

$$\int_{B_r(x_0)} u_{k,V}\, dx = \int_{\mathbb{R}^N} V u_{k,V}^p \mathbb{G}[\chi_{B_r(x_0)}]dx + k\mathbb{G}[\chi_{B_r(x_0)}](0).$$

By (2.14) and the fact that

$$u_{k,V} \ge k\mathbb{G}[\delta_0] \quad \text{and} \quad \mathbb{G}[\chi_{B_r(x_0)}] > 0,$$

we obtain then

$$\begin{aligned}\int_{B_r(x_0)} u_{k,V}\, dx &\ge k^{p-1} \int_{\mathbb{R}^N} V u_{k,V}\mathbb{G}[\delta_0]^{p-1}\mathbb{G}[\chi_{B_r(x_0)}]dx + k\mathbb{G}[\chi_{B_r(x_0)}](0)\\ &\ge c_{20}k^{p-1}\int_{B_r(x_0)} u_{k,V} dx + k\mathbb{G}[\chi_{B_r(x_0)}](0)\end{aligned}$$

with $c_{20} > 0$, which is impossible if k is sufficient large, from where the assertion follows.

Next we prove (ii). Let $V_1 \ge V_2$, then we have that u_{k,V_1} is a super-solution of (P_k) with $V = V_2$, whose minimal solution is u_{k,V_2}. Thus we have $u_{k,V_2} \le u_{k,V_1}$.

Finally, we show (iii) is valid. In fact, if V is radially symmetric, so is v_n, which is defined in (1.7) since v_0 is radially symmetric. It follows that the limit $u_{k,V}$ of v_n is radially symmetric too. □

Remark. For future reference, we remark that for $p > 1$ and $k \in (0, k_p]$ with $k_p := \frac{p-1}{p}(\sigma_2 p)^{-\frac{1}{p-1}}$, the minimal solution $u_{k,V}$ verifies

$$u_{k,V} \le w_{t_p} \le c_{21}k\mathbb{G}[\delta_0] \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}, \tag{2.16}$$

for some $c_{21} > 0$ depending only on k_p. Thus, $Vu_{k,V}$ is locally bounded in $\mathbb{R}^N \setminus \{0\}$, which allows us to show that $u_{k,V}$ is a classical solution of (1.10).

3. Properties of minimal solutions

In this section, we establish regularity and decay estimate for weak solutions, as well as the stability for the minimal solution. First we consider our regularity result.

Proposition 3.1. *Assume that the function V satisfies (1.5) with $a_\infty > a_0$ and $a_0 \in \mathbb{R}$, and*

$$p \in \left(0, \frac{N}{N-2}\right). \tag{3.1}$$

Then, any positive weak solution u of (P_k) is a classical solution of (1.10).

Proof. Let u be a weak solution of (P_k). Since $Vu^p \in L^1(\mathbb{R}^N)$, u can be rewritten as

$$u = \mathbb{G}[Vu^p] + k\mathbb{G}[\delta_0].$$

For any $x_0 \in \mathbb{R}^N \setminus \{0\}$, let $r_0 = \frac{1}{4}|x_0|$ and write $B_i = B_{2^{-i}r_0}(x_0)$. Then we have that, for any $i \in \mathbb{N}$,

$$u = \mathbb{G}[\chi_{B_{i-1}} Vu^p] + \mathbb{G}[\chi_{\mathbb{R}^N \setminus B_{i-1}} Vu^p] + k\mathbb{G}[\delta_0]$$

and $V \in L^\infty_{loc}(\mathbb{R}^N \setminus \{0\})$. For $x \in B_i$, we have that

$$\mathbb{G}[\chi_{\mathbb{R}^N \setminus B_{i-1}} Vu^p](x) = \int\limits_{\mathbb{R}^N \setminus B_{i-1}} \frac{c_N V(y)u^p(y)}{|x-y|^{N-2}} dy,$$

then, for some $C_i > 0$, we have

$$\|\mathbb{G}[\chi_{\mathbb{R}^N \setminus B_i} Vu^p]\|_{C^2(B_{i-1})} \le C_i \|Vu^p\|_{L^1(B_{2r_0}(x_0))}. \tag{3.2}$$

From (2.2), we easily see that, for some constant $c_i > 0$ depending on i, we have

$$\|\mathbb{G}[\delta_0]\|_{C^2(B_{i-1})} \le c_i |x_0|^{2-N}. \tag{3.3}$$

By (iii) of Proposition 5.1, $Vu^p \in L^{q_0}(B_{2r_0}(x_0))$ with $q_0 = \frac{1}{2}(1 + \frac{1}{p}\frac{N}{N-2}) > 1$. By Proposition 5.1 again we find

$$\mathbb{G}[\chi_{B_{2r_0}(x_0)} Vu^p] \in L^{p_1}(B_{2r_0}(x_0)) \text{ with } \quad p_1 = \frac{Nq_0}{N-2q_0}.$$

Similarly,

$$Vu^p \in L^{q_1}(B_{r_0}(x_0)) \text{ with } \quad q_1 = \frac{p_1}{p},$$

and

$$\mathbb{G}[\chi_{B_{r_0}(x_0)} Vu^p] \in L^{p_2}(B_{r_0}(x_0)) \text{ with } \quad p_2 = \frac{Nq_1}{N-2q_1}.$$

Let $q_i = \frac{p_i}{p}$ and $p_{i+1} = \frac{Nq_i}{N-2q_i}$ if $N - 2q_i > 0$. Then we obtain inductively that

$$Vu^p \in L^{q_i}(B_i) \quad \text{and} \quad \mathbb{G}[\chi_{B_i} Vu^p] \in L^{p_{i+1}}(B_i).$$

We may verify that

$$\frac{q_{i+1}}{q_i} = \frac{1}{p}\frac{N}{N-2q_i} > \frac{1}{p}\frac{N}{N-2q_1} > 1.$$

Therefore, $\lim_{i\to+\infty} q_i = +\infty$, so there exists i_0 such that $N - 2q_{i_0} > 0$, but $N - 2q_{i_0+1} < 0$, and we deduce that

$$\mathbb{G}[\chi_{B_{i_0}} Vu^p] \in L^\infty(B_{i_0}).$$

As a result,

$$u(x_0) \le c_{i_0} \|\mathbb{G}[\delta_0]\|_{L^\infty(B_{2r_0})(x_0)} + c_{i_0} \|Vu^p\|_{L^1(B_{2r_0})(x_0)} \to 0 \quad \text{as} \quad |x_0| \to +\infty$$

and

$$Vu^p \in L^\infty(B_{i_0}).$$

On the other hand, by Proposition 5.2,

$$|\nabla \mathbb{G}[\chi_{B_{i_0}} Vu^p]| \in L^\infty(B_{i_0}).$$

By elliptic regularity, we know from (3.3) that u is Hölder continuous in B_{i_0} and so is Vu^p. Hence, u is a classical solution of (1.10). □

Next, we study the singularity of the weak solution of (P_k) at the origin and the decay at infinity.

Proposition 3.2. *Suppose that the function V satisfies (1.5), with a_0 and a_∞ given in (1.19), and $p>1$ satisfies (1.18) and (3.1). Let u be a weak solution of (P_k), then*

$$\sup_{x\in\mathbb{R}^N\setminus\{0\}} u(x)|x|^{N-2}<+\infty. \tag{3.4}$$

Proof. From Proposition 3.1 u is a classical solution of (1.10), so we only consider the singularity of u at the origin and the decay at infinity. We first consider the singularity at the origin. We claim that

$$\lim_{|x|\to 0^+} u(x)|x|^{N-2}=c_N k. \tag{3.5}$$

Indeed, since $\mathbb{G}[Vu^p\chi_{\mathbb{R}^N\setminus B_1(0)}]\in C^2(B_{\frac12}(0))$ and $k\mathbb{G}[\delta_0](x)=c_Nk|x|^{2-N}$, we see from

$$u=\mathbb{G}[Vu^p\chi_{B_1(0)}]+k\mathbb{G}[\delta_0]+\mathbb{G}[Vu^p\chi_{\mathbb{R}^N\setminus B_1(0)}] \tag{3.6}$$

that it is sufficient to estimate $\mathbb{G}[Vu^p\chi_{B_1(0)}]$ in $B_{\frac12}(0)$. Let

$$u_1=\mathbb{G}[Vu^p\chi_{B_1(0)}].$$

Since $Vu^p\in L^{s_0}(B_{\frac12}(0))$ with $s_0=\frac12[1+\frac1p\frac{N}{N-2}]>1$ from Proposition 5.1 we see that $u_1\in L^{s_1p}(B_{\frac12}(0))$, that is, $u_1^p\in L^{s_1}(B_{\frac12}(0))$ with

$$s_1=\frac1p\frac{N}{N-2s_0}s_0.$$

By (3.6),

$$u^p\le c_{22}(u_1^p+k^p\mathbb{G}^p[\delta_{x_0}]+1)\quad\text{in}\ \ B_1(0), \tag{3.7}$$

where $c_{22}>0$. By the definition of u_1 and (3.7), we obtain

$$u_1\le c_{22}(\mathbb{G}[u_1^p]+k^p\mathbb{G}[V\mathbb{G}^p[\delta_0]]+\mathbb{G}[\chi_{B_1(0)}]), \tag{3.8}$$

where

$$\mathbb{G}[\chi_{B_1(0)}]]\in L^\infty(B_{\frac12}(0)),\quad k^p\mathbb{G}[V\mathbb{G}^p[\delta_0]](x)\le c_{23}|x|^{(2-N)p-a_0+2}$$

and

$$(2-N)p-a_0+2>2-N.$$

If $s_1>\frac12Np$, by Proposition 5.1, $u_1\in L^\infty(B_{2^{-1}}(0))$. Hence, we know from (3.8) that

$$u_1(x)\le c_{24}|x|^{(2-N)p-a_0+2} \tag{3.9}$$

in $B_{2^{-1}}(0)$. Since $(2-N)p-a_0+2>2-N$, we deduce from (3.6) and (3.9) that (3.5) holds. On the other hand, if $s_1<\frac12Np$, we proceed as above. Let

$$u_2=\mathbb{G}[\chi_{B_{2^{-1}}(0)}u_1^p].$$

By Proposition 5.1, $u_2\in L^{s_2p}(B_{2^{-1}}(0))$, where

$$s_2=\frac1p\frac{Ns_1}{Np-2s_1}>\frac{N}{N-s_0}s_1>\left(\frac1p\frac{N}{N-2s_0}\right)^2s_0.$$

Inductively, we define

$$s_m=\frac1p\frac{Ns_{m-1}}{Np-2s_{m-1}}>\left(\frac1p\frac{N}{N-2s_0}\right)^m s_0.$$

So there is $m_0\in\mathbb{N}$ such that

$$s_{m_0} > \frac{1}{2}Np \quad \text{and} \quad u_{m_0} \in L^\infty(B_{2^{-m_0-1}}(0)).$$

Therefore, (3.5) holds. Next, we establish the decay at infinity, that is,

$$\limsup_{|x|\to+\infty} u(x)|x|^{N-2} < +\infty. \tag{3.10}$$

We know from Proposition 3.1 that $u \in L^\infty_{loc}(\mathbb{R}^N \setminus \{0\})$ and

$$\lim_{|x|\to+\infty} u(x) = 0. \tag{3.11}$$

We divide the proof into three parts: (a) $a_\infty > N$; (b) $a_\infty \in (2, N]$; (c) $a_\infty \in (0, 2]$.
Case (a) $a_\infty > N$. Let $\psi_0(x) = |x|^{2-N} - |x|^{2-a_\infty}$ for $|x| \geq 2$. There exists $c_{25} > 0$ such that

$$-\Delta\psi_0(x) \geq c_{25}|x|^{-a_\infty}.$$

By (3.11) and the assumption on V, there exist constants $A, B \geq 1$ such that

$$V(x)u^p(x) \leq A|x|^{-a_\infty} \quad \text{if} \quad |x| \geq 2 \quad \text{and} \quad u(x) \leq B(2^{2-N} - 2^{2-a_\infty}) \quad \text{if} \quad |x| = 2.$$

The, by the comparison principle, we find

$$u(x) \leq AB\psi_0 \leq AB|x|^{2-N} \quad \text{if } \ |x| \geq 2.$$

Case (b) $a_\infty \in (2, N]$. Let

$$\tau_1 = \begin{cases} 2 - a_\infty & \text{if} \quad a_\infty \in (2, N), \\ \frac{1}{p}(2 - N) & \text{if} \quad a_\infty = N, \end{cases}$$

and define $\psi_1(x) = |x|^{\tau_1}$. Hence, there exists $c_{26} > 0$ such that

$$-\Delta\psi_1(x) \geq c_{26}|x|^{-a_\infty}, \qquad x \neq 0.$$

We may find constants $A, B \geq 1$ such that

$$V(x)u^p(x) \leq A|x|^{\tau_1-2} \quad \text{if} \quad |x| \geq 1 \quad \text{and} \quad u(x) \leq B \quad \text{if} \quad |x| \geq 1.$$

Then, by the comparison principle again,

$$u(x) \leq AB\psi_1(x) \quad \text{for } \ |x| \geq 1.$$

Next we define $\tau_2 = 2 - a_\infty + p\tau_1$. If $\tau_2 \in [-N, -2)$, we define $\psi_2(x) = |x|^{\tau_2}$ and we repeat the above argument to obtain

$$u(x) \leq c_{26}\psi_2(x) \quad \text{if } \ |x| \geq 1. \tag{3.12}$$

Inductively, we define

$$\tau_j = 2 - a_\infty + p\tau_{j-1}.$$

Then there exists $j_0 \in \mathbb{N}$ such that $\tau_{j_0-1} > -N$ and $\tau_{j_0} < -N$. If $\tau_{j_0-1} > -N$, we proceed as above. If $\tau_{j_0} < -N$, set

$$\psi_{\tau_{j_0}}(x) = |x|^{2-N} - |x|^{2+\tau_{j_0}}$$

and we reduce the problem to the case (a) $a_\infty > N$. Then, (3.10) holds.

Finally, we consider the case (c) $a_\infty \in (0, 2]$. For $|x| > 2$ fixed, let $r_0 = \frac{1}{2}|x|^{\frac{a_\infty}{N}}$, where $\frac{a_\infty}{N} \in (0, 1)$. Therefore,

$$\begin{aligned}\mathbb{G}[Vu^p](x) &= \int\limits_{B_{r_0}(x)} \frac{c_N}{|x-y|^{N-2}} V(y)u^p(y)dy + \int\limits_{\mathbb{R}^N \setminus B_{r_0}(x)} \frac{c_N}{|x-y|^{N-2}} V(y)u^p(y)dy \\ &\leq c_{27}(|x| - r_0)^{-a_\infty} \|u\|^p_{L^\infty(B_r(x))} r_0^2 + r_0^{2-N} \|Vu^p\|_{L^1(\mathbb{R}^N)} \\ &\leq c_{28}|x|^{-(1+\frac{2}{N})a_\infty}.\end{aligned}$$

Recalling

$$u = \mathbb{G}[Vu^p] + k\mathbb{G}[\delta_0] \quad \text{and} \quad \mathbb{G}[\delta_0](x) = c_N|x|^{2-N},$$

from above we have

$$u \le c_{29}|x|^{2-N} \quad \text{if} \quad \gamma_0 := (1+\frac{2}{N})a_\infty \ge N-2$$

and we conclude the proof. Or we have

$$u(x) \le c_{30}|x|^{-\gamma_0} \quad \text{if} \quad \gamma_0 < N-2.$$

In the case $a_\infty + p\gamma_0 \le 2$, let $r_1 = \frac{1}{2}|x|^{\frac{a_\infty+p\gamma_0}{N}}$, where $\frac{a_\infty+p\gamma_0}{N} \in (0,1)$. Then we have

$$\begin{aligned}\mathbb{G}[Vu^p](x) &= \int\limits_{B_{r_1}(x)} \frac{c_N}{|x-y|^{N-2}}V(y)u^p(y)dy + \int\limits_{\mathbb{R}^N\setminus B_{r_1}(x)} \frac{c_N}{|x-y|^{N-2}}V(y)u^p(y)dy \\ &\le c_{31}(|x|-r_1)^{-a_\infty-p\gamma_0}r_1^2 + c_N r^{2-N}\|Vu^p\|_{L^1(\mathbb{R}^N)} \\ &\le c_{32}|x|^{-(1+\frac{2}{N})(a_\infty+p\gamma_0)},\end{aligned}$$

which implies that

$$u(x) \le c_{33}|x|^{-\gamma_1},$$

where $\gamma_1 = (1+\frac{2}{N})(a_\infty+p\gamma_0)$. Inductively, we define $r_j = |x|^{\gamma_j}$ with $\gamma_j = (1+\frac{2}{N})(a_\infty+p\gamma_j)$. There exists $j_0 \in \mathbb{N}$ such that $a_\infty + p\gamma_{j_0-1} \le 2$ and $a_\infty + p\gamma_{j_0} > 2$. In the former case, we iterate as above; in the latter case, we have

$$V(x)u^p(x) \le c_{34}|x|^{a_\infty+p\gamma_{j_0}}.$$

By the proof of (b), (3.10) holds and the proof is complete. □

Now, we consider the stability of the minimal solution of (P_k).

Proposition 3.3. *Assume that the function V satisfies (1.5) with $a_0 < \min\{2, a_\infty\}$, $p > 1$ satisfies (1.18) and $k \in (0, k^*)$. Then, any minimal positive solution $u_{k,V}$ of (P_k) is stable. Moreover, $u_{k,V}$ satisfies* (1.12).

Proof. We start proving the stability of the minimal solution for $k > 0$ small, and next we prove it for all $k < k^*$. By (2.16), for $k > 0$ small,

$$u_{k,V}(x) \le c_{35}k|x|^{2-N} \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},$$

where $c_{35} > 0$ is independent of k. Therefore,

$$V(x)u_{k,V}^{p-1}(x) \le c_{35}^{p-1}\sigma_1 k^{p-1}\frac{|x|^{(2-N)(p-1)-a_0}}{1+|x|^{a_\infty-a_0}}. \tag{3.13}$$

By our assumption on p we have

$$(2-N)(p-1) - a_0 \ge -2 \quad \text{and} \quad (2-N)(p-1) - a_\infty < -N.$$

From here and (3.13) we find

$$V(x)u_{k,V}^{p-1}(x) \le c_{35}^{p-1}\sigma_1\frac{k^{p-1}}{|x|^2}. \tag{3.14}$$

Hence, for any $\xi \in C_c^{1,1}(\mathbb{R}^N)$, by (3.14) and the Hardy–Sobolev inequality, we deduce for $k > 0$ small that

$$\int\limits_{\mathbb{R}^N} Vu_{k,V}^{p-1}\xi^2\,dx \le c_{35}^{p-1}\sigma_1 k^{p-1}\int\limits_{\mathbb{R}^N} \frac{\xi^2(x)}{|x|^2}\,dx \le \frac{1}{p}\int\limits_{\mathbb{R}^N} |\nabla\xi|^2\,dx. \tag{3.15}$$

By density, (3.15) holds also for $\xi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, which means that $u_{k,V}$ is a semi-stable solution of (P_k), for $k > 0$ small.

Next, we prove the stability of minimal solutions for all $k \in (0, k^*)$. Suppose that if u_k is not stable, then we have that

$$\lambda_1 := \inf_{\xi \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla \xi|^2 \, dx}{p \int_{\mathbb{R}^N} V u_{k,V}^{p-1} \xi^2 \, dx} \le 1. \tag{3.16}$$

By compact embedding (Lemma 4.1), λ_1 is achieved by a non-negative function ξ_1 satisfying

$$-\Delta \xi_1 = \lambda_1 p V u_{k,V}^{p-1} \xi_1.$$

Choosing $\hat{k} \in (k, k^*)$ and letting $w = u_{\hat{k},V} - u_{k,V} > 0$, we have that

$$w = \mathbb{G}[V u_{\hat{k},V}^p - V u_{k,V}^p] + (\hat{k} - k)\mathbb{G}[\delta_0].$$

By the elementary inequality $(a+b)^p \ge a^p + p a^{p-1} b$ for $a, b \ge 0$, we infer that

$$w \ge \mathbb{G}[p V u_{k,V}^{p-1} w] + (\hat{k} - k)\mathbb{G}[\delta_0].$$

Multiplying this inequality by $-\Delta \xi_1$, we obtain

$$\begin{aligned} \lambda_1 \int_{\mathbb{R}^N} p V u_{k,V}^{p-1} w \xi_1 \, dx &= \int_{\mathbb{R}^N} (-\Delta) w \xi_1 \, dx \\ &\ge \int_{\mathbb{R}^N} p V u_{k,V}^{p-1} w \xi_1 \, dx + (\hat{k} - k)\xi_1(0) > \int_{\mathbb{R}^N} p V u_{k,V}^{p-1} w \xi_1 \, dx, \end{aligned}$$

which is impossible. Consequently,

$$p \int_{\mathbb{R}^N} V u_{k,V}^{p-1} \xi^2 dx < \int_{\mathbb{R}^N} |\nabla \xi|^2 dx, \quad \forall \xi \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\},$$

proving stability of $u_{k,V}$. In what follows we prove (1.12). For any $k \in (0, k^*)$, let $k' = \frac{k+k^*}{2} > k$ and $l_0 = (\frac{k}{k'})^{\frac{1}{p}} < 1$. Then there exists a minimal solution $u_{k',V}$ of (P_k), which is stable. Since $k - k' l_0^p = 0$, we deduce that

$$\begin{aligned} l_0 u_{k',V} \ge l_0^p u_{k',V} &= l_0^p \left(\mathbb{G}[V u_{k',V}^p] + k' \mathbb{G}[\delta_0] \right) + (k - k' l_0^p)\mathbb{G}[\delta_0] \\ &= \mathbb{G}[V (l_0 u_{k',V})^p] + k \mathbb{G}[\delta_0], \end{aligned}$$

that is, $l_0 u_{k',V}$ is a super-solution of (P_k). Therefore,

$$l_0 u_{k',V} \ge u_{k,V}.$$

Thus, for $\xi \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}$,

$$\begin{aligned} 0 < \int_{\mathbb{R}^N} |\nabla \xi|^2 dx - p \int_{\mathbb{R}^N} V u_{k',V}^{p-1} \xi^2 \, dx &\le \int_{\mathbb{R}^N} |\nabla \xi|^2 \, dx - p l_0^{1-p} \int_{\mathbb{R}^N} V u_{k,V}^{p-1} \xi^2 \, dx \\ &= l_0^{1-p} \left[l_0^{p-1} \int_{\mathbb{R}^N} |\nabla \xi|^2 \, dx - p \int_{\mathbb{R}^N} V u_{k,V}^{p-1} \xi^2 dx \right], \end{aligned}$$

which implies that

$$\int_{\mathbb{R}^N} |\nabla\xi|^2 dx - p\int_{\mathbb{R}^N} V u_{k,V}^{p-1}\xi^2 dx$$
$$= (1-l_0^{p-1})\int_{\mathbb{R}^N} |\nabla\xi|^2 dx + \left[l_0^{p-1}\int_{\mathbb{R}^N} |\nabla\xi|^2 dx - p\int_{\mathbb{R}^N} V u_{k,V}^{p-1}\xi^2 dx \right]$$
$$\geq (1-l_0^{p-1})\int_{\mathbb{R}^N} |\nabla\xi|^2 dx.$$

This inequality together with the fact

$$1-l_0^{p-1} \geq c_{36}[(k^*)^{\frac{p-1}{p}} - k^{\frac{p-1}{p}}],$$

implies (1.12), completing the proof. □

Corollary 3.1. *Assume that $p>1$, the function V satisfies (1.5) with $a_0 < \min\{2, a_\infty\}$. Then, for $k \in (0, k_p)$, the minimal solution $u_{k,V}$ of (P_k) is stable and satisfies (1.10) as well as (1.12).*

Proof. Since for $k \leq k_p$, the minimal solution $u_{k,V}$ of (P_k) is controlled by w_{t_p}, which implies that $Vu_{k,V} \in L^\infty_{loc}(\mathbb{R}^N \setminus \{0\})$. It follows by Proposition 5.1 and 5.2 that $u_{k,V}$ is a classical solution of (1.10). The proof is completed by the proof of Proposition 3.3 and (2.16). □

Proof of Theorem 1.2. The theorem follows by Propositions 3.1 and 3.3, and Corollary 3.1. □

4. Mountain Pass solution

In order to find the second solution of (P_k), we look for a non-trivial function u so that $u_{k,V}+u$ is a solution of (P_k), which is different from the minimal solution $u_{k,V}$ of (P_k). We are then led to consider the problem

$$-\Delta u = V(u_{k,V}+u_+)^p - Vu_{k,V}^p \quad \text{in} \quad \mathbb{R}^N, \qquad \lim_{|x|\to+\infty} u(x) = 0. \tag{4.1}$$

Intuitively, the cancellation of the singularity of $u_{k,V}$ in the nonlinear term on the right hand side of (4.1) allows us to find a solution of (4.1) as a critical point of the functional

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx - \int_{\mathbb{R}^N} VF(u_{k,V}, v_+) dx \tag{4.2}$$

defined on $\mathcal{D}^{1,2}(\mathbb{R}^N)$, where

$$F(s,t) = \frac{1}{p+1}\Big[(s+t_+)^{p+1} - s^{p+1} - (p+1)s^p t_+\Big]. \tag{4.3}$$

Let V_0 be given in (1.5) and denote by $L^q(\mathbb{R}^N, V_0 dx)$ the weighted L^q space defined by

$$L^q(\mathbb{R}^N, V_0 dx) = \{u : \int_{\mathbb{R}^N} V_0|u|^q dx < +\infty\}.$$

The following lemma implies that the functional E is well-defined on $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

Lemma 4.1. *Let $a_0 < 2$, $a_\infty > \max\{a_0, 0\}$ and $p>1$ satisfy (1.18). Then the inclusion $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N, V_0 dx)$ is continuous and compact.*

Proof. For $\beta \in (0,2)$, it follows by Hölder, Hardy and Sobolev's inequalities that

$$\int_{\mathbb{R}^N} \frac{\xi^{2^*(\beta)}}{|x|^\beta} dx \le \left(\int_{\mathbb{R}^N} \frac{\xi^2}{|x|^2} dx\right)^{\frac{\beta}{2}} \left(\int_{\mathbb{R}^N} \xi^{2^*} dx\right)^{\frac{2-\beta}{2}}$$
$$\le c_{36} \left(\int_{\mathbb{R}^N} |\nabla \xi|^2 dx\right)^{\frac{2^*(\beta)}{2}}. \tag{4.4}$$

We claim that the inclusion $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N, V_0 dx)$ is continuous if

$$\max\{2^*(a_\infty), 1\} \le q \le \min\{2^*(a_0), 2^*\}. \tag{4.5}$$

For $\xi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, if $0 \le a_0 < 2$, by (4.4) we have,

$$\|\xi\|_{L^{2^*(a_0)}(B_1(0),|x|^{a_0}dx)} \le \|\xi\|_{L^{2^*(a_0)}(\mathbb{R}^N,|x|^{a_0}dx)} \le c_{36}\|\xi\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}.$$

If $a_0 < 0$, by Sobolev inequality we have,

$$\|\xi\|_{L^{2^*}(B_1(0),|x|^{a_0}dx)} \le \|\xi\|_{L^{2^*}(B_1(0))} \le c_{36}\|\xi\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}$$

and using Hölder's inequality, we obtain

$$\|\xi\|_{L^q(B_1(0),|x|^{a_0}dx)} \le c_{36}\|\xi\|_{L^{2^*(a_0)}(B_1(0),|x|^{a_0}dx)} \le c_{36}\|\xi\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}.$$

On the other hand, for $a_\infty \in (0,2]$, we have $2^*(a_\infty) < q \le 2^*$, by Hölder's inequality, (4.4) and considering $\tau = N - \frac{q(N-2)}{2} < a_\infty$, we have

$$\int_{\mathbb{R}^N\setminus B_1(0)} |\xi|^q |x|^{-a_\infty} dx \le \int_{\mathbb{R}^N\setminus B_1(0)} |\xi|^q |x|^{-\tau} dx \le \|\xi\|^{\frac{q}{2}}_{\mathcal{D}^{1,2}(\mathbb{R}^N)}.$$

The case $a_\infty > 2$ can be reduced to $a_\infty \in (0,2]$. In conclusion, for all the cases we have

$$\|\xi\|_{L^q(B_1(0),|x|^{a_0}dx)} \le c_{36}\|\xi\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}, \qquad \|\xi\|_{L^{2^*}(\mathbb{R}^N\setminus B_1(0),|x|^{a_\infty}dx)} \le \|\xi\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}. \tag{4.6}$$

Combining this with the fact that

$$\lim_{t\to 0^+} V_0(t)t^{a_0} = \sigma_1 \quad \text{and} \quad \lim_{t\to +\infty} V_0(t)t^{a_\infty} = \sigma_1,$$

yields the claim. Next we show that the inclusion $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N, V_0 dx)$ is compact if (4.5) holds. Let $\{\xi_n\}_n$ be a bounded sequence in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. For any $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{\mathbb{R}^N\setminus B_R(0)} |\xi_n|^q |x|^{-a_\infty} dx \le R^{-a_\infty+\tau} \int_{\mathbb{R}^N\setminus B_R(0)} |\xi_n|^q |x|^{-\tau} dx \le c_{37} R^{-a_\infty+\tau} \le \frac{\varepsilon}{2}, \tag{4.7}$$

where τ was defined above. By Sobolev embedding, $\xi_n \to \xi$ in $H^1(B_R(0))$ up to a subsequence. This, together with (4.7), yields the result. □

Corollary 4.1. *The inclusion $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N, V_0 u_{k,V}^{p-1} dx)$ is continuous and compact if $k \le k_p$.*

Proof. Since $u_{k,V}(x) \le c_{21} c_N k|x|^{2-N}$ if $k \le k_p$, there exists $c_{38} > 0$ such that

$$\limsup_{|x|\to 0^+} u_{k,V}^{p-1}(x) V_0(x) |x|^{a_0+(p-1)(N-2)} \le c_{38}$$

and

$$\limsup_{|x|\to+\infty} u_{k,V}^{p-1}(x)V_0(x)|x|^{a_\infty+(p-1)(N-2)} \le c_{38}.$$

By the proof of Lemma 4.1, we see that the inclusion is continuous and compact if

$$\max\{2^*(a_\infty+(p-1)(N-2)),1\} < q < \min\{2^*(a_0+(p-1)(N-2)),2^*\}. \tag{4.8}$$

This is the case if $q=2$, so the assertion of the corollary follows. □

Proof of Theorem 1.3. We use the Mountain Pass Theorem. For any $\epsilon>0$ we have

$$0\le F(s,t)\le (p+\epsilon)s^{p-1}t^2+C_\epsilon t^{p+1}, \quad s,\, t\ge 0, \tag{4.9}$$

where $C_\epsilon>0$. By (1.5) and Lemma 4.1, for any $v\in\mathcal{D}^{1,2}(\mathbb{R}^N)$ we have

$$\begin{aligned}\int_{\mathbb{R}^N} VF(u_{k,V},v_+)\,dx &\le (p+\epsilon)\int_{\mathbb{R}^N} Vu_{k,V}^{p-1}v_+^2\,dx + C_\epsilon\int_{\mathbb{R}^N} Vv_+^{p+1}\,dx \\ &\le c_\epsilon(\|v\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}+\|v\|^{p+1}_{\mathcal{D}^{1,2}(\mathbb{R}^N)}),\end{aligned} \tag{4.10}$$

where $c_\epsilon>0$. Thus E is well-defined. We may also verify that E is C^1 on $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

Next we prove the geometric assumption of the Mountain Pass Theorem. Let $v\in\mathcal{D}^{1,2}(\mathbb{R}^N)$ be such that $\|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}=1$, $k\in(0,k_p)$ and $\epsilon>0$ small enough, then from Corollary 3.1 and (4.9) we have

$$\begin{aligned}E(tv)&=\frac12\|tv\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}-\int_{\mathbb{R}^N} VF(u_{k,V},tv_+)\,dx \\ &\ge t^2\left(\frac12\|v\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}-(p+\epsilon)\int_{\mathbb{R}^N} V_0v_k^{p-1}v^2dx\right)-C_\epsilon t^{p+1}\int_{\mathbb{R}^N} V_0|v|^{p+1}dx \\ &\ge c_{39}t^2\|v\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}-c_{40}t^{p+1}\|v\|^{p+1}_{\mathcal{D}^{1,2}(\mathbb{R}^N)}=c_{39}t^2-c_{40}t^{p+1},\end{aligned}$$

where $c_{39},c_{40}>0$. So there exists $t_0>0$ small such that

$$E(t_0v)\ge\frac{c_{39}}{4}t_0^2=:\beta>0.$$

On the other hand, we fix a non-negative function $v_0\in\mathcal{D}^{1,2}(\mathbb{R}^N)$ with $\|v_0\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}=1$ and its support is a subset of the supp$V$. Since $(a+b)^p\ge a^p+b^p$ for $a,b>0$ and $p>1$,

$$F(u_{k,V},tv_0)\ge\frac{1}{p+1}\left(t^{p+1}v_0^{p+1}-(p+1)u_{k,V}^p tv_0\right).$$

Next we see that there exists $T>0$ such that for $t\ge T$,

$$\begin{aligned}E(tv_0)&=\frac{t^2}{2}\|v_0\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}-\int_{\mathbb{R}^N} VF(u_{k,V},tv_0)dx \\ &\le\frac{t^2}{2}\|v_0\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}-\frac{1}{p+1}t^{p+1}\int_{\mathbb{R}^N} Vv_0^{p+1}dx+t\int_{\mathbb{R}^N} Vu_{k,V}^p v_0dx\le 0.\end{aligned}$$

Choosing $e=Tv_0$, we have $E(e)\le 0$. Next, we verify that E satisfies the Palais–Smale condition at level c. Let $\{v_n\}\subset\mathcal{D}^{1,2}(\mathbb{R}^N)$ satisfies $E(v_n)\to c$ and $E'(v_n)\to 0$ as $n\to\infty$, where c is the mountain pass level

$$c=\inf_{\gamma\in\Gamma}\max_{s\in[0,1]}E(\gamma(s)), \tag{4.11}$$

$\Gamma=\{\gamma\in C([0,1];\mathcal{D}^{1,2}(\mathbb{R}^N)):\gamma(0)=0,\ \gamma(1)=e\}$, and $c\ge\beta$. Next we consider the inequality

$$f(s,t)t-(2+c_p)F(s,t)\ge -\frac{c_p p}{2}s^{p-1}t^2,\quad s,t\ge 0$$

proved in [20, C.2 (iv)], where $f(s,t)=(s+t_+)^p-s^p$ and $c_p=\min\{1,p-1\}$. From here, (1.12), $E(v_n)\to c$ and $E'(v_n)\to 0$ we find constants $c_{41}, c_{42}>0$ such that

$$\begin{aligned}c_{41}+c_{41}\|v_n\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} &\ge \frac{c_p}{2}\|v_n\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}-\int_{\mathbb{R}^N}(2+c_p)VF(u_{k,V},(v_n)_+)dx\\ &\quad+\int_{\mathbb{R}^N}(2+c_p)Vf(u_{k,V},(v_n)_+)(v_n)_+dx\\ &\ge \frac{c_p}{2}\left[\|v_n\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}-p\int_{\mathbb{R}^N}Vu_{k,V}^{p-1}v_n^2dx\right]\\ &\ge c_{42}\frac{c_p}{2}\|v_n\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}.\end{aligned}$$

Therefore, v_n is uniformly bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ for $k\in(0,k^*)$. We may assume that there exists $v\in\mathcal{D}^{1,2}(\mathbb{R}^N)$ such that

$$v_n\rightharpoonup v\quad\text{in}\quad \mathcal{D}^{1,2}(\mathbb{R}^N)\quad\text{and}\quad v_n\to v\quad\text{a.e. in }\mathbb{R}^N.$$

By Lemma 4.1,

$$v_n\to v\quad\text{in }\ L^2(\mathbb{R}^N,V_0u_{k,V}^{p-1}dx)\quad\text{and in }L^{p+1}(\mathbb{R}^N,V_0dx),\quad\text{as }n\to\infty.$$

Invoking the inequality

$$\begin{aligned}&|F(u_{k,V},v_n)-F(u_{k,V},v)|\\ &\quad=\frac{1}{p+1}|(u_{k,V}+(v_n)_+)^p-(u_{k,V}+v_+)^p-(p+1)u_{k,V}^p((v_n)_+-v_+)|\\ &\quad\le (p+\epsilon)u_{k,V}^{p-1}((v_n)_+-v_+)^2+C_\epsilon((v_n)_+-v_+)^{p+1},\end{aligned}$$

we have

$$F(u_{k,V},v_n)\to F(u_{k,V},v)\quad\text{a.e. in }\mathbb{R}^N\quad\text{and in }L^1(\mathbb{R}^N,V_0dx).$$

This, together with $\lim_{n\to\infty}E(v_n)=c$, implies $\|v_n\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}\to\|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}$ as $n\to\infty$. Hence, $v_n\to v$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ as $n\to\infty$.

Now we use the Mountain Pass Theorem in [2] to find a non-trivial, non-negative critical point $v_k\in\mathcal{D}^{1,2}(\mathbb{R}^N)$ of E, which is a weak solution of (4.1). Hence,

$$\int_{\mathbb{R}^N}\nabla(u_{k,V}+v_k)\cdot\nabla\varphi\,dx=\int_{\mathbb{R}^N}V(u_{k,V}+v_k)^p\varphi\,dx,\tag{4.12}$$

for all $\varphi\in\mathcal{D}^{1,2}(\mathbb{R}^N)$ with $0\notin\operatorname{supp}\varphi$. Next we show as in (2.3) of [10], that for any $x_0\neq 0$ and $r<\frac12|x_0|$, there holds

$$\sup_{|x-x_0|<r}|u(x)|=\lim_{q\to+\infty}\Big(\int_{B_r(x_0)}V(x)|u(x)|^q\,dx\Big)^{\frac1q}.$$

Then, by the assumption that $p<\frac{N-a_0}{N-2}$ and $u_{k,V}(x)\le c_{21}k|x|^{2-N}$, there exists $q>\frac N2$ such that

$$Vu_{k,V}^{p-1}\in L^q_{loc}(\mathbb{R}^N\setminus\{0\}).$$

Therefore, by the Moser–Nash iteration as, for instance in [10,13], that $u_{k,V}+v_k\in L^\infty_{loc}(\mathbb{R}^N\setminus\{0\})$, from where $u_{k,V}+v_k\in C^2_{loc}(\mathbb{R}^N\setminus\{0\})$. Moreover, by Theorem 2 in [10] we have

$$\limsup_{|x|\to+\infty}(u_{k,V}(x)+v_k(x))|x|^{N-2}<+\infty,$$

from where

$$V(u_{k,V}+v_k)^p\in L^1(\mathbb{R}^N).$$

Now we may conclude that

$$\int_{\mathbb{R}^N}(u_{k,V}+v_k)(-\Delta)\xi dx=\int_{\mathbb{R}^N}V(u_{k,V}+v_k)^p\xi dx+k\xi(0),\quad \forall\xi\in C_c^{1,1}(\mathbb{R}^N). \tag{4.13}$$

This means that $v_k+u_{k,V}$ is weak solution of (P_k) and also implies $v_k+u_{k,V}$ is a classical solution of (1.10). The maximum principle yields $v_k>0$ and then $v_k+u_{k,V}>u_{k,V}$, completing the proof of the theorem. □

Finally, we consider the case that V is radially symmetric. Denote by $\mathcal{D}_r^{1,2}(\mathbb{R}^N)$ the closure of all the radially symmetric functions in $C_c^\infty(\mathbb{R}^N)$ under the norm

$$\|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}=\left(\int_{\mathbb{R}^N}|\nabla v|^2dx\right)^{\frac{1}{2}}.$$

Suppose $a_0<2$, $a_\infty>\max\{a_0,0\}$ and p satisfy (1.20), we may show that the inclusion $\mathcal{D}_r^{1,2}(\mathbb{R}^N)\hookrightarrow L^{p+1}(\mathbb{R}^N,Vdx)$ is continuous and compact.

Proof of Theorem 1.4. Since V is radially symmetric, so is the minimal solution $u_{k,V}$ of (P_k), which is also stable, for $k\in(0,k_p]$. By the Mountain Pass Theorem, we may find a critical point of the functional

$$E_r(v)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx-\int_{\mathbb{R}^N}VF(u_{k,V},v_+)dx \tag{4.14}$$

in $\mathcal{D}_r^{1,2}(\mathbb{R}^N)$. The rest of the proof is similar to the proof of Theorem 1.3. □

5. Appendix: regularities

We recall that $G(x,y)=\frac{c_N}{|x-y|^{N-2}}$ is the Green kernel of $-\Delta$ in $\mathbb{R}^N\times\mathbb{R}^N$ and $\mathbb{G}[\cdot]$ is the Green operator defined as

$$\mathbb{G}[f](x)=\int_{\mathbb{R}^N}G(x,y)f(y)dy.$$

Proposition 5.1. *Suppose that $\Omega\subset\mathbb{R}^N$ is a bounded domain and $h\in L^s(\Omega)$. Then, there exists $c_{43}>0$ such that*

$$\|\mathbb{G}[h]\|_{L^\infty(\Omega)}\le c_{43}\|h\|_{L^s(\Omega)}\quad \textit{if}\quad \frac{1}{s}<\frac{2}{N}, \tag{5.1}$$

$$\|\mathbb{G}[h]\|_{L^r(\Omega)}\le c_{43}\|h\|_{L^s(\Omega)}\quad \textit{if}\quad \frac{1}{s}\le\frac{1}{r}+\frac{2}{N}\quad \textit{and}\quad s>1 \tag{5.2}$$

and

$$\|\mathbb{G}[h]\|_{L^r(\Omega)}\le c_{43}\|h\|_{L^1(\Omega)}\quad \textit{if}\quad 1<\frac{1}{r}+\frac{2}{N}. \tag{5.3}$$

Proof. First we prove (5.1). By Hölder's inequality, for any $x \in \Omega$,

$$\Big\| \int_\Omega G(x,y)h(y)dy \Big\|_{L^\infty(\Omega)} \le \Big\| \Big(\int_\Omega G(x,y)^{s'}dy \Big)^{\frac{1}{s'}} \Big(\int_\Omega |h(y)|^s dy \Big)^{\frac{1}{s}} \Big\|_{L^\infty(\Omega)}$$
$$\le c_N \|h\|_{L^s(\Omega)} \Big\| \int_\Omega \frac{1}{|x-y|^{(N-2)s'}} dy \Big\|_{L^\infty(\Omega)},$$

where $s' = \frac{s}{s-1}$. Since $\frac{1}{s} < \frac{2}{N}$ and $(N-2)s' < N$, we have

$$\int_\Omega \frac{1}{|x-y|^{(N-2)s'}}dy \le \int_{B_d(x)} \frac{1}{|x-y|^{(N-2)s'}}dy = c_{44} \int_0^d r^{N-1-(N-2)s'}dr \le c_{45} d^{N-(N-2)s'},$$

where $c_{44}, c_{45} > 0$ and $d = \sup\{|x-y| : \ x, y \in \Omega\}$, from where (5.1) holds.

Next, we prove (5.2) for $r \le s$ and (5.3) for $r = 1$. There holds

$$\left\{ \int_\Omega \left[\int_\Omega G(x,y)h(y)dy \right]^r dx \right\}^{\frac{1}{r}} = \left\{ \int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^N} G(x,y)h(y)\chi_\Omega(x)\chi_\Omega(y)dy \right]^r dx \right\}^{\frac{1}{r}}$$
$$\le c_N \left\{ \int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^N} \frac{h(y)\chi_\Omega(x)\chi_\Omega(y)}{|x-y|^{N-2}}dy \right]^r dx \right\}^{\frac{1}{r}}$$
$$= c_N \left\{ \int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^N} \frac{h(x-y)\chi_\Omega(x)\chi_\Omega(x-y)}{|y|^{N-2}}dy \right]^r dx \right\}^{\frac{1}{r}}.$$

By Minkowski's inequality, we have that

$$\left\{ \int_\Omega \left[\int_\Omega G(x,y)h(y)dy \right]^r dx \right\}^{\frac{1}{r}} \le c_N \int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^N} \frac{h^r(x-y)\chi_\Omega(x)\chi_\Omega(x-y)}{|y|^{(N-2)r}}dx \right]^{\frac{1}{r}} dy$$
$$\le c_N \int_{\tilde\Omega} \left[\int_{\mathbb{R}^N} h^r(x-y)\chi_\Omega(x)\chi_\Omega(x-y)dx \right]^{\frac{1}{r}} \frac{1}{|y|^{N-2}}dy$$
$$\le c_N \|h\|_{L^r(\Omega)} \le c_{46}\|h\|_{L^s(\Omega)},$$

where $c_{46} > 0$ and $\tilde\Omega = \{x - y : x, y \in \Omega\}$ is bounded. Finally we prove (5.2) in the case $r > s \ge 1$ and $\frac{1}{s} \le \frac{1}{r} + \frac{2}{N}$, and (5.3) for $r > 1$, $1 < \frac{1}{r} + \frac{2}{N}$. We claim that if $r > s$ and $\frac{1}{r^*} = \frac{1}{s} - \frac{2}{N}$, the mapping $h \to \mathbb{G}(h)$ is of weak-type (s, r^*) in the sense that

$$|\{x \in \Omega : |\mathbb{G}[h](x)| > t\}| \le \Big(A_{s,r^*} \frac{\|h\|_{L^s(\Omega)}}{t}\Big)^{r^*}, \quad h \in L^s(\Omega) \text{ and all } \ t > 0, \tag{5.4}$$

where A_{s,r^*} is a positive constant. Defining

$$G_0(x,y) = \begin{cases} G(x,y), & \text{if } \ |x-y| \le \nu, \\ 0, & \text{if } \ |x-y| > \nu, \end{cases}$$

for $\nu > 0$ and $G_\infty(x,y) = G(x,y) - G_0(x,y)$. Then we have

$$|\{x \in \Omega : |\mathbb{G}[h](x)| > 2t\}| \le |\{x \in \Omega : |\mathbb{G}_0[h](x)| > t\}| + |\{x \in \Omega : |\mathbb{G}_\infty[h](x)| > t\}|,$$

where $\mathbb{G}_0[h]$ and $\mathbb{G}_\infty[h]$ are defined similar to $\mathbb{G}[h]$. By Minkowski's inequality, we find that

$$\begin{aligned}|\{x\in\Omega:|\mathbb{G}_0[h](x)|>t\}| &\le \frac{\|\mathbb{G}_0(h)\|^s_{L^s(\Omega)}}{t^s}\\ &\le \frac{\|\int_\Omega \chi_{B_\nu(x-y)}|x-y|^{2-N}|h(y)|dy\|^s_{L^s(\Omega)}}{t^s}\\ &\le \frac{[\int_\Omega(\int_\Omega|h(x-y)|^s dx)^{\frac{1}{s}}|y|^{2-N}\chi_{B_\nu}(y)dy]^s}{t^s}\\ &\le \frac{\|h\|^s_{L^s(\Omega)}\int_{B_\nu}|x|^{-N+2}dx}{t^s}=c_{47}\frac{\|h\|^s_{L^s(\Omega)}\nu^2}{t^s}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|\mathbb{G}_\infty[h]\|_{L^\infty(\Omega)} &\le \|\int_\Omega \chi_{B^c_\nu}(x-y)|x-y|^{2-N}|h(y)|dy\|_{L^\infty(\Omega)}\\ &\le (\int_\Omega |h(y)|^s dy)^{\frac{1}{s}}\|(\int_{\Omega\setminus B_\nu(y)}|x-y|^{(2-N)s'}dy)^{\frac{1}{s'}}\|_{L^\infty(\Omega)}\\ &\le c_{48}\|h\|_{L^s(\Omega)}\nu^{2-\frac{N}{s}},\end{aligned}$$

where $s'=\frac{s}{s-1}$ if $s>1$, and if $s=1$, $s'=\infty$. Choosing $\nu=(\frac{t}{c_{48}\|h\|_{L^s(\Omega)}})^{\frac{1}{2-\frac{N}{s}}}$, we obtain

$$\|\mathbb{G}_\infty[h]\|_{L^\infty(\Omega)}\le t,$$

which means that

$$|\{x\in\Omega:|\mathbb{G}_\infty[h](x)|>t\}|=0.$$

With this choice of ν, we have that

$$|\{x\in\Omega:|\mathbb{G}[h]|>2t\}|\le c_{49}\frac{\|h\|^s_{L^s(\Omega)}\nu^{2s}}{t^s}\le c_{50}\left(\frac{\|h\|_{L^s(\Omega)}}{t}\right)^{r^*}.$$

The claim for $r>s$ follows from the Marcinkiewicz interpolation theorem. □

Proposition 5.2. *Suppose that $\Omega\subset\mathbb{R}^N$ is a bounded domain and $h\in L^s(\Omega)$. Then, there exists $c_{51}>0$ such that:*

$$\|\nabla\mathbb{G}[h]\|_{L^\infty(\Omega)}\le c_{51}\|h\|_{L^s(\Omega)} \quad \text{if} \quad \frac{1}{s}<\frac{1}{N}, \tag{5.5}$$

$$\|\nabla\mathbb{G}[h]\|_{L^r(\Omega)}\le c_{51}\|h\|_{L^s(\Omega)} \quad \text{if} \quad \frac{1}{s}\le\frac{1}{r}+\frac{1}{N} \quad \text{and} \quad s>1 \tag{5.6}$$

and

$$\|\nabla\mathbb{G}[h]\|_{L^r(\Omega)}\le c_{51}\|h\|_{L^1(\Omega)} \quad \text{if} \quad 1<\frac{1}{r}+\frac{1}{N}. \tag{5.7}$$

Proof. Since

$$|\nabla\mathbb{G}[h](x)|=|\int_\Omega \nabla_x G(x,y)h(y)\,dy|\le\int_\Omega|\nabla_x G(x,y)|\,|h(y)|\,dy$$

and

$$|\nabla_x G(x,y)|=c_N(N-2)|x-y|^{1-N}.$$

Then conclusion follows as the proof of Proposition 5.1. □

Conflict of interest statement

Authors declare no conflict of interest.

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