# Plates with incompatible prestrain of high order 

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#### Abstract

We study the elastic behaviour of incompatibly prestrained thin plates of thickness $h$ whose internal energy $E^{h}$ is governed by an imposed three-dimensional smooth Riemann metric $G$ only depending on the variable in the midsurface $\omega$. It is already known that $h^{-2} \inf E^{h}$ converges to a finite value $c$ when the metric $G$ restricted to the midsurface has a sufficiently regular immersion, namely $W^{2,2}\left(\omega, \mathbb{R}^{3}\right)$. The obtained limit model generalizes the bending (Kirhchoff) model of Euclidean elasticity. In the present paper, we deal with the case when $c$ equals 0 . Then, equivalently, three independent entries of the three-dimensional Riemann curvature tensor associated with $G$ are null. We prove that, in such regime, necessarily inf $E^{h} \leq C h^{4}$. We identify the $\Gamma$-limit of the scaled energies $h^{-4} E^{h}$ and show that it consists of a von Kármán-like energy. The unknowns in this energy are the first order incremental displacements with respect to the deformation defined by the bending model and the second order tangential strains. In addition, we prove that when $\inf h^{-4} E^{h} \rightarrow 0$, then $G$ is realizable and hence $\min E^{h}=0$ for every $h$.


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## Résumé

On s'intéresse au comportement de structures minces d'épaisseur $h$ dont l'énergie interne $E^{h}$ est régie par une métrique riemannienne tridimensionnelle $G$ imposée, constante dans l'épaisseur, n'admettant pas nécessairement d'immersion isométrique. On sait que lorsque la restriction de $G$ à la surface moyenne $\omega$ possède une immersion isométrique suffisamment régulière, c'est-à-dire appartenant à $W^{2,2}\left(\omega, \mathbb{R}^{3}\right)$, alors $h^{-2} \inf E^{h}$ admet une limite finie $c$ quand $h$ tend vers 0 . Le modèle limite correspondant généralise le modèle de flexion non linéaire, classique pour la métrique euclidienne. Nous nous plaçons ici dans le cas où $c$ vaut 0 , ce qui équivaut à la nullité de trois des six coeffiecients du tenseur de courbure associé à $G$. Nous montrons qu'alors inf $E^{h} \leq C h^{4}$. Nous identifions la $\Gamma$-limite de $h^{-4} E^{h}$ et montrons qu'elle généralise l'énergie de von Kármán. Elle s'exprime en fonction des déplacements incrémentaux par rapport à la surface définie par le modèle de flexion et de déformations tangentielles généralisées. De plus, nous montrons que l'infimum de ce modèle limite à l'ordre 4 n'est nul que si $G$ admet une immersion isométrique, auquel cas $\min E^{h}=0$ pour tout $h$.
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[^0]
## 1. Introduction

The purpose of this paper is to study the elastic behaviour of pre-stressed thin plates $\Omega^{h}$, characterized by nonimmersable Riemannian metrics $G$ on their reference configurations. To such metrics (and the prestrain they generate), we refer as "incompatible"; the incompatibility is measured by infimizing the energy $E^{h}$ given below, sometimes called the "non-Euclidean" elastic energy. We will be concerned with the regimes of curvatures of $G$ resulting in the incompatibility of "high order". By this we mean that inf $E^{h} \sim h^{\beta}$ as the plate's thickness $h$ goes to 0 , and that the scaling exponent $\beta$ satisfies: $\beta>2$.

In paper [5] we analyzed the scaling $\inf E^{h} \sim h^{2}$ and proved that it only occurs when the metric $G_{2 \times 2}$ on the mid-plate has an isometric immersion in $\mathbb{R}^{3}$ with the regularity $W^{2,2}$ and when, at the same time, the three tangential Riemann curvatures of $G$ do not vanish identically. The two-dimensional limiting energy, obtained from the sequence $h^{-2} E^{h}$ via $\Gamma$-convergence, as $h \rightarrow 0$, was an extension of the classical nonlinear bending energy.

In the present paper we assume that:

$$
\begin{equation*}
h^{-2} \inf E^{h} \rightarrow 0 \tag{1.1}
\end{equation*}
$$

and prove that the only nontrivial two-dimensional limiting theory in this regime is a von Kármán-like energy, valid when $\inf E^{h} \sim h^{4}$. It further turns out that this scaling is automatically implied by (1.1) and $\inf E^{h} \neq 0$. Indeed, we show that (1.1) implies $h^{-4} \inf E^{h} \leq C$, and that $h^{-4} \inf E^{h} \rightarrow 0$ if and only if $G$ is immersable, in which case $\min E^{h}=0$ for all $h$.

Let us observe that this scale separation is different from the findings of [29] valid in the Euclidean case of $G=\mathrm{Id}_{3}$, where the possible limiting energies are distinguished by the scaling of the applied forces $f^{h} \sim h^{\alpha}$. In that context, three distinct limiting theories have been obtained for $\inf E^{h} \sim h^{\beta}$ with $\beta>2$ (corresponding to $\alpha>2$ ). Namely: $\beta \in(2,4)$ yielded the linearized bending model subject to a nonlinear constraint on the limiting displacements; $\beta=4$ yielded the classical von Kármán model; and $\beta>4$ corresponded to the linear elasticity.

The present results differ as well from the higher order hierarchy of scalings and the elastic theories of shells, as derived through an asymptotic expansion in [46]. These differences are due to the fact that while the magnitude of external forces is adjustable at will, it is not so for the six sectional curvatures (together with their covariant derivatives) of a given metric $G$. As we show, the six curvatures of $G=G\left(x^{\prime}\right)$ depending only on the mid-plate variables fall into two categories: including or excluding the thin direction variable. Then, the simultaneous vanishing of curvatures in the first category or in both categories correspond to the two scenarios at hand in terms of the scaling of inf $E^{h}$.

### 1.1. Some background in dimension reduction for thin structures

Early attempts for replacing the three-dimensional model of a thin elastic structure with planar mid-surface at rest, by a two-dimensional model, were based on a priori simplifying assumptions on the deformations and on the stresses. Later, the natural idea of using the thickness as a small parameter and of establishing a limit model was largely explored; we refer in particular to the works by Ciarlet and Destuynder who set the method in the appropriate framework of the weak formulation of boundary value problems [11,12], proved convergence to the linear plate model [22] in the context of linearized elasticity, and obtained formally the von Kármán plate model from finite elasticity [8]. See also [55,56] for the time-dependent case and [9] for a comprehensive list of references.

The issue of deriving two-dimensional models valid for large deformations, by means of an asymptotic formalism, was subsequently tackled by Fox, Simo and the second author in [26]. They showed, in the context of the Saint Venant-Kirchhoff materials subject to appropriate boundary conditions, how to recover a hierarchy of four models. This hierarchy, driven by the order of magnitude of the applied loads, consisted of: the nonlinear membrane model, the inextensional bending model, the von Kármán model and the linear plate model. The models thus obtained still required a justification through rigorous convergence results. In [35], Le Dret and the second author used the variational point of view and proved $\Gamma$-convergence of the 3 -dimensional elastic energies to a nonlinear membrane energy, valid for loads of magnitude of order 1 . We remark that the expression of the limiting stored energy therein consisted of quasiconvexification of the 3d energy, first minimized with respect to normal stretches. This allowed to recover the degeneracy under compression; a feature that is otherwise missed by formal expansions. We further mention that for $3 \mathrm{~d} \rightarrow 1 \mathrm{~d}$ reduction, a similar point of view had been introduced by Acerbi, Buttazzo and Percivale in [1].

A key-point for deriving rigorously the above mentioned nonlinear bending model has been the geometric rigidity result due to Friesecke, James and Müller [28]. In a similar spirit, the same authors justified the von Kármán model, the linear model [29] and also they introduced novel intermediate models, in particular in the range of energies - or equivalently of loadings - between the scaling responsible for bending $(\beta=2)$ and the von Kármán scaling $(\beta=4)$. In this range of models, the three-dimensional stored energy appears in the limit stored energy through its second derivative at rest. Scaling the energy with exponents $\beta$ other than integers had been explored for the membrane to bending range in [18] leading to convergence results for $0<\beta<5 / 3$ while the regime $5 / 3 \leq \beta<2$ remains open and is conjectured to be relevant for crumpling of elastic sheets. Other significant extensions concern derivation of limit theories for incompressible materials [16,17,64,47], for heterogeneous materials [61], through establishing convergence of equilibria rather than strict minimizers [51,53,37,52,38], and finally for shallow shells [40].

Extension of the above variational method valid in the framework of the large deformation model was conducted in parallel for slender structures whose midsurface at rest is non-planar. The first result by the second author and Le Dret [36] relates to scaling $\beta=0$ and models membrane shells: the limit stored energy depends then only on the stretching and shearing produced by the deformation on the midsurface. Another study is due to Friesecke, James, Mora and Müller [27] who analyzed the case $\beta=2$. This scaling corresponds to a flexural shell model, where the only admissible deformations are those preserving the midsurface metric. The limit energy depends then on the change of curvature produced by the deformation. Further, the first author, Mora and Pakzad derived the relevant linear theories ( $\beta>4$ ) and the von Kármán-like theories $(\beta=4)$ in [42], and subsequently proceeded to finalize the analysis for elliptic shells in the full regime $\beta>2$ in [43]. A similar analysis has been performed in the case of developable shells in [31] leading to the proof of the collapse of all two-dimensional limiting theories to the linear theory when $\beta>2$. Following these findings, a conjecture was made in [46] about the infinite hierarchy of shell models and the various possible limiting scenarios differentiated by rigidity properties of shells. Let us recall that a comprehensive body of work had been previously devoted to the asymptotic derivation of shell models in the small displacement regime under clear hypotheses on the model taken for granted, three-dimensional or already two-dimensional and containing the thickness as a parameter. Several models were recovered by Ciarlet and coauthors [13-15], by Destuynder [21,23] and by Sanchez-Palencia and coauthors [59,60,7,6,50]. Sanchez-Palencia, in particular, theorized the role and interplay of the midsurface geometry and of the boundary conditions [58], as well as underlined the singular perturbation behaviour. We refer to [10] for additional references.

Most recently, there has been a sustained interest in studying similar problems where the shape formation is not driven by exterior forces but rather by the internal prestrain caused by e.g. growth, swelling, shrinkage or plasticity [33, $25,63]$. Variants of a thin plate theory can be used in the self-similar structures which form due to variations in an intrinsic metric that is asymptotically flat at infinity [2], and also in the case of a circular disk with edge-localized growth [25], or in the shape of a long leaf [48]. Ben Amar and coauthors formally derived a variant of the Föppl-von Kármán equilibrium equations from finite incompressible elasticity [19,20]: they use the multiplicative decomposition of the gradient proposed in [57] similar to ours, and study cockling of paper, grass blades and sympetalous flowers [20,4].

Experimentalists, working in close connection with mathematicians, are presently building devices that produce thin deformable structures with target metrics exhibiting a high level of inhomogeneities. One of the first efforts to reproduce the effect of the prestrain on the shape of thin films in an artificial setting was reported in [33]. The authors manufactured thin gel films that underwent nonuniform lateral radially symmetric shrinkage when activated in a hot bath. Large-scale buckling, multi-scale wrinkling structures and symmetry-breaking patterns appeared in the sheets, depending on the nature of the "programmed in" metric and of the thickness. Another approach to controlling of shape through prestrain was suggested in [32]: by photopatterning polymer films, the authors produce temperatureresponsive flat gel sheets that can transform into prescribed curved surfaces when the in-built metric is activated. Finally, an experimental setup for shape control of thin films of liquid crystal elastomers is described in [65]. For other experimental results, see $[62,63,34,30,2,54]$.

A systematic study of the possible limits when a target metric is prescribed was undertaken by the first author and collaborators: a generalized version of the nonlinear bending model was rigorously derived in [45] under the assumption that the target metric is independent of thickness and block-diagonal. This analysis was completed in [5] by removing the block-diagonal assumption and by giving a necessary and sufficient condition for $E^{h}$ to scale as $h^{2}$. The objective of the present paper is to study higher order prestrains.

Let us also mention that in $[40,41,44]$ similar derivations were carried out under a different assumption on the asymptotic behaviour of the prescribed metric, which also implied energy scaling $h^{\beta}$ in different regimes of $\beta>2$.

In [40] it was shown that the resulting equations are identical to those postulated to account for the effects of growth in elastic plates [48] and used to describe the shape of a long leaf. In [44] a model with a Monge-Ampère constraint was derived and analyzed from various aspects. Other results concerning the energy scaling for the materials with prescribed metric are derived in [3], where by imposing suitable boundary data, conditions of [45,5] are not satisfied and hence the energy scales larger than $h^{2}$ (see also [63]).

### 1.2. The set-up and notation

Let $\omega$ be an open, bounded, connected and simply connected subset of $\mathbb{R}^{2}$, with Lipschitz continuous boundary. ${ }^{1}$ For $0<h \ll 1$ we consider thin films $\Omega^{h}$ with midsurface $\omega$ :

$$
\begin{equation*}
\Omega^{h}=\left\{x=\left(x^{\prime}, x_{3}\right) ; \quad x^{\prime} \in \omega, \quad x_{3} \in(-h / 2, h / 2)\right\} . \tag{1.2}
\end{equation*}
$$

Let $G: \bar{\Omega}^{h} \rightarrow \mathbb{R}^{3 \times 3}$ be a given smooth Riemann metric on $\Omega^{h}$, uniform through the thickness:

$$
G\left(x^{\prime}, x_{3}\right)=G\left(x^{\prime}\right) \quad \text { for every }\left(x^{\prime}, x_{3}\right) \in \Omega^{h},
$$

and let $A=\sqrt{G}$ denote the unique positive definite symmetric square root of $G$. Consider the energy functional $E^{h}: W^{1,2}\left(\Omega^{h}, \mathbb{R}^{3}\right) \rightarrow \overline{\mathbb{R}}_{+}$defined as:

$$
\begin{equation*}
E^{h}\left(u^{h}\right)=\frac{1}{h} \int_{\Omega^{h}} W\left(\nabla u^{h} A^{-1}\right) \mathrm{d} x . \tag{1.3}
\end{equation*}
$$

The nonlinear elastic energy density $W: \mathbb{R}^{3 \times 3} \rightarrow \overline{\mathbb{R}}_{+}$is a Borel measurable function, assumed to be $\mathcal{C}^{2}$ in a neighbourhood of $\operatorname{SO}(3)$ and to satisfy, for every $F \in \mathbb{R}^{3 \times 3}$, every $R \in \mathrm{SO}(3)$ and with a uniform constant $c>0$, the conditions:

$$
\begin{equation*}
W(R)=0, \quad W(R F)=W(F), \quad W(F) \geq c \operatorname{dist}^{2}(F, \mathrm{SO}(3)) . \tag{1.4}
\end{equation*}
$$

The first condition states that the energy of a rigid motion is 0 , while the second is the frame invariance. They imply that $D W\left(\mathrm{Id}_{3}\right)=0$ and that $D^{2} W\left(\mathrm{Id}_{3}\right)(T, \cdot)=0$ for all skew symmetric matrices $T \in \operatorname{so}(3)$. The third assumption above reflects the quadratic growth of the density $W$ away from the energy well $\mathrm{SO}(3)$. Note that these assumptions are not contradictory with the physical condition $W(F)=\infty$ for $\operatorname{det} F \leq 0$.

Throughout the paper, we use the following notation. Given a matrix $F \in \mathbb{R}^{3 \times 3}$, we denote its transpose by $F^{t}$, its symmetric part by $\operatorname{sym} F=\frac{1}{2}\left(F+F^{t}\right)$, and its skew part by skew $F=F-\operatorname{sym} F$. By $\operatorname{SO}(n)=\left\{R \in \mathbb{R}^{n \times n} ; R^{t}=\right.$ $R^{-1}$ and $\left.\operatorname{det} R=1\right\}$ we denote the group of special rotations, while $\operatorname{so}(n)=\left\{F \in \mathbb{R}^{n \times n} ; \operatorname{sym} F=0\right\}$ is the space of skew-symmetric matrices. We use the matrix norm $|F|=\left(\operatorname{trace}\left(F^{t} F\right)\right)^{1 / 2}$, which is induced by the inner product $\left\langle F_{1}: F_{2}\right\rangle=\operatorname{trace}\left(F_{1}^{t} F_{2}\right)$. The $2 \times 2$ principal minor of a matrix $F \in \mathbb{R}^{3 \times 3}$ is denoted by $F_{2 \times 2}$. All limits are taken as the thickness parameter $h$ vanishes, i.e. when $h \rightarrow 0$. Finally, by $C$ we denote any universal constant, independent of $h$.

### 1.3. Some previous directly related results

Recall that, according to a classical theorem of Riemannian geometry, the given metric $G$ on $\Omega^{h}$ admits an (automatically smooth) isometric immersion if and only if the Riemann curvature tensor of $G$ vanishes identically. In other words, condition: $\min _{u^{h} \in W^{1,2}\left(\Omega^{h}, \mathbb{R}^{3}\right)} E^{h}\left(u^{h}\right)=0$ is equivalent to: $\operatorname{Riem}(G) \equiv 0$.

In [45], it has been proved that already the vanishing of the infimum:

$$
\inf _{u^{h} \in W^{1,2}\left(\Omega^{h}, \mathbb{R}^{3}\right)} E^{h}\left(u^{h}\right)=0
$$

is sufficient for (and thus equivalent to): $\operatorname{Riem}(G) \equiv 0$. But, even when $\operatorname{Riem}(G) \not \equiv 0$, the non-zero sequence $\left\{\inf E^{h}\left(u^{h}\right)\right\}$ still converges to 0 with $h \rightarrow 0$. Even more, it can be checked that the generalized membrane energy

[^1]obtained as the $\Gamma$-limit of $E^{h}$, vanishes on $W^{1,2}$ immersions of $\omega$ in $\mathbb{R}^{3}$, isometric with respect to $G_{2 \times 2}$. The Nash and Kuiper's famous results assert then that such immersions (even with regularity $\mathcal{C}^{1, \alpha}$ where $\alpha<1 / 5$, see [24]) exist for every $G$. In particular, it follows that: $\lim _{h \rightarrow 0} \inf E^{h}=0$.

Investigating a higher order level of the energy, it has been proved in [5] that the $\Gamma$-limit of the rescaled energies $h^{-2} E^{h}$ is given by:

$$
\mathcal{I}_{2}(y)=\frac{1}{24} \int_{\omega} \mathcal{Q}_{2, A}\left(x^{\prime},(\nabla y)^{t} \nabla \vec{b}\right) \mathrm{d} x^{\prime},
$$

on the set of all $y \in W^{2,2}\left(\omega, \mathbb{R}^{3}\right)$ such that $(\nabla y)^{t} \nabla y=G_{2 \times 2}$; when no $W^{2,2}\left(\omega, \mathbb{R}^{3}\right)$ isometric immersion of $G_{2 \times 2}$ exists, then the $\Gamma$-limit is $+\infty$. The quadratic forms $\mathcal{Q}_{2, A}\left(x^{\prime}, \cdot\right)$ are given in terms of the second order derivative of the energy density $W$ at $\mathrm{Id}_{3}$, as in (3.4). The Cosserat vector $\vec{b} \in\left(W^{1,2} \cap L^{\infty}\right)\left(\omega, \mathbb{R}^{3}\right)$ is uniquely determined from the isometric immersion $y$ by:

$$
\begin{equation*}
Q^{t} Q=G \quad \text { where } \quad Q e_{1}=\partial_{1} y, \quad Q e_{2}=\partial_{2} y, \quad Q e_{3}=\vec{b}, \quad \text { with } \operatorname{det} Q>0 . \tag{1.5}
\end{equation*}
$$

The functional $\mathcal{I}_{2}$ is a fully nonlinear bending energy. In case of $G e_{3}=e_{3}$, it reduces to the classical bending content quantifying the second fundamental form $(\nabla y)^{t} \nabla \vec{b}=(\nabla y)^{t} \nabla \vec{N}$ on the deformed surface $y(\omega)$ with the unit normal vector $\vec{N}$; this classical bending energy is also known as the nonlinear Kirchhoff energy. It has also been proved that

$$
\begin{equation*}
\lim _{h \rightarrow 0} \inf _{u^{h} \in W^{1,2}\left(\Omega^{h}, \mathbb{R}^{3}\right)} h^{-2} E^{h}\left(u^{h}\right)=0 \tag{1.6}
\end{equation*}
$$

if and only if three specific Riemann curvatures of $G$ vanish, namely:

$$
\begin{equation*}
R_{1212}=R_{1213}=R_{1223} \equiv 0 \quad \text { in } \Omega^{h} . \tag{1.7}
\end{equation*}
$$

Finally, it has been shown that (1.7) (or equivalently (1.6)) is satisfied if and only if there exists a mapping $y_{0}: \omega \rightarrow \mathbb{R}^{3}$ such that:

$$
\left\{\begin{array}{l}
\left(\nabla y_{0}\right)^{t} \nabla y_{0}=G_{2 \times 2},  \tag{1.8}\\
\operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla \vec{b}_{0}\right)=0,
\end{array}\right.
$$

where the vector field $\vec{b}_{0}$ is defined in terms of $y_{0}$ as in (1.5):

$$
\begin{equation*}
Q_{0}^{t} Q_{0}=G, \quad Q_{0} e_{1}=\partial_{1} y_{0}, \quad Q_{0} e_{2}=\partial_{2} y_{0} \quad \text { and } Q_{0} e_{3}=\vec{b}_{0} \quad \text { with } \operatorname{det} Q_{0}>0 \tag{1.9}
\end{equation*}
$$

The isometric immersion $y_{0}$ is smooth, the vector field $b_{0}$ is smooth and $y_{0}$ is unique up to overall rigid motions. This is a consequence of the observation that under (1.8), the second fundamental form of the surface $y_{0}(\omega)$ is uniquely given in terms of $G$. The second equation in (1.8) follows from the fact that the kernel of each quadratic form $\mathcal{Q}_{2, A}$ coincides with so(2).

For future use, let us remark that, denoting the inverse matrix $G^{-1}=\left[G^{i j}\right]_{i, j: 1 . .3}$, we have:

$$
\begin{equation*}
\vec{b}_{0}=-\frac{1}{G^{33}}\left(G^{13} \partial_{1} y_{0}+G^{23} \partial_{2} y_{0}\right)+\frac{1}{\sqrt{G^{33}}} \vec{N} . \tag{1.10}
\end{equation*}
$$

### 1.4. New results of this work

In this paper we investigate the higher energy scalings and the limiting behaviour of the minimizing configurations to $E^{h}$ under condition (1.7). We first prove (in Lemma 2.1) that (1.7), which as we recall is equivalent to (1.1), implies:

$$
\inf E^{h} \leq C h^{4} .
$$

We then derive (in Theorem 3.1 and Theorem 4.1) the $\Gamma$-limit $\mathcal{I}_{4}$ of the rescaled energies $h_{\overrightarrow{-4}} E^{h}$, together with their compactness properties. Namely, let $y_{0}$ be the unique immersion satisfying (1.8), where $\vec{b}_{0}$ is as in (1.9). Let $\vec{d}_{0}: \omega \rightarrow \mathbb{R}^{3}$ be the smooth vector field given in terms of $y_{0}$ by:

$$
\begin{equation*}
\left\langle Q_{0}^{t} \vec{d}_{0}, e_{1}\right\rangle=-\left\langle\partial_{1} \vec{b}_{0}, \vec{b}_{0}\right\rangle, \quad\left\langle Q_{0}^{t} \vec{d}_{0}, e_{2}\right\rangle=-\left\langle\partial_{2} \vec{b}_{0}, \vec{b}_{0}\right\rangle, \quad\left\langle Q_{0}^{t} \vec{d}_{0}, e_{3}\right\rangle=0 . \tag{1.11}
\end{equation*}
$$

The limit $\mathcal{I}_{4}$ is then given by the following energy functional:

$$
\begin{align*}
\mathcal{I}_{4}(V, \mathrm{e})=\frac{1}{2} \int_{\omega} & \mathcal{Q}_{2, A}\left(x^{\prime}, \mathrm{e}+\frac{1}{2}(\nabla V)^{t} \nabla V+\frac{1}{24}\left(\nabla \vec{b}_{0}\right)^{t} \nabla \vec{b}_{0}\right) \mathrm{d} x^{\prime}+\frac{1}{24} \int_{\omega} \mathcal{Q}_{2, A}\left(x^{\prime},\left(\nabla y_{0}\right)^{t} \nabla \vec{p}+(\nabla V)^{t} \nabla \vec{b}_{0}\right) \mathrm{d} x^{\prime} \\
& +\frac{1}{1440} \int_{\omega} \mathcal{Q}_{2, A}\left(x^{\prime},\left(\nabla y_{0}\right)^{t} \nabla \vec{d}_{0}+\left(\nabla \vec{b}_{0}\right)^{t} \nabla \vec{b}_{0}\right) \mathrm{d} x^{\prime} \tag{1.12}
\end{align*}
$$

acting on the space of extended strains ${ }^{2}$ :

$$
\begin{equation*}
\mathrm{e} \in \mathrm{cl}_{L^{2}}\left\{\operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla w\right) ; w \in W^{1,2}\left(\omega, \mathbb{R}^{3}\right)\right\} \tag{1.13}
\end{equation*}
$$

and the space of first order infinitesimal isometries:

$$
\begin{equation*}
V \in W^{2,2}\left(\omega, \mathbb{R}^{3}\right) \quad \text { such that: } \quad \operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla V\right)=0 \tag{1.14}
\end{equation*}
$$

In (1.13), the notation $\mathrm{cl}_{L^{2}}$ stands for the closure of the indicated set of $2 \times 2$ tensors on $\omega$, in the space $L^{2}\left(\omega, \mathbb{R}_{s y m}^{2 \times 2}\right)$. In (1.12), the vector field $\vec{p} \in W^{1,2}\left(\omega, \mathbb{R}^{3}\right)$ depends linearly on the gradient of $V$ and is uniquely associated with it by:

$$
\begin{equation*}
\left(\nabla y_{0}\right)^{t} \vec{p}=-(\nabla V)^{t} \vec{b}_{0} \quad \text { and } \quad\left\langle\vec{p}, \vec{b}_{0}\right\rangle=0 \tag{1.15}
\end{equation*}
$$

The spaces consisting of e and $V$ contain the information about the admissible error displacements, relative to the leading order immersion $y_{0}$, under the energy scaling $E^{h} \sim h^{4}$. We discuss their geometrical significance, together with the bending and stretching tensors in the first two terms of $\mathcal{I}_{4}(V, \mathrm{e})$, in section 5 . We further prove in Theorem 6.2 that the last term in (1.12), which is constant and as such does not play a role in the minimization process, is precisely given by the only potentially nonzero (in view of (1.7)) curvatures of $G$, namely:

$$
\operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla \vec{d}_{0}\right)+\left(\nabla \vec{b}_{0}\right)^{t} \nabla \vec{b}_{0}=\left[\begin{array}{ll}
R_{1313} & R_{1323} \\
R_{1323} & R_{2323}
\end{array}\right] .
$$

We may thus write, informally:

$$
\begin{gathered}
\mathcal{I}_{4}(V, \mathrm{e})=\frac{1}{2} \int_{\omega} \mathcal{Q}_{2, A}\left(x^{\prime}, \text { stretching of order } h^{2}\right) \mathrm{d} x^{\prime}+\frac{1}{24} \int_{\omega} \mathcal{Q}_{2, A}\left(x^{\prime}, \text { bending of order } h\right) \mathrm{d} x^{\prime} \\
\\
+\frac{1}{1440} \int_{\omega} \mathcal{Q}_{2, A}\left(x^{\prime}, \text { Riemann curvature of } G\right) \mathrm{d} x^{\prime} .
\end{gathered}
$$

In particular, since all three terms above are nonnegative, this directly implies that the condition $\lim _{h \rightarrow 0} \frac{1}{h^{4}} \inf E^{h}=0$, which is equivalent to $\min \mathcal{I}_{4}=0$, is further equivalent to the immersability $G$, i.e. the vanishing of all its Riemann curvatures $\operatorname{Riem}(G) \equiv 0$ in $\Omega^{h}$.

### 1.5. An asymptotic expansion argument

We now sketch a heuristic derivation of the energy $\mathcal{I}_{4}(V$, e $)$, by matching the asymptotic expansion of the metric $\left(\nabla u^{h}\right)^{t} \nabla u^{h}$ generated by the deformation $u^{h}: \Omega^{h} \rightarrow \mathbb{R}^{3}$ below, with the prescribed target metric $G$.

Given the fields $V, w, \vec{p}$ as in (1.13), (1.14) and (1.15), we make an ansatz:

$$
u^{h}\left(x^{\prime}, x_{3}\right)=y_{0}\left(x^{\prime}\right)+h V\left(x^{\prime}\right)+h^{2} w\left(x^{\prime}\right)+x_{3} \vec{b}_{0}\left(x^{\prime}\right)+\frac{x_{3}^{2}}{2} \vec{d}_{0}\left(x^{\prime}\right)
$$

and compute the tangential minor of the pull-back metric:

[^2]\[

$$
\begin{aligned}
\left(\left(\nabla u^{h}\right)^{t} \nabla u^{h}\right)_{2 \times 2}= & \left(\nabla y_{0}\right)^{t} \nabla y_{0}+2 h \operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla V\right)+2 h^{2}\left(\operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla w\right)+\frac{1}{2}(\nabla V)^{t} \nabla V\right) \\
& +2 x_{3} \operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla \vec{b}_{0}\right)+2 h x_{3}\left(\operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla \vec{p}\right)+\frac{1}{2}(\nabla V)^{t} \nabla \vec{b}_{0}\right) \\
& +2 x_{3}^{2}\left(\frac{1}{2} \operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla \vec{d}_{0}\right)+\frac{1}{2}\left(\nabla \vec{b}_{0}\right)^{t} \nabla \vec{b}_{0}\right)+O\left(h^{3}\right) .
\end{aligned}
$$
\]

Note that the first term on the right hand side above equals $G_{2 \times 2}$, while the second and fourth terms are 0 , in view of (1.8) and (1.14), so that:

$$
\begin{aligned}
& \left(\left(\nabla u^{h}\right)^{t} \nabla u^{h}-G\right)_{2 \times 2}\left(x^{\prime}, x_{3}\right)=2 h^{2} I_{1}\left(x^{\prime}\right)+2 h x_{3} I_{3}\left(x^{\prime}\right)+2 x_{3}^{2} I_{2}\left(x^{\prime}\right)+O\left(h^{3}\right), \\
& \text { where: } I_{1}=\operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla w\right)+\frac{1}{2}(\nabla V)^{t} \nabla V, \quad I_{2}=\frac{1}{2} \operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla \vec{d}_{0}\right)+\frac{1}{2}\left(\nabla \vec{b}_{0}\right)^{t} \nabla \vec{b}_{0}, \\
& \text { and: } I_{3}=\operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla \vec{p}\right)+(\nabla V)^{t} \nabla \vec{b}_{0} .
\end{aligned}
$$

Writing:

$$
\begin{aligned}
\operatorname{dist}^{2}\left(\nabla u^{h} A^{-1}, \mathrm{SO}(3)\right) & =\left|\left(A^{-1}\left(\nabla u^{h}\right)^{t} \nabla u^{h} A^{-1}\right)^{1 / 2}-\mathrm{Id}\right|^{2} \\
& \sim\left|\frac{1}{2}\left(A^{-1}\left(\nabla u^{h}\right)^{t} \nabla u^{h} A^{-1}-\mathrm{Id}\right)\right|^{2} \sim \frac{1}{4}\left|\left(\nabla u^{h}\right)^{t} \nabla u^{h}-G\right|^{2},
\end{aligned}
$$

and recalling the minimization over the normal portion of the strain in the two-dimensional energy density (3.4), we arrive at:

$$
\begin{aligned}
\frac{1}{h^{4}} E^{h}\left(u^{h}\right) & \sim \frac{1}{4 h^{4}} \frac{1}{h} \int_{\Omega^{h}}\left|\left(\left(\nabla u^{h}\right)^{t} \nabla u^{h}\right)_{2 \times 2}-G_{2 \times 2}\right|^{2} \\
& \sim \frac{1}{h^{4}}\left(h^{4} \int_{\omega}\left|I_{1}\right|^{2}+h^{2} f_{-h / 2}^{h / 2} x_{3}^{2} \int_{\omega}\left|I_{3}\right|^{2}+f_{-h / 2}^{h / 2} x_{3}^{4} \int_{\omega}\left|I_{2}\right|^{2}+2 h^{2} f_{-h / 2}^{h / 2} x_{3}^{2} \int_{\omega}\left\langle I_{1}: I_{2}\right\rangle\right) \\
& =\int_{\omega}\left(\left|I_{1}+\frac{1}{12} I_{2}\right|^{2}+\frac{1}{2}\left|I_{3}\right|^{2}+\frac{1}{180}\left|I_{2}\right|^{2}\right) \mathrm{d} x^{\prime} .
\end{aligned}
$$

The three integral terms above correspond, in order of appearance, to the three terms in $\mathcal{I}_{4}(V$, e), after setting:

$$
\mathrm{e}=\operatorname{sym}\left(\left(\nabla y_{0}\right)^{t}\left(\nabla w+\frac{1}{24} \nabla \vec{d}_{0}\right)\right)
$$

The same calculation, with all details regarding the error terms, is carried out in the construction of the recovery sequence in the proof of Theorem 4.1.

## 2. The scaling and approximation lemmas

Let $B_{0}\left(x^{\prime}\right)$ be the matrix field defined by:

$$
\begin{equation*}
B_{0} e_{1}=\partial_{1} \vec{b}_{0}, \quad B_{0} e_{2}=\partial_{2} \vec{b}_{0} \text { and } B_{0} e_{3}=\vec{d}_{0}, \tag{2.1}
\end{equation*}
$$

where $\vec{d}_{0}$ is as in (1.11). Observe that $Q_{0}^{t} B_{0}$ is skew symmetric and that it has the form:

$$
Q_{0}^{t} B_{0}=\left[\begin{array}{c|c}
\left(\nabla y_{0}\right)^{t} \nabla \vec{b}_{0} & \left(\nabla y_{0}\right)^{t} \vec{d}_{0}  \tag{2.2}\\
\hline\left(\vec{b}_{0}\right)^{t} \nabla \vec{b}_{0} & \left\langle\vec{b}_{0}, \vec{d}_{0}\right\rangle
\end{array}\right] .
$$

Indeed, $\left(\nabla y_{0}\right)^{t} \nabla \vec{b}_{0} \in \operatorname{so}(2)$ by (1.8), while by (1.11): $\left(\nabla y_{0}\right)^{t} \vec{d}_{0}=-\left(\nabla \vec{b}_{0}\right)^{t} \vec{b}_{0}$ and $\left\langle\vec{b}_{0}, \vec{d}_{0}\right\rangle=0$.
Lemma 2.1. Condition (1.7) implies: $\inf _{W^{1,2}\left(\Omega^{h}, \mathbb{R}^{3}\right)} E^{h} \leq C h^{4}$.

Proof. Let us construct a sequence $u^{h} \in W^{1,2}\left(\Omega^{h}, \mathbb{R}^{3}\right)$ that has low energy. Define:

$$
\begin{equation*}
u^{h}\left(x^{\prime}, x_{3}\right)=y_{0}\left(x^{\prime}\right)+x_{3} \vec{b}_{0}\left(x^{\prime}\right)+\frac{x_{3}^{2}}{2} \vec{d}_{0}\left(x^{\prime}\right) \tag{2.3}
\end{equation*}
$$

in fact each $u^{h}$ is the restriction to $\Omega^{h}$ of the same deformation. We have:

$$
\nabla u^{h}\left(x^{\prime}, x_{3}\right)=Q_{0}\left(x^{\prime}\right)+x_{3} B_{0}\left(x^{\prime}\right)+\frac{x_{3}^{2}}{2} D_{0}\left(x^{\prime}\right),
$$

where the matrix field $D_{0}\left(x^{\prime}\right) \in \mathbb{R}^{3 \times 3}$ is given through:

$$
D_{0}\left(x^{\prime}\right) e_{1}=\partial_{1} \vec{d}_{0}, \quad D_{0}\left(x^{\prime}\right) e_{2}=\partial_{2} \vec{d}_{0}, \quad D_{0}\left(x^{\prime}\right) e_{3}=0
$$

so that:

$$
\nabla u^{h} A^{-1}=Q_{0} A^{-1}+x_{3} B_{0} A^{-1}+\frac{x_{3}^{2}}{2} D_{0} A^{-1} .
$$

For brevity, denote $F^{h}=\nabla u^{h} A^{-1}$ and further:

$$
\begin{equation*}
F^{h}\left(x^{\prime}, x_{3}\right)=Q_{0} A^{-1}\left(x^{\prime}\right)\left(\operatorname{Id}_{3}+x_{3} T_{1}\left(x^{\prime}\right)+x_{3}^{2} T_{2}\left(x^{\prime}\right)\right)=\left(Q_{0} A^{-1}\left(x^{\prime}\right)\right) G^{h}\left(x^{\prime}, x_{3}\right) \tag{2.4}
\end{equation*}
$$

with $T_{1}=A^{-1} Q_{0}^{t} B_{0} A^{-1}, T_{2}=\frac{1}{2} A^{-1} Q_{0}^{t} D_{0} A^{-1}$ and $G^{h}=\operatorname{Id}_{3}+x_{3} T_{1}+x_{3}^{2} T_{2}$. Since $Q_{0} A^{-1} \in \mathrm{SO}(3)$, frame indifference implies that $W\left(F^{h}\right)=W\left(G^{h}\right)$. Thanks to the boundedness and smoothness of $T_{1}$ and $T_{2}$, we have $\operatorname{det} G^{h}>0$ in $\Omega^{h}$ for $h \ll 1$, and thus polar factorization of $G^{h}$ further yields: $W\left(F^{h}\right)=W\left(\left(\left(G^{h}\right)^{t} G^{h}\right)^{1 / 2}\right)$. Note that $T_{1}$ is skew symmetric, by the same property of $Q_{0}^{t} B_{0}$. Consequently, $\left(G^{h}\right)^{t} G^{h}$ and the expansion of its square root do not contain terms linear in $x_{3}$. Indeed, letting $K=T_{2}+T_{2}^{t}-T_{1}^{2}$ :

$$
\left(\left(G^{h}\right)^{t} G^{h}\right)\left(x^{\prime}, x_{3}\right)=\operatorname{Id}_{3}+x_{3}^{2} K\left(x^{\prime}\right)+\mathcal{O}\left(x_{3}^{3}\right) \quad \text { and } \quad\left(\left(G^{h}\right)^{t} G^{h}\right)^{1 / 2}\left(x^{\prime}, x_{3}\right)=\operatorname{Id}_{3}+\frac{x_{3}^{2}}{2} K\left(x^{\prime}\right)+\mathcal{O}\left(x_{3}^{3}\right) .
$$

As a consequence, by $W\left(\mathrm{Id}_{3}\right)=0$ and $D W\left(\mathrm{Id}_{3}\right)=0$, we obtain:

$$
W\left(F^{h}\right)=W\left(\left(\left(G^{h}\right)^{t} G^{h}\right)^{1 / 2}\right)=\frac{x_{3}^{4}}{8} D^{2} W\left(\operatorname{Id}_{3}\right)(K, K)+\mathcal{O}\left(x_{3}^{5}\right)
$$

Using (1.3), we get:

$$
E^{h}\left(u^{h}\right)=\frac{1}{h} \int_{\Omega^{h}} W\left(F^{h}\right) \mathrm{d} x \leq C h^{4},
$$

which concludes the proof.
In Lemma 2.1 we constructed deformations whose gradient was close enough to $Q_{0}+x_{3} B_{0}$ to result in the energy order $h^{4}$. Conversely, in Corollary 2.4, we establish that the gradients of deformations $u^{h}$ whose energy scales like $h^{4}$ are close to $Q_{0}+x_{3} B_{0}$ modulo local multiplications by $R^{h}\left(x^{\prime}\right) \in \mathrm{SO}(3)$. Corollary 2.4 makes this statement precise and also gives an estimate of $\nabla R^{h}$. We first give local estimates in Lemma 2.3. To this end, recall the geometric rigidity estimate [29]:

Theorem 2.2. ([29]) Let $\Omega \subset \mathbb{R}^{3}$ be a bounded, connected, Lipschitz domain. Then:

$$
\forall u \in W^{1,2}\left(\Omega, \mathbb{R}^{3}\right) \quad \exists \bar{R} \in \mathrm{SO}(3) \quad \int_{\Omega}|\nabla u-\bar{R}|^{2} \leq C \int_{\Omega} \operatorname{dist}^{2}(\nabla u, \mathrm{SO}(3)) .
$$

The constant C is uniform for all bi-Lipschitz equivalent domains $\Omega$ with controlled Lipschitz constants.

Lemma 2.3. Assume (1.7). For all sufficiently small $h \ll 1$ and all open sets $\mathcal{U} \subset \omega$, there exists a constant $C=$ $C(\mathcal{U} \times(-h / 2, h / 2))$ with the following property. Calling $\mathcal{U}^{h}=\mathcal{U} \times(-h / 2, h / 2)$, for any $u^{h} \in W^{1,2}\left(\mathcal{U}^{h}, \mathbb{R}^{3}\right)$ there exists $\bar{R}^{h} \in \mathrm{SO}$ (3) such that:

$$
\begin{equation*}
\frac{1}{h} \int_{\mathcal{U}^{h}}\left|\nabla u^{h}(x)-\bar{R}^{h}\left(Q_{0}\left(x^{\prime}\right)+x_{3} B_{0}\left(x^{\prime}\right)\right)\right|^{2} \mathrm{~d} x \leq C\left(\frac{1}{h} \int_{\mathcal{U}^{h}} \operatorname{dist}^{2}\left(\nabla u^{h} A^{-1}, \mathrm{SO}(3)\right) \mathrm{d} x+h^{3}\left|\mathcal{U}^{h}\right|\right) . \tag{2.5}
\end{equation*}
$$

The constant $C$ is uniform for all bi-Lipschitz equivalent $\mathcal{U}^{h}$ with controlled Lipschitz constants.
Proof. In order to use the geometric rigidity estimate, we perform a change of variables. For any $u^{h} \in W^{1,2}\left(\mathcal{U}^{h}, \mathbb{R}^{3}\right)$, we let $v^{h}=u^{h} \circ Y^{-1}$ with $Y: \mathcal{U}^{h} \rightarrow Y\left(\mathcal{U}^{h}\right)=\mathcal{V}^{h} \subset \mathbb{R}^{3}$ given as in (2.3):

$$
Y\left(x^{\prime}, x_{3}\right)=y_{0}\left(x^{\prime}\right)+x_{3} \vec{b}_{0}\left(x^{\prime}\right)+\frac{x_{3}^{2}}{2} \vec{d}_{0}\left(x^{\prime}\right) .
$$

We note that $v^{h} \in W^{1,2}\left(\mathcal{V}^{h}, \mathbb{R}^{3}\right)$ and:

$$
\begin{equation*}
\nabla u^{h} A^{-1}\left(x^{\prime}, x_{3}\right)=\nabla v^{h}(z)\left(\nabla Y A^{-1}\right)\left(x^{\prime}, x_{3}\right), \quad z:=Y\left(x^{\prime}, x_{3}\right) . \tag{2.6}
\end{equation*}
$$

Let $T_{1}^{\prime}=B_{0} Q_{0}^{-1}$ and $T_{2}^{\prime}=\frac{1}{2} D_{0} Q_{0}^{-1}$. From $Q_{0}^{t} B_{0} \in \operatorname{so}(3)$, it is immediate that $T_{1}^{\prime}$ is skew symmetric as well, since $T_{1}^{\prime}=Q_{0}^{-t}\left(Q_{0}^{t} B_{0}\right) Q_{0}^{-1}$. Computations as in Lemma 2.1 now give:

$$
\begin{equation*}
\nabla Y\left(x^{\prime}, x_{3}\right)=Q_{0}\left(x^{\prime}\right)+x_{3} B_{0}\left(x^{\prime}\right)+\frac{x_{3}^{2}}{2} D_{0}\left(x^{\prime}\right) \tag{2.7}
\end{equation*}
$$

and:

$$
\nabla Y A^{-1}=\left(\operatorname{Id}_{3}+x_{3} T_{1}^{\prime}\left(x^{\prime}\right)+x_{3}^{2} T_{2}^{\prime}\left(x^{\prime}\right)\right)\left(Q_{0} A^{-1}\right)
$$

We see that $\operatorname{det}\left(\nabla Y A^{-1}\right)>0$ for $h \ll 1$. Further, $\nabla Y A^{-1}=\left(\nabla Y A^{-1}\left(\nabla Y A^{-1}\right)^{t}\right)^{1 / 2} R$ by the left polar decomposition, and hence for some symmetric matrix field $M=\mathcal{O}(1)$ and a rotation $R \in S O(3)$, we get:

$$
\nabla Y A^{-1}=\left(\operatorname{Id}_{3}+x_{3}^{2} M\left(x^{\prime}, x_{3}\right)\right) R\left(x^{\prime}, x_{3}\right) .
$$

Therefore:

$$
\begin{aligned}
\operatorname{dist}\left(\nabla v^{h} \nabla Y A^{-1}, \mathrm{SO}(3)\right) & =\operatorname{dist}\left(\nabla v^{h}\left(\operatorname{Id}_{3}+x_{3}^{2} M\right) R, \mathrm{SO}(3)\right)=\operatorname{dist}\left(\nabla v^{h}\left(\operatorname{Id}_{3}+x_{3}^{2} M\right), \mathrm{SO}(3)\right) \\
& \geq c \operatorname{dist}\left(\nabla v^{h}, \mathrm{SO}(3)\left(\operatorname{Id}_{3}+x_{3}^{2} M\right)^{-1}\right) \geq c \operatorname{dist}\left(\nabla v^{h}, \mathrm{SO}(3)\right)+\mathcal{O}\left(x_{3}^{2}\right) .
\end{aligned}
$$

Here and in the remainder of the proof, $c$ denotes a constant that only depends on $\omega$ and $G$. Let $J=\left|(\operatorname{det} \nabla Y) \circ Y^{-1}\right|^{-1}$. By (2.6) and the above computation, we obtain:

$$
\int_{\mathcal{U}^{h}} \operatorname{dist}^{2}\left(\nabla u^{h} A^{-1}, \mathrm{SO}(3)\right) \mathrm{d} x \geq c \int_{\mathcal{V}^{h}} \operatorname{dist}^{2}\left(\nabla v^{h}, \mathrm{SO}(3)\right) J \mathrm{~d} z-c \int_{\mathcal{U}^{h}} x_{3}^{4} \mathrm{~d} x .
$$

In other words, since $J \geq c>0$, we get:

$$
\frac{1}{h} \int_{\mathcal{U}^{h}} \operatorname{dist}^{2}\left(\nabla u^{h} A^{-1}, \mathrm{SO}(3)\right) \mathrm{d} x+h^{3}\left|\mathcal{U}^{h}\right| \geq \frac{c}{h} \int_{\mathcal{V}^{h}} \operatorname{dist}^{2}\left(\nabla v^{h}, \mathrm{SO}(3)\right) \mathrm{d} z .
$$

By Theorem 2.2, there exists $C>0$ such that for any $v^{h} \in W^{1,2}\left(\mathcal{V}^{h}, \mathbb{R}^{3}\right)$, there is $\bar{R}^{h} \in \operatorname{SO}(3)$ with:

$$
C \int_{\mathcal{V}^{h}} \operatorname{dist}^{2}\left(\nabla v^{h}, \mathrm{SO}(3)\right) \mathrm{d} z \geq \int_{\mathcal{V}^{h}}\left|\nabla v^{h}-\bar{R}^{h}\right|^{2} \mathrm{~d} z
$$

By the reverse change of variables which satisfies $J^{-1} \geq c>0$ and $|\nabla Y| \leq C$, we obtain, again with a uniform constant $C$ :

$$
C\left(\frac{1}{h} \int_{\mathcal{U}^{h}} \operatorname{dist}^{2}\left(\nabla u^{h} A^{-1}, \operatorname{SO}(3)\right) \mathrm{d} x+h^{3}\left|\mathcal{U}^{h}\right|\right) \geq \frac{1}{h} \int_{\mathcal{U}^{h}}\left|\nabla u^{h}-\bar{R}^{h} \nabla Y\right|^{2} \mathrm{~d} x
$$

This accomplishes the proof of the lemma in view of (2.7).
Using the third assumption in (1.4), we now pass to global estimates.
Corollary 2.4. Assume (1.7) and let $u^{h}$ be a sequence of deformations such that:

$$
E^{h}\left(u^{h}\right) \leq C h^{4} .
$$

Then, there exist matrix fields $R^{h} \in W^{1,2}(\omega, \mathrm{SO}(3))$ such that:

$$
\begin{equation*}
\frac{1}{h} \int_{\Omega^{h}}\left|\nabla u^{h}(x)-R^{h}\left(x^{\prime}\right)\left(Q_{0}\left(x^{\prime}\right)+x_{3} B_{0}\left(x^{\prime}\right)\right)\right|^{2} \mathrm{~d} x \leq C h^{4} \tag{2.8}
\end{equation*}
$$

and:

$$
\begin{equation*}
\int_{\omega}\left|\nabla R^{h}\left(x^{\prime}\right)\right|^{2} \mathrm{~d} x^{\prime} \leq C h^{2} \tag{2.9}
\end{equation*}
$$

The proof follows the lines of $[29,45,39]$, with necessary modifications. For completeness, we present the details in the Appendix.

## 3. The lower bound

In the theorem below, we prove that the deformations $u^{h}$ with energy of order $h^{4}$ converge, together with their properly defined increments of order 1 or 2.

Theorem 3.1. Let $u^{h} \in W^{1,2}\left(\Omega^{h}, \mathbb{R}^{3}\right)$ be a sequence of deformations satisfying $E^{h}\left(u^{h}\right) \leq C h^{4}$. Then there exists a sequence of translations $c^{h} \in \mathbb{R}^{3}$ and rotations $\bar{R}^{h} \in \mathrm{SO}(3)$ such that the associated renormalizations:

$$
\begin{equation*}
y^{h}\left(x^{\prime}, x_{3}\right)=\left(\bar{R}^{h}\right)^{t} u^{h}\left(x^{\prime}, h x_{3}\right)-c^{h} \in W^{1,2}\left(\Omega^{1}, \mathbb{R}^{3}\right) \tag{3.1}
\end{equation*}
$$

have the properties below. Recall that $y_{0}$ and $\vec{b}_{0}$ are the unique solutions to (1.8), (1.9). All convergences hold up to a subsequence.
(i) $y^{h} \rightarrow y_{0}$ in $W^{1,2}\left(\Omega^{1}, \mathbb{R}^{3}\right)$ and $\frac{1}{h} \partial_{3} y^{h} \rightarrow \vec{b}_{0}$ in $L^{2}\left(\Omega^{1}, \mathbb{R}^{3}\right)$;
(ii) the scaled average displacements:

$$
\begin{equation*}
V^{h}\left(x^{\prime}\right)=\frac{1}{h} \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(y^{h}\left(x^{\prime}, x_{3}\right)-\left(y_{0}\left(x^{\prime}\right)+h x_{3} \vec{b}_{0}\left(x^{\prime}\right)\right)\right) \mathrm{d} x_{3} \tag{3.2}
\end{equation*}
$$

converge in $W^{1,2}\left(\omega, \mathbb{R}^{3}\right)$ to a limiting field $V \in W^{2,2}\left(\omega, \mathbb{R}^{3}\right)$, satisfying the constraint:

$$
\begin{equation*}
\operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla V\right)=0 ; \tag{3.3}
\end{equation*}
$$

(iii) the scaled tangential strains:

$$
\frac{1}{h} \operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla V^{h}\right)
$$

converge weakly in $L^{2}\left(\omega, \mathbb{R}^{2 \times 2}\right)$ to some $\mathrm{e} \in L^{2}\left(\omega, \mathbb{R}_{s y m}^{2 \times 2}\right)$.
(iv) Further, defining the quadratic forms $\mathcal{Q}_{3}$ and $\mathcal{Q}_{2, A}$ by:

$$
\begin{align*}
\mathcal{Q}_{3}(F) & =D^{2} W\left(\mathrm{Id}_{3}\right)(F, F), \\
\mathcal{Q}_{2, A}\left(x^{\prime}, F_{2 \times 2}\right) & =\min \left\{\mathcal{Q}_{3}\left(A\left(x^{\prime}\right)^{-1} \tilde{F} A\left(x^{\prime}\right)^{-1}\right) ; \quad \tilde{F} \in \mathbb{R}^{3 \times 3} \text { with } \tilde{F}_{2 \times 2}=F_{2 \times 2}\right\}, \tag{3.4}
\end{align*}
$$

we have:

$$
\begin{align*}
\liminf _{h \rightarrow 0} \frac{1}{h^{4}} E^{h}\left(u^{h}\right) \geq \mathcal{I}_{4}(V, \mathrm{e})= & \frac{1}{2} \int_{\omega} \mathcal{Q}_{2, A}\left(x^{\prime}, \mathrm{e}+\frac{1}{2}(\nabla V)^{t} \nabla V+\frac{1}{24}\left(\nabla \vec{b}_{0}\right)^{t} \nabla \vec{b}_{0}\right) \mathrm{d} x^{\prime} \\
& +\frac{1}{24} \int_{\omega} \mathcal{Q}_{2, A}\left(x^{\prime},\left(\nabla y_{0}\right)^{t} \nabla \vec{p}+(\nabla V)^{t} \nabla \vec{b}_{0}\right) \mathrm{d} x^{\prime}  \tag{3.5}\\
& +\frac{1}{1440} \int_{\omega} \mathcal{Q}_{2, A}\left(x^{\prime},\left(\nabla y_{0}\right)^{t} \nabla \vec{d}_{0}+\left(\nabla \vec{b}_{0}\right)^{t} \nabla \vec{b}_{0}\right) \mathrm{d} x^{\prime}
\end{align*}
$$

where the vector field $\vec{p} \in W^{1,2}\left(\omega, \mathbb{R}^{3}\right)$ is uniquely associated with $V$ by:

$$
\begin{equation*}
\left(\nabla y_{0}\right)^{t} \vec{p}=-(\nabla V)^{t} \vec{b}_{0} \quad \text { and } \quad\left\langle\vec{p}, \vec{b}_{0}\right\rangle=0 \tag{3.6}
\end{equation*}
$$

Proof. We split the proof into several steps. In the first four steps, we establish the existence of convergent subsequences for the quantities under consideration and obtain properties of their limits. In the final step, we prove the lower bound in (3.5).

Step 1. To prove the claimed convergence properties for (3.1), we first set:

$$
\bar{R}^{h}=\mathbb{P}_{\mathrm{SO}(3)} \int_{\Omega^{h}} \nabla u^{h}(x) Q_{0}\left(x^{\prime}\right)^{-1} \mathrm{~d} x .
$$

Note that the projection above is well defined, because for every $x^{\prime} \in \omega$ we have, in view of (2.8):

$$
\begin{aligned}
\operatorname{dist}^{2}\left(f_{\Omega^{h}} \nabla u^{h} Q_{0}^{-1} \mathrm{~d} x, \mathrm{SO}(3)\right) & \leq\left|f_{\Omega^{h}} \nabla u^{h} Q_{0}^{-1} \mathrm{~d} x-R^{h}\left(x^{\prime}\right)\right|^{2} \\
& \leq C\left|{\underset{\Omega^{h}}{ }}\left(\nabla u^{h} Q_{0}^{-1}-R^{h}\right) \mathrm{d} x\right|^{2}+C\left|f_{\Omega^{h}} R^{h} \mathrm{~d} x-R^{h}\left(x^{\prime}\right)\right|^{2} \\
& \leq C\left|f_{\Omega^{h}}\left(\nabla u^{h}-R^{h}\left(Q_{0}+x_{3} B_{0}\right)\right) Q_{0}^{-1}\right|^{2} \mathrm{~d} x+C\left|R^{h}\left(x^{\prime}\right)-f_{\omega} R^{h}\right|^{2} \\
& \leq C{\underset{\Omega^{h}}{ }\left|\nabla u^{h}-R^{h}\left(Q_{0}+x_{3} B_{0}\right)\right|^{2} \mathrm{~d} x+C\left|R^{h}\left(x^{\prime}\right)-f_{\omega} R^{h}\right|^{2}}
\end{aligned}
$$

so that, taking the average on $\omega$, by the Poincaré-Wirtinger inequality and (2.9), we get:

$$
\operatorname{dist}^{2}\left(f_{\Omega^{h}} \nabla u^{h} Q_{0}^{-1} \mathrm{~d} x, \mathrm{SO}(3)\right) \leq C h^{4}+C \int_{\omega}\left|\nabla R^{h}\right|^{2} \leq C h^{2}
$$

In particular, we observe that:

$$
\begin{equation*}
\left|f_{\Omega^{h}} \nabla u^{h} Q_{0}^{-1} \mathrm{~d} x-\bar{R}^{h}\right|^{2} \leq C h^{2} \tag{3.7}
\end{equation*}
$$

Moreover, by (2.8), (2.9) and (3.7):

$$
\begin{align*}
f_{\omega}\left|R^{h}-\bar{R}^{h}\right|^{2} \mathrm{~d} x & ={\underset{\Omega^{h}}{f}\left|R^{h}-\bar{R}^{h}\right|^{2} \mathrm{~d} x} \quad \begin{array}{rl}
C & f \\
\Omega_{\Omega^{h}} & \left.\left|R^{h}-\underset{\omega}{f} R^{h}\right|^{2}+\left|\left(\underset{\omega}{f} R^{h}\right)-\underset{\Omega^{h}}{f} \nabla u^{h} Q_{0}^{-1}\right|^{2}\right)+\underset{\Omega^{h}}{f}\left|\bar{R}^{h}-\underset{\Omega^{h}}{f} \nabla u^{h} Q_{0}^{-1}\right|^{2} \\
& \leq C \underset{\Omega^{h}}{f}\left|\nabla R^{h}\right|^{2} \mathrm{~d} x+C \underset{\Omega^{h}}{f}\left|\nabla u^{h}-R^{h}\left(Q_{0}+x_{3} B_{0}\right)\right|^{2} \mathrm{~d} x+C h^{2} \leq C h^{2}
\end{array} .
\end{align*}
$$

Let now $c^{h} \in \mathbb{R}^{3}$ be such that $\int_{\omega} V^{h}=0$ where $V^{h}$ is defined as in (3.2). Denote by $\nabla_{h} y^{h}$ the matrix whose columns are given by $\partial_{1} y^{h}, \partial_{2} y^{h}$ and $\partial_{3} y^{h} / h$, so that:

$$
\begin{equation*}
\nabla_{h} y^{h}\left(x^{\prime}, x_{3}\right)=\left(\bar{R}^{h}\right)^{t} \nabla u^{h}\left(x^{\prime}, h x_{3}\right) . \tag{3.9}
\end{equation*}
$$

Observe that by (2.8) and (3.8):

$$
\begin{aligned}
\int_{\Omega^{1}}\left|\nabla_{h} y^{h}-Q_{0}\right|^{2} \mathrm{~d} x & \leq C f_{\Omega^{h}}\left|\nabla u^{h}-\bar{R}^{h} Q_{0}\right|^{2} \mathrm{~d} x \\
& \leq C\left(f_{\Omega^{h}}\left|\nabla u^{h}-R^{h}\left(Q_{0}+x_{3} B_{0}\right)\right|^{2} \mathrm{~d} x+\int_{\Omega^{h}}\left|x_{3} R^{h} B_{0}\right|^{2} \mathrm{~d} x+f_{\Omega^{h}}\left|R^{h}-\bar{R}^{h}\right|^{2} \mathrm{~d} x\right) \leq C h^{2} .
\end{aligned}
$$

Therefore, $\nabla_{h} y^{h}$ converges in $L^{2}\left(\Omega^{1}\right)$ to $Q_{0}$. Observe that the sequence $\left\{y^{h}\right\}$ is bounded in $W^{1,2}\left(\Omega^{1}\right)$, by the choice of $c^{h}$. Passing to a subsequence, if necessary, we get that $y^{h}$ converges weakly in $W^{1,2}\left(\Omega^{1}\right)$ and so, in fact:

$$
y^{h} \rightarrow y_{0} \quad \text { in } \quad W^{1,2}\left(\Omega^{1}, \mathbb{R}^{3}\right) \quad \text { and } \quad \frac{1}{h} \partial_{3} y^{h} \rightarrow \vec{b}_{0} \quad \text { in } \quad L^{2}\left(\Omega^{1}, \mathbb{R}^{3}\right)
$$

Step 2. Note that, for every $x^{\prime} \in \omega$ :

$$
\begin{align*}
\nabla V^{h}\left(x^{\prime}\right) & =\frac{1}{h}\left(f_{-1 / 2}^{1 / 2} \nabla_{h} y^{h}(x)-Q_{0}\left(x^{\prime}\right) \mathrm{d} x_{3}\right)_{3 \times 2} \\
& =\frac{1}{h}\left(f_{-1 / 2}^{1 / 2} \nabla_{h} y^{h}-\left(\bar{R}^{h}\right)^{t} R^{h}\left(Q_{0}+h x_{3} B_{0}\right) \mathrm{d} x_{3}\right)_{3 \times 2}+\frac{1}{h}\left(\left(\left(\bar{R}^{h}\right)^{t} R^{h}-\mathrm{Id}_{3}\right) Q_{0}\right)_{3 \times 2}=I_{1}^{h}+I_{2}^{h} \tag{3.10}
\end{align*}
$$

The first term above converges to 0 . Indeed:

$$
\begin{align*}
\left\|I_{1}^{h}\right\|_{L^{2}(\omega)}^{2} & \leq \frac{C}{h^{2}} f_{\Omega^{1}}\left|\left(\bar{R}^{h}\right)^{t} \nabla u^{h}\left(x^{\prime}, h x_{3}\right)-\left(\bar{R}^{h}\right)^{t} R^{h}\left(Q_{0}\left(x^{\prime}\right)+h x_{3} B_{0}\right)\right|^{2} \mathrm{~d} x \\
& \leq \frac{C}{h^{2}} f_{\Omega^{h}}\left|\nabla u^{h}\left(x^{\prime}, x_{3}\right)-R^{h}\left(Q_{0}+x_{3} B_{0}\right)\right|^{2} \mathrm{~d} x \leq C h^{2} . \tag{3.11}
\end{align*}
$$

Towards estimating the second term in (3.10), denote:

$$
S^{h}=\frac{1}{h}\left(\left(\bar{R}^{h}\right)^{t} R^{h}-\mathrm{Id}_{3}\right) .
$$

By (3.8) and (2.9), it follows that:

$$
\left\|S^{h}\right\|_{L^{2}(\omega)}^{2} \leq \frac{C}{h^{2}} \int_{\omega}\left|R^{h}-\bar{R}^{h}\right|^{2} \leq C \quad \text { and } \quad\left\|\nabla S^{h}\right\|_{L^{2}(\omega)}^{2} \leq \frac{C}{h^{2}} \int_{\omega}\left|\nabla R^{h}\right|^{2} \leq C .
$$

Passing to a subsequence, we can assume that:

$$
\begin{equation*}
S^{h} \rightharpoonup S \quad \text { weakly in } W^{1,2}(\omega) \tag{3.12}
\end{equation*}
$$

which implies:

$$
\begin{equation*}
I_{2}^{h} \rightarrow\left(S Q_{0}\right)_{3 \times 2} \quad \text { in } L^{2}\left(\omega, \mathbb{R}^{3 \times 2}\right) \tag{3.13}
\end{equation*}
$$

Consequently, by (3.10):

$$
\begin{equation*}
\nabla V^{h} \rightarrow\left(S Q_{0}\right)_{3 \times 2} \quad \text { in } L^{2}\left(\omega, \mathbb{R}^{3 \times 2}\right) \tag{3.14}
\end{equation*}
$$

As before, we conclude that $V^{h}$ converges in $W^{1,2}(\omega)$ and that its limit $V$ belongs to $W^{2,2}\left(\omega, \mathbb{R}^{3}\right)$, since $\nabla V=$ $\left(S Q_{0}\right)_{3 \times 2} \in W^{1,2}(\omega)$. We now prove (3.3). By definition of $S^{h}$ :

$$
\begin{equation*}
\operatorname{sym} S^{h}=-\frac{h}{2}\left(S^{h}\right)^{t} S^{h}, \tag{3.15}
\end{equation*}
$$

so in view of the boundedness of $\left\{S^{h}\right\}$ in $W^{1,2}$ :

$$
\left\|\operatorname{sym} S^{h}\right\|_{L^{2}(\omega)} \leq C h\left\|S^{h}\right\|_{L^{4}(\omega)}^{2} \leq C h\left\|S^{h}\right\|_{W^{1,2}(\omega)}^{2} \leq C h .
$$

Consequently, $S$ is a skew symmetric field. But $\left(\nabla y_{0}\right)^{t} \nabla V=\left(Q_{0}^{t} S Q_{0}\right)_{2 \times 2}$, hence (3.3) follows.
For future use, let us define $\vec{p} \in W^{1,2}\left(\omega, \mathbb{R}^{3}\right)$ by:

$$
\begin{equation*}
[\nabla V \mid \vec{p}]=S Q_{0} \tag{3.16}
\end{equation*}
$$

Since $Q_{0}^{t}[\nabla V \mid p] \in \operatorname{so}(3)$, it is easily checked that $\vec{p}$ is given solely in terms of $V$ by:

$$
\left\{\begin{array}{l}
\left(\nabla y_{0}\right)^{t} \vec{p}=-(\nabla V)^{t} \vec{b}_{0}  \tag{3.17}\\
\left\langle\vec{p}, \vec{b}_{0}\right\rangle=0 .
\end{array}\right.
$$

Step 3. We now want to establish convergence in (iii). In view of (3.10) we write:

$$
\begin{equation*}
\frac{1}{h} \operatorname{sym}\left(Q_{0}^{t} \nabla V^{h}\right)_{2 \times 2}\left(x^{\prime}\right)=\frac{1}{h} \operatorname{sym}\left(Q_{0}^{t} I_{1}^{h}\right)_{2 \times 2}+\frac{1}{h} \operatorname{sym}\left(Q_{0}^{t} S^{h} Q_{0}\right)_{2 \times 2}=J_{1}^{h}+J_{2}^{h} \tag{3.18}
\end{equation*}
$$

We first deal with the sequence $J_{2}^{h}$. By (3.12), $S^{h} \rightarrow S$ in $L^{4}(\Omega)$ and so (3.15) implies:

$$
\frac{1}{h} \operatorname{sym} S^{h} \rightarrow-\frac{1}{2} S^{t} S=\frac{1}{2} S^{2} \quad \text { in } L^{2}(\omega)
$$

Therefore:

$$
\begin{equation*}
J_{2}^{h} \rightarrow-\frac{1}{2}\left(Q_{0}^{t} S^{t} S Q_{0}\right)_{2 \times 2}=-\frac{1}{2}(\nabla V)^{t} \nabla V \quad \text { in } \quad L^{2}(\omega) \tag{3.19}
\end{equation*}
$$

We now prove that $J_{1}^{h}$ converges. Recall that by (3.18), (3.10) and (3.9):

$$
\begin{equation*}
J_{1}^{h}=\frac{1}{h} \operatorname{sym}\left(Q_{0}^{t} I_{1}^{h}\right)_{2 \times 2}=\operatorname{sym}\left(Q_{0}^{t}\left(\bar{R}^{h}\right)^{t} \int_{-1 / 2}^{1 / 2} Z^{h}\left(x^{\prime}, x_{3}\right) \mathrm{d} x_{3}\right)_{2 \times 2} \tag{3.20}
\end{equation*}
$$

where the rescaled strains $Z^{h}$ are defined by:

$$
\begin{equation*}
Z^{h}\left(x^{\prime}, x_{3}\right)=\frac{1}{h^{2}}\left(\nabla u^{h}\left(x^{\prime}, h x_{3}\right)-R^{h}\left(x^{\prime}\right)\left(Q_{0}\left(x^{\prime}\right)+h x_{3} B_{0}\left(x^{\prime}\right)\right)\right) . \tag{3.21}
\end{equation*}
$$

By (2.8), the sequence $\left\{Z^{h}\right\}$ is bounded in $L^{2}\left(\Omega^{1}, \mathbb{R}^{3}\right)$. Therefore, up to a subsequence:

$$
\begin{equation*}
Z^{h} \rightharpoonup Z \quad \text { weakly in } L^{2}\left(\Omega^{1}, \mathbb{R}^{3}\right) . \tag{3.22}
\end{equation*}
$$

The following convergence yields (iii) by (3.18) and (3.19):

$$
\begin{equation*}
J_{1}^{h} \rightharpoonup J_{1}:=\operatorname{sym}\left(Q_{0}^{t}(\bar{R})^{t} \int_{-1 / 2}^{1 / 2} Z\left(x^{\prime}, x_{3}\right) \mathrm{d} x_{3}\right)_{2 \times 2} \quad \text { weakly in } L^{2}(\omega) \tag{3.23}
\end{equation*}
$$

Step 4. We now aim at giving the structure of the weak limit e of $\frac{1}{h} \operatorname{sym}\left(Q_{0}^{t} \nabla V^{h}\right)_{2 \times 2}$ in terms of the limiting fields $V$ and $Z$. We have just seen that:

$$
\begin{equation*}
\mathrm{e}=J_{1}-\frac{1}{2}(\nabla V)^{t} \nabla V \tag{3.24}
\end{equation*}
$$

where $J_{1}$ is given by (3.23). As a tool, consider the difference quotients $f^{s, h}$ :

$$
f^{s, h}\left(x^{\prime}, x_{3}\right)=\frac{1}{h^{2} s}\left(y^{h}\left(x^{\prime}, x_{3}+s\right)-y^{h}\left(x^{\prime}, x_{3}\right)-h s\left(\vec{b}_{0}+h\left(x_{3}+\frac{s}{2}\right) \vec{d}_{0}\right)\right) .
$$

We will show that $f^{s, h} \rightharpoonup \vec{p}$ weakly in $W^{1,2}\left(\Omega^{1}, \mathbb{R}^{3}\right)$, as $h \rightarrow 0$, for any $s$. Write:

$$
f^{s, h}\left(x^{\prime}, x_{3}\right)=\frac{1}{h^{2}} \int_{0}^{s} \partial_{3} y^{h}\left(x^{\prime}, x_{3}+t\right)-h\left(\vec{b}_{0}+h\left(x_{3}+t\right) \vec{d}_{0}\right) \mathrm{d} t
$$

and observe that:

$$
\begin{aligned}
\frac{1}{h^{2}}\left(\partial_{3} y^{h}-h\left(\vec{b}_{0}+h x_{3} \vec{d}_{0}\right)\right) & =\frac{1}{h}\left(\left(\bar{R}^{h}\right)^{t} \nabla u^{h}\left(x^{\prime}, h x_{3}\right)-\left(Q_{0}+h x_{3} B_{0}\right)\right) e_{3} \\
& =\frac{1}{h}\left(\bar{R}^{h}\right)^{t}\left(\nabla u^{h}\left(x^{\prime}, h x_{3}\right)-R^{h}\left(Q_{0}+h x_{3} B_{0}\right)\right) e_{3}+S^{h}\left(Q_{0}+h x_{3} B_{0}\right) e_{3} \\
& =h\left(\bar{R}^{h}\right)^{t} Z^{h}\left(x^{\prime}, x_{3}\right) e_{3}+S^{h}\left(Q_{0}+h x_{3} B_{0}\right) e_{3} .
\end{aligned}
$$

The first term on the right-hand side above converges to 0 in $L^{2}\left(\Omega^{1}\right)$ because $\left\{Z^{h}\right\}$ is bounded in $L^{2}\left(\Omega^{1}, \mathbb{R}^{3}\right)$, while the second term converges to $S Q_{0} e_{3}=S \vec{b}_{0}$ in $L^{2}\left(\Omega^{1}\right)$ by (3.12). Note that $S Q_{0} e_{3}=\vec{p}$ by (3.16). Therefore, $f^{s, h} \rightarrow \vec{p}$ in $L^{2}\left(\Omega^{1}\right)$.

We now deal with the derivatives of the studied sequence. Firstly:

$$
\partial_{3} f^{s, h}\left(x^{\prime}, x_{3}\right)=\frac{1}{s}\left(\frac{1}{h^{2}}\left(\partial_{3} y^{h}\left(x^{\prime}, x_{3}+s\right)-h\left(\vec{b}_{0}+h\left(x_{3}+s\right) \vec{d}_{0}\right)\right)-\frac{1}{h^{2}}\left(\partial_{3} y^{h}\left(x^{\prime}, x_{3}\right)-h\left(\vec{b}_{0}+h x_{3} \vec{d}_{0}\right)\right)\right)
$$

converges to 0 in $L^{2}\left(\Omega^{1}\right)$. For $i=1,2$, the in-plane derivatives read as:

$$
\begin{aligned}
\partial_{i} f^{s, h}\left(x^{\prime}, x_{3}\right)= & \frac{1}{h^{2} s}\left(\left(\bar{R}^{h}\right)^{t} \partial_{i} u^{h}\left(x^{\prime}, h\left(x_{3}+s\right)\right)-\left(\bar{R}^{h}\right)^{t} \partial_{i} u^{h}\left(x^{\prime}, h x_{3}\right)-h s\left(\partial_{i} \vec{b}_{0}+h\left(x_{3}+\frac{s}{2}\right)\right) \partial_{i} \vec{d}_{0}\right) \\
= & \frac{1}{s}\left(\left(\bar{R}^{h}\right)^{t} Z^{h}\left(x^{\prime}, x_{3}+s\right)-\left(\bar{R}^{h}\right)^{t} Z^{h}\left(x^{\prime}, x_{3}\right)\right) e_{i} \\
& +\frac{1}{h^{2} s}\left(\left(\bar{R}^{h}\right)^{t} R^{h}\left(Q_{0}+h\left(x_{3}+s\right) B_{0}\right)-\left(\bar{R}^{h}\right)^{t} R^{h}\left(Q_{0}+h x_{3} B_{0}\right)\right) e_{i}-\frac{1}{h}\left(B_{0} e_{i}+h\left(x_{3}+\frac{s}{2}\right) \partial_{i} \vec{d}_{0}\right) .
\end{aligned}
$$

The last two terms above can be written as: $S^{h} B_{0} e_{i}-\left(x_{3}+\frac{s}{2}\right) \partial_{i} \vec{d}_{0}$, hence by (3.22):

$$
\partial_{i} f^{s, h}\left(x^{\prime}, x_{3}\right)-\frac{1}{s}(\bar{R})^{t}\left(Z\left(x^{\prime}, x_{3}+s\right)-Z\left(x^{\prime}, x_{3}\right)\right) e_{i}+S B_{0} e_{i}-\left(x_{3}+\frac{s}{2}\right) \partial_{i} \vec{d}_{0} \quad \text { weakly in } L^{2}\left(\Omega^{1}, \mathbb{R}^{3}\right)
$$

where $\bar{R} \in \mathrm{SO}(3)$ is an accumulation point of the rotations $\bar{R}^{h}$.
Consequently, $f^{s, h} \rightharpoonup \vec{p}$ weakly in $W^{1,2}\left(\Omega^{1}, \mathbb{R}^{3}\right)$ and, for $i=1,2$ :

$$
\begin{equation*}
s \partial_{i} \vec{p}=(\bar{R})^{t}\left(Z\left(x^{\prime}, x_{3}+s\right)-Z\left(x^{\prime}, x_{3}\right)\right) e_{i}+s S B_{0} e_{i}-s\left(x_{3}+\frac{s}{2}\right) \partial_{i} \vec{d}_{0} \tag{3.25}
\end{equation*}
$$

which proves that $Z\left(x^{\prime}, \cdot\right) e_{i}$ has polynomial form and that:

$$
\begin{equation*}
\left(\bar{R}^{t} Z\left(x^{\prime}, x_{3}\right)\right)_{3 \times 2}=\left(\bar{R}^{t} Z\left(x^{\prime}, 0\right)\right)_{3 \times 2}+x_{3}\left(\nabla \vec{p}-\left(S B_{0}\right)_{3 \times 2}\right)+\frac{x_{3}^{2}}{2} \nabla \vec{d}_{0} . \tag{3.26}
\end{equation*}
$$

By (3.22), it follows that:

$$
J_{1}=\operatorname{sym}\left(Q_{0}^{t}(\bar{R})^{t} Z\left(x^{\prime}, 0\right)\right)_{2 \times 2}+\frac{1}{24} \operatorname{sym}\left(Q_{0}^{t} \nabla \vec{d}_{0}\right)_{2 \times 2}
$$

With (3.24), we finally arrive at the following identity that links e, $V$ and $Z$ :

$$
\begin{equation*}
\mathrm{e}\left(x^{\prime}\right)=\operatorname{sym}\left(Q_{0}^{t}(\bar{R})^{t} Z\left(x^{\prime}, 0\right)\right)_{2 \times 2}+\frac{1}{24} \operatorname{sym}\left(Q_{0}^{t} \nabla \vec{d}_{0}\right)_{2 \times 2}-\frac{1}{2}(\nabla V)^{t} \nabla V . \tag{3.27}
\end{equation*}
$$

Step 5. We now prove the lower bound in (iv). Recall that by (3.21):

$$
\nabla u^{h}\left(x^{\prime}, h x_{3}\right)=R^{h}\left(x^{\prime}\right)\left(Q_{0}\left(x^{\prime}\right)+h x_{3} B_{0}\left(x^{\prime}\right)\right)+h^{2} Z^{h}\left(x^{\prime}, x_{3}\right) .
$$

Since $Q_{0} A^{-1} \in \mathrm{SO}(3)$ we have:

$$
W\left(\nabla u^{h} A^{-1}\right)=W\left(\left(Q_{0} A^{-1}\right)^{t}\left(R^{h}\right)^{t} \nabla u^{h} A^{-1}\right)=W\left(\operatorname{Id}_{3}+h \mathcal{J}+h^{2} \mathcal{G}^{h}\right),
$$

where:

$$
\mathcal{J}\left(x^{\prime}, x_{3}\right)=x_{3} A^{-1}\left(Q_{0}^{t} B_{0}\right) A^{-1}\left(x^{\prime}\right) \in \operatorname{so}(3), \quad \mathcal{G}^{h}\left(x^{\prime}, x_{3}\right)=A^{-1} Q_{0}^{t}\left(R^{h}\right)^{t} Z^{h}\left(x^{\prime}, x_{3}\right) A^{-1}
$$

Note that by (3.22):

$$
\mathcal{G}^{h}\left(x^{\prime}, x_{3}\right) \rightharpoonup \mathcal{G}=A^{-1} Q_{0}^{t}\left(\bar{R}^{t}\right) Z\left(x^{\prime}, x_{3}\right) A^{-1} \quad \text { weakly in } L^{2}\left(\Omega^{1}, \mathbb{R}^{3 \times 3}\right) .
$$

Define the "good sets":

$$
\Omega_{h}=\left\{x \in \Omega^{1} ; h\left|\mathcal{G}^{h}\right|<1\right\} .
$$

By the above, the characteristic functions $\mathbb{1}_{\Omega_{h}}$ converge to $\mathbb{1}$ in $L^{1}\left(\Omega^{1}\right)$. Further, by frame invariance and Taylor expanding $W$ on $\Omega_{h}$ :

$$
\begin{aligned}
W\left(\operatorname{Id}_{3}+h \mathcal{J}+h^{2} \mathcal{G}^{h}\right) & =W\left(e^{-h \mathcal{J}}\left(\operatorname{Id}_{3}+h \mathcal{J}+h^{2} \mathcal{G}^{h}\right)\right)=W\left(\operatorname{Id}_{3}+h^{2}\left(\mathcal{G}^{h}-\frac{1}{2} \mathcal{J}^{2}\right)+o\left(h^{2}\right)\right) \\
& =\frac{1}{2} \mathcal{Q}_{3}\left(h^{2}\left(\mathcal{G}^{h}-\frac{1}{2} \mathcal{J}^{2}\right)\right)+o\left(h^{4}\right),
\end{aligned}
$$

where the remainder $o\left(h^{4}\right)$ is uniform in the set $\Omega_{h}$. Therefore:

$$
\begin{align*}
\liminf _{h \rightarrow 0} \frac{1}{h^{4}} E^{h}\left(u^{h}\right) & \geq \liminf _{h \rightarrow 0} \frac{1}{h^{4}} \int_{\Omega^{1}} \mathbb{1}_{\Omega_{h}} W\left(\operatorname{Id}_{3}+h \mathcal{J}+h^{2} \mathcal{G}^{h}\right) \mathrm{d} x \\
& =\liminf _{h \rightarrow 0} \frac{1}{2} \int_{\Omega^{1}} \mathcal{Q}_{3}\left(\mathbb{1}_{\Omega_{h}} \operatorname{sym}\left(\mathcal{G}^{h}-\frac{1}{2} \mathcal{J}^{2}\right)\right) \mathrm{d} x \geq \frac{1}{2} \int_{\Omega^{1}} \mathcal{Q}_{3}\left(\operatorname{sym}\left(\mathcal{G}-\frac{1}{2} \mathcal{J}^{2}\right)\right) \mathrm{d} x, \tag{3.28}
\end{align*}
$$

by the weak sequential lower semi-continuity of the quadratic form $\mathcal{Q}_{3}$ in $L^{2}$ and in view of:

$$
\mathbb{1}_{\Omega^{h}} \operatorname{sym}\left(\mathcal{G}^{h}-\frac{1}{2} \mathcal{J}^{2}\right) \rightharpoonup \operatorname{sym} \mathcal{G}-\frac{1}{2} \mathcal{J}^{2} \quad \text { weakly in } L^{2}\left(\Omega^{1}\right)
$$

Note that by (3.16) we have: $\left(Q_{0}^{t} S B_{0}\right)_{2 \times 2}=-(\nabla V)^{t} \nabla \vec{b}_{0}$ and that:

$$
\mathcal{J}^{2}=-\mathcal{J}^{t} \mathcal{J}=-x_{3}^{2} A^{-1} B_{0}^{t} B_{0} A^{-1}
$$

Thus, using (3.26), the right-hand side of (3.28) is bounded below by:

$$
\begin{aligned}
& \frac{1}{2} \int_{\Omega^{1}} \mathcal{Q}_{2, A}\left(x^{\prime}, \operatorname{sym}\left(Q_{0}^{t}(\bar{R})^{t} Z\left(x^{\prime}, 0\right)+x_{3}\left(Q_{0}^{t} \nabla \vec{p}+(\nabla V)^{t} \nabla \vec{b}_{0}\right)+\frac{x_{3}^{2}}{2}\left(Q_{0}^{t} \nabla \vec{d}_{0}+\left(\nabla \vec{b}_{0}\right)^{t} \nabla \vec{b}_{0}\right)\right)_{2 \times 2}\right) \mathrm{d} x \\
& \quad=\frac{1}{2} \int_{\Omega^{1}} \mathcal{Q}_{2, A}\left(x^{\prime}, I\left(x^{\prime}\right)+x_{3} I I I\left(x^{\prime}\right)+x_{3}^{2} I I\left(x^{\prime}\right)\right) \mathrm{d} x
\end{aligned}
$$

Above we used (3.27) and we denoted:

$$
\begin{align*}
I\left(x^{\prime}\right) & =\mathrm{e}-\frac{1}{24} \operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla \vec{d}_{0}\right)+\frac{1}{2}(\nabla V)^{t} \nabla V \\
I I\left(x^{\prime}\right) & =\frac{1}{2} \operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla \vec{d}_{0}\right)+\frac{1}{2}\left(\nabla \vec{b}_{0}\right)^{t} \nabla \vec{b}_{0}  \tag{3.29}\\
I I I\left(x^{\prime}\right) & =\operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla \vec{p}\right)+\operatorname{sym}\left((\nabla V)^{t} \nabla \vec{b}_{0}\right) .
\end{align*}
$$

Let $\mathcal{L}_{2, A}\left(x^{\prime}\right)$ be the symmetric bilinear form generating the quadratic form $\mathcal{Q}_{2, A}\left(x^{\prime}\right)$. Since the odd powers of $x_{3}$ integrate to 0 on the symmetric interval ( $-1 / 2,1 / 2$ ), we get:

$$
\begin{aligned}
\int_{\Omega^{1}} & \mathcal{Q}_{2, A}\left(x^{\prime}, I\left(x^{\prime}\right)+x_{3} I I I\left(x^{\prime}\right)+x_{3}^{2} I I\left(x^{\prime}\right)\right) \mathrm{d} x \\
= & \int_{\omega} \mathcal{Q}_{2, A}\left(x^{\prime}, I\left(x^{\prime}\right)\right) \mathrm{d} x^{\prime}+\left(\int_{-1 / 2}^{1 / 2} x_{3}^{2} \mathrm{~d} x_{3}\right) \int_{\omega} Q_{2, A}\left(x^{\prime}, I I I\left(x^{\prime}\right)\right) \mathrm{d} x^{\prime} \\
\quad & +\left(\int_{-1 / 2}^{1 / 2} x_{3}^{4} \mathrm{~d} x_{3}\right) \int_{\omega} Q_{2, A}\left(x^{\prime}, I I\left(x^{\prime}\right)\right) \mathrm{d} x^{\prime}+2\left(\int_{-1 / 2}^{1 / 2} x_{3}^{2} \mathrm{~d} x_{3}\right) \int_{\omega} \mathcal{L}_{2, A}\left(x^{\prime}, I\left(x^{\prime}\right), I I\left(x^{\prime}\right)\right) \mathrm{d} x^{\prime} \\
= & \int_{\omega} \mathcal{Q}_{2, A}\left(x^{\prime}, I\right)+\frac{1}{12} \int_{\omega} Q_{2, A}\left(x^{\prime}, I I I\right)+\frac{1}{80} \int_{\omega} Q_{2, A}\left(x^{\prime}, I I\right)+\frac{2}{12} \int_{\omega} \mathcal{L}_{2, A}\left(x^{\prime}, I, I I\right) \mathrm{d} x^{\prime} \\
= & \int_{\omega} \mathcal{Q}_{2, A}\left(x^{\prime}, I+\frac{1}{12} I I\right) \mathrm{d} x^{\prime}+\frac{1}{12} \int_{\omega} Q_{2, A}\left(x^{\prime}, I I I\right) \mathrm{d} x^{\prime}+\frac{1}{180} \int_{\omega} Q_{2, A}\left(x^{\prime}, I I\right) \mathrm{d} x^{\prime}=2 \mathcal{I}_{4}(V, \mathrm{e}),
\end{aligned}
$$

by a direct calculation. This completes the proof of Theorem 3.1 in view of (3.28).

## 4. The upper bound

We now complete the proof of $\mathcal{I}_{4}$ being the $\Gamma$-limit of $h^{-4} E^{h}$, by showing the optimality of (3.5).
Theorem 4.1. Let $V \in W^{2,2}\left(\omega, \mathbb{R}^{3}\right)$ and $\mathrm{e} \in L^{2}\left(\omega, \mathbb{R}_{s y m}^{2 \times 2}\right)$ satisfy:

$$
\begin{align*}
& \operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla V\right)=0, \\
& \mathrm{e} \in \mathcal{S}:=\operatorname{cl}_{L^{2}}\left\{\operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla w\right) ; w \in W^{1,2}\left(\omega, \mathbb{R}^{3}\right)\right\} . \tag{4.1}
\end{align*}
$$

Then there exists a sequence $u^{h} \in W^{1,2}\left(\Omega^{h}, \mathbb{R}^{3}\right)$ such that assertions (i), (ii) and (iii) of Theorem 3.1 are satisfied with $R^{h}=\operatorname{Id}$ and $c^{h}=0$, and:

$$
\begin{equation*}
\limsup _{h \rightarrow 0} \frac{1}{h^{4}} E^{h}\left(u^{h}\right) \leq \mathcal{I}_{4}(V, \mathrm{e}) . \tag{4.2}
\end{equation*}
$$

Proof. In the construction below, we will use the following notation. Let $F_{2 \times 2}^{*}$ denote the $\mathbb{R}^{3 \times 3}$ matrix whose principal $2 \times 2$ minor equals $F_{2 \times 2}$ and other entries are 0 . By (3.4), for every $F_{2 \times 2} \in \mathbb{R}^{2 \times 2}$ one can write:

$$
\begin{equation*}
\mathcal{Q}_{2, A}\left(x^{\prime}, F_{2 \times 2}\right)=\min _{c \in \mathbb{R}^{3}}\left\{\mathcal{Q}_{3}\left(A^{-1}\left(F_{2 \times 2}^{*}+\operatorname{sym}\left(c \otimes e_{3}\right)\right) A^{-1}\right)\right\} . \tag{4.3}
\end{equation*}
$$

We will denote by $c\left(x^{\prime}, F_{2 \times 2}\right)$ the unique minimizer in (4.3). Note that $c\left(x^{\prime}, \cdot\right)$ is a linear function of $F_{2 \times 2}$ and it depends only on its symmetric part ( $\operatorname{sym} F_{2 \times 2}$ ). We divide the proof into several steps.
Step 1. Since $\mathrm{e} \in \mathcal{S}$, there exists a sequence $w^{h} \in W^{1,2}\left(\omega, \mathbb{R}^{3}\right)$ such that:

$$
\begin{equation*}
\operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla\left(w^{h}+\frac{1}{24} \vec{d}_{0}\right)\right) \rightarrow \mathrm{e} \quad \text { in } L^{2}\left(\omega, \mathbb{R}^{2 \times 2}\right) \tag{4.4}
\end{equation*}
$$

and without loss of generality we can assume that each $w^{h}$ is smooth up to $\partial \omega$, together with:

$$
\begin{equation*}
\lim _{h \rightarrow 0} \sqrt{h}\left\|w^{h}\right\|_{W^{2, \infty}}=0 \tag{4.5}
\end{equation*}
$$

Fix a small $\epsilon_{0} \in(0,1)$ and let $v^{h} \in W^{2, \infty}\left(\omega, \mathbb{R}^{3}\right)$ be a sequence of Lipschitz deformations satisfying:

$$
\begin{equation*}
v^{h} \rightarrow V \quad \text { in } W^{2,2}\left(\omega, \mathbb{R}^{3}\right), \quad h\left\|v^{h}\right\|_{W^{2, \infty}} \leq \epsilon_{0}, \quad \lim _{h \rightarrow 0} \frac{1}{h^{2}}\left|\left\{x^{\prime} \in \omega ; v^{h}\left(x^{\prime}\right) \neq V\left(x^{\prime}\right)\right\}\right|=0 . \tag{4.6}
\end{equation*}
$$

We refer to [49] and [29] for the construction of such truncated sequence $v^{h}$. Define $\vec{p}^{h} \in W^{1, \infty}\left(\omega, \mathbb{R}^{3}\right)$ :

$$
\vec{p}^{h}=\left(Q_{0}^{t}\right)^{-1}\left[\begin{array}{c}
-\left(\nabla v^{h}\right)^{t} \vec{b}_{0}  \tag{4.7}\\
0
\end{array}\right]
$$

and also define the fields $\vec{q}^{h} \in W^{1, \infty}\left(\omega, \mathbb{R}^{3}\right), \vec{k}_{0}$ smooth and $\tilde{r}^{h} \in L^{\infty}\left(\omega, \mathbb{R}^{3}\right)$ such that:

$$
\begin{aligned}
& Q_{0}^{t} \vec{q}^{h}=\frac{1}{2} c\left(x^{\prime}, 2\left(\nabla y_{0}\right)^{t} \nabla w^{h}+\left(\nabla v^{h}\right)^{t} \nabla v^{h}\right)-\left[\begin{array}{c}
\left(\nabla w^{h}\right)^{t} \vec{b}_{0} \\
0
\end{array}\right]-\left[\begin{array}{c}
\left(\nabla v^{h}\right)^{t} \vec{p}^{h} \\
\frac{1}{2}\left|\vec{p}^{h}\right|^{2}
\end{array}\right], \\
& Q_{0}^{t} \vec{k}_{0}=c\left(x^{\prime},\left(\nabla y_{0}\right)^{t} \nabla \vec{d}_{0}+\left(\nabla \vec{b}_{0}\right)^{t} \nabla \vec{b}_{0}\right)-\left[\begin{array}{c}
\left(\nabla \vec{b}_{0}\right)^{t} \vec{d}_{0} \\
\left|\vec{d}_{0}\right|^{2}
\end{array}\right], \\
& Q_{0}^{t} \tilde{r}^{h}=c\left(x^{\prime},\left(\nabla y_{0}\right)^{t} \nabla \vec{p}^{h}+\left(\nabla v^{h}\right)^{t} \nabla \vec{b}_{0}\right)-\left[\begin{array}{c}
\left(\nabla v^{h}\right)^{t} \vec{d}_{0} \\
\left\langle\vec{p}^{h}, \vec{d}_{0}\right\rangle
\end{array}\right] .
\end{aligned}
$$

Finally, let $\vec{r}^{h} \in W^{1, \infty}\left(\omega, \mathbb{R}^{3}\right)$ be such that:

$$
\begin{equation*}
\lim _{h \rightarrow 0}\left\|\vec{r}^{h}-\tilde{r}^{h}\right\|_{L^{2}}=0, \quad \lim _{h \rightarrow 0} \sqrt{h}\left\|\vec{r}^{h}\right\|_{W^{1, \infty}}=0 \tag{4.8}
\end{equation*}
$$

It follows from the definition of the minimizing map $c$, that:

$$
\begin{align*}
& \mathcal{Q}_{3}\left(A^{-1}\left(2 Q_{0}^{t}\left[\nabla w^{h} \mid \vec{q}^{h}\right]+\left[\nabla v^{h} \mid \vec{p}^{h}\right]^{t}\left[\nabla v^{h} \mid \vec{p}^{h}\right]\right) A^{-1}\right)=\mathcal{Q}_{2, A}\left(x^{\prime}, 2\left(\nabla y_{0}\right)^{t} \nabla w^{h}+\left(\nabla v^{h}\right)^{t} \nabla v^{h}\right), \\
& \mathcal{Q}_{3}\left(A^{-1}\left(Q_{0}^{t}\left[\nabla \vec{d}_{0} \mid \vec{k}_{0}\right]+\left[\nabla \vec{b}_{0} \mid \vec{d}_{0}\right]^{t}\left[\nabla \vec{b}_{0} \mid \vec{d}_{0}\right]\right) A^{-1}\right)=\mathcal{Q}_{2, A}\left(x^{\prime},\left(\nabla y_{0}\right)^{t} \nabla \vec{d}_{0}+\left(\nabla \vec{b}_{0}\right)^{t} \nabla \vec{b}_{0}\right),  \tag{4.9}\\
& \mathcal{Q}_{3}\left(A^{-1}\left(2 Q_{0}^{t}\left[\nabla \vec{p}^{h} \mid \tilde{r}^{h}\right]+2\left[\nabla v^{h} \mid \vec{p}^{h}\right]^{t}\left[\nabla \vec{b}_{0} \mid \vec{d}_{0}\right]^{t}\right) A^{-1}\right)=\mathcal{Q}_{2, A}\left(x^{\prime},\left(\nabla y_{0}\right)^{t} \nabla \vec{p}^{h}+\left(\nabla v^{h}\right)^{t} \nabla \vec{b}_{0}\right) .
\end{align*}
$$

Moreover, we have the following pointwise bounds:

$$
\begin{align*}
\left|\vec{p}^{h}\right| & \leq C\left|\nabla v^{h}\right|, \quad\left|\nabla \vec{p}^{h}\right| \leq C\left(\left|\nabla v^{h}\right|+\left|\nabla^{2} v^{h}\right|\right), \\
\left|\vec{q}^{h}\right| & \leq C\left(\left|\nabla w^{h}\right|+\left|\nabla v^{h}\right|^{2}+\left|\nabla v^{h}\right|\left|\vec{p}^{h}\right|+\left|\vec{p}^{h}\right|^{2}\right) \leq C\left(\left|\nabla w^{h}\right|+\left|\nabla v^{h}\right|^{2}\right),  \tag{4.10}\\
\left|\nabla \vec{q}^{h}\right| & \leq C\left(\left|\nabla w^{h}\right|+\left|\nabla^{2} w^{h}\right|+\left|\nabla^{2} v^{h}\right|\left|\nabla v^{h}\right|+\left|\nabla v^{h}\right|^{2}\right) .
\end{align*}
$$

Step 2. Consider the sequence $u^{h} \in W^{1, \infty}\left(\Omega^{h}, \mathbb{R}^{3}\right)$ defined as:

$$
\begin{aligned}
u^{h}\left(x^{\prime}, x_{3}\right)= & y_{0}\left(x^{\prime}\right)+h v^{h}\left(x^{\prime}\right)+h^{2} w^{h}\left(x^{\prime}\right)+x_{3} \vec{b}_{0}\left(x^{\prime}\right)+\frac{x_{3}^{2}}{2} \vec{d}_{0}\left(x^{\prime}\right) \\
& +\frac{x_{3}^{3}}{6} \vec{k}_{0}\left(x^{\prime}\right)+h x_{3} \vec{p}^{h}\left(x^{\prime}\right)+h^{2} x_{3} \vec{q}^{h}\left(x^{\prime}\right)+\frac{h x_{3}^{2}}{2} \vec{r}^{h}\left(x^{\prime}\right)
\end{aligned}
$$

For every $\left(x^{\prime}, x_{3}\right) \in \Omega^{1}$ we write:

$$
\nabla u^{h}\left(x^{\prime}, h x_{3}\right)=Q_{0}\left(x^{\prime}\right)+Z_{1}^{h}\left(x^{\prime}, x_{3}\right)+Z_{2}^{h}\left(x^{\prime}, x_{3}\right),
$$

where:

$$
\begin{aligned}
& Z_{1}^{h}\left(x^{\prime}, x_{3}\right)=h\left[\nabla v^{h} \mid \vec{p}^{h}\right]+h^{2}\left[\nabla w^{h} \mid \vec{q}^{h}\right]+h x_{3}\left[\nabla \vec{b}_{0} \mid \vec{d}_{0}\right]+\frac{h^{2} x_{3}^{2}}{2}\left[\nabla \vec{d}_{0} \mid \vec{k}_{0}\right]+h^{2} x_{3}\left[\nabla \vec{p}^{h} \mid \vec{r}^{h}\right], \\
& Z_{2}^{h}\left(x^{\prime}, x_{3}\right)=\frac{h^{3} x_{3}^{3}}{6}\left[\nabla \vec{k}_{0} \mid 0\right]+h^{3} x_{3}\left[\nabla \vec{q}^{h} \mid 0\right]+\frac{h^{3} x_{3}}{2}\left[\nabla \vec{r}^{h} \mid 0\right] .
\end{aligned}
$$

Since $Q_{0} A^{-1} \in \mathrm{SO}(3)$, we get:

$$
\nabla u^{h} A^{-1}\left(x^{\prime}, h x_{3}\right)=Q_{0} A^{-1}\left(\operatorname{Id}_{3}+A^{-1} Q_{0}^{t} Z_{1}^{h} A^{-1}+A^{-1} Q_{0}^{t} Z_{2}^{h} A^{-1}\right)
$$

and, in view of (4.6), (4.8) and (4.10), there follows for $h$ sufficiently small:

$$
\begin{aligned}
& \left\|A^{-1} Q_{0}^{t} Z_{1}^{h} A^{-1}+A^{-1} Q_{0}^{t} Z_{2}^{h} A^{-1}\right\|_{L^{\infty}} \\
& \leq C\left(h\left\|\nabla v^{h}\right\|_{L^{\infty}}+h\left\|\vec{p}^{h}\right\|_{L^{\infty}}+h^{2}\left\|\nabla w^{h}\right\|_{L^{\infty}}+h^{2}\left\|\vec{q}^{h}\right\|_{L^{\infty}}+h\left\|\nabla \vec{b}_{0}\right\|_{L^{\infty}}+h\left\|\vec{d}_{0}\right\|_{L^{\infty}}\right. \\
& \quad+h^{2}\left\|\nabla \vec{d}_{0}\right\|_{L^{\infty}}+h^{2}\left\|\vec{k}_{0}\right\|_{L^{\infty}}+h^{2}\left\|\nabla \vec{p}^{h}\right\|_{L^{\infty}}+h^{2}\left\|\vec{r}^{h}\right\|_{L^{\infty}}+h^{3}\left\|\nabla \vec{k}_{0}\right\|_{L^{\infty}} \\
& \left.\quad+h^{3}\left\|\nabla \vec{q}^{h}\right\|_{L^{\infty}}+h^{3}\left\|\nabla \vec{r}^{h}\right\|_{L^{\infty}}\right) \leq C \epsilon_{0} .
\end{aligned}
$$

By the left polar decomposition, there exists a further rotation $R \in \mathrm{SO}(3)$ such that:

$$
\begin{aligned}
R \nabla u^{h} A^{-1}= & \left(\left(\operatorname{Id}_{3}+A^{-1} Q_{0}^{t} Z_{1}^{h} A^{-1}+A^{-1} Q_{0}^{t} Z_{2}^{h} A^{-1}\right)^{t}\left(\operatorname{Id}_{3}+A^{-1} Q_{0}^{t} Z_{1}^{h} A^{-1}+A^{-1} Q_{0}^{t} Z_{2}^{h} A^{-1}\right)\right)^{1 / 2} \\
= & \left(\operatorname{Id}_{3}+2 A^{-1} \operatorname{sym}\left(Q_{0}^{t} Z_{1}^{h}\right) A^{-1}+A^{-1}\left(Z_{1}^{h}\right)^{t} Z_{1}^{h} A^{-1}+\mathcal{O}\left(\left|Z_{2}^{h}\right|\right)\right)^{1 / 2} \\
= & \operatorname{Id}_{3}+A^{-1} \operatorname{sym}\left(Q_{0}^{t} Z_{1}^{h}\right) A^{-1}+\frac{1}{2} A^{-1}\left(Z_{1}^{h}\right)^{t} Z_{1}^{h} A^{-1} \\
& \quad+\mathcal{O}\left(\left|\operatorname{sym}\left(Q_{0}^{t} Z_{1}^{h}\right)+\left(Z_{1}^{h}\right)^{t} Z_{1}^{h}\right|^{2}\right)+\mathcal{O}\left(\left|Z_{2}^{h}\right|\right) .
\end{aligned}
$$

Step 3. Consider the set:

$$
\Omega_{h}=\left\{\left(x^{\prime}, x_{3}\right) \in \Omega^{1} ; v^{h}\left(x^{\prime}\right)=V\left(x^{\prime}\right)\right\} .
$$

Note that on $\Omega_{h}$ we have: $\vec{p}^{h}=\vec{p}$ and $Q_{0}^{t}\left[\nabla v^{h} \mid \vec{p}^{h}\right] \in \operatorname{so}(3)$. By Taylor's expansion, it follows that:

$$
\frac{1}{h^{4}} \int_{\Omega_{h}} W\left(\nabla u\left(x^{\prime}, h x_{3}\right) A^{-1}\right) \mathrm{d} x=\frac{1}{2 h^{4}} \int_{\Omega_{h}} \mathcal{Q}_{3}\left(A^{-1}\left(Q_{0}^{t} Z_{1}^{h}+\frac{1}{2}\left(Z_{1}^{h}\right)^{t} Z_{1}^{h}\right) A^{-1}\right) \mathrm{d} x+\mathcal{E}_{1}^{h}
$$

where the error term $\mathcal{E}_{1}^{h}$ can be estimated by:

$$
\left|\mathcal{E}_{1}^{h}\right| \leq \frac{C}{h^{4}} \int_{\Omega_{h}}\left|2 \operatorname{sym}\left(Q_{0}^{t} Z_{1}^{h}\right)+\left(Z_{1}^{h}\right)^{t} Z_{1}^{h}\right|^{3}+\left|Z_{2}^{h}\right|^{2}+\left|2 \operatorname{sym}\left(Q_{0}^{t} Z_{1}^{h}\right)+\left(Z_{1}^{h}\right)^{t} Z_{1}^{h}\right|\left|Z_{2}^{h}\right| \mathrm{d} x .
$$

Now on $\Omega_{h}$ we also have, by (4.10):

$$
\begin{aligned}
\left|2 \operatorname{sym}\left(Q_{0}^{t} Z_{1}^{h}\right)+\left(Z_{1}^{h}\right)^{t} Z_{1}^{h}\right| & \leq C\left(h^{2}\left|\nabla w^{h}\right|+h^{2}\left|\nabla v^{h}\right|^{2}+h^{2}+h^{2}\left|\nabla v^{h}\right|+h^{2}\left|\nabla^{2} v^{h}\right|+h^{2}\left|\vec{r}^{h}\right|\right), \\
\left|Z_{2}^{h}\right| & \leq C h^{3}\left(1+\left|\nabla \vec{q}^{h}\right|+\left|\nabla \vec{r}^{h}\right|\right) \\
& \leq C h^{3}\left(1+\left|\nabla w^{h}\right|+\left|\nabla^{2} w^{h}\right|+\left|\nabla^{2} v^{h}\right|\left|\nabla v^{h}\right|+\left|\nabla v^{h}\right|^{2}+\left|\nabla \vec{r}^{h}\right|\right),
\end{aligned}
$$

and therefore, in view of (4.5), (4.8), (4.6) and $V \in W^{2,2}$ :

$$
\begin{aligned}
& \frac{1}{h^{4}} \int_{\Omega_{h}}\left|2 \operatorname{sym}\left(Q_{0}^{t} Z_{1}^{h}\right)+\left(Z_{1}^{h}\right)^{t} Z_{1}^{h}\right|^{3} \mathrm{~d} x \\
& \leq \\
& \leq \frac{C}{h^{4}} \int_{\Omega_{h}} h^{6}\left|\nabla w^{h}\right|^{3}+h^{6}\left|\nabla v^{h}\right|^{6}+h^{6}+h^{6}\left|\nabla v^{h}\right|^{3}+h^{6}\left|\nabla^{2} v^{h}\right|^{3}+h^{6}\left|\vec{r}^{h}\right|^{3} \mathrm{~d} x \\
& \leq \frac{C}{h^{4}}\left(h^{2}\left\|\nabla w^{h}\right\|_{L^{\infty}}\left(h^{2}\left\|\nabla w^{h}\right\|_{L^{2}}\right)^{2}+h^{6}\|\nabla V\|_{L^{6}}^{6}+h^{6}|\omega|+h^{6}\|\nabla V\|_{L^{3}}^{3}\right. \\
& \left.\quad \quad+h^{6}\left\|\nabla^{2} v^{h}\right\|_{L^{\infty}}\left\|\nabla^{2} V\right\|_{L^{2}}^{2}+\left(\sqrt{h}\left\|\vec{r}^{h}\right\|_{L^{\infty}}\right)^{3} h^{9 / 2}\right) \rightarrow 0 \quad \text { as } h \rightarrow 0
\end{aligned}
$$

In a similar manner:

$$
\begin{aligned}
& \frac{1}{h^{4}} \int_{\Omega_{h}}\left|Z_{2}^{h}\right|^{2} \mathrm{~d} x \leq \frac{C}{h^{4}} \int_{\Omega_{h}} h^{5}+\left(h\left\|\nabla v^{h}\right\|_{L^{\infty}}\right)^{2} h^{4}\left|\nabla^{2} v^{h}\right|^{2}+h^{6}\left|\nabla v^{h}\right|^{4} \mathrm{~d} x \rightarrow 0 \quad \text { as } h \rightarrow 0, \\
& \frac{1}{h^{4}} \int_{\Omega_{h}}\left|2 \operatorname{sym}\left(Q_{0}^{t}\right) Z_{1}^{h}+\left(Z_{1}^{h}\right)^{t} Z_{1}^{h}\right|\left|Z_{2}^{h}\right| \mathrm{d} x \\
& \quad \leq \frac{C}{h^{4}} \int_{\Omega_{h}}\left(h^{5}\left|\nabla w^{h}\right|^{2}+h^{5}\left|\nabla^{2} w^{h}\right|^{2}+h^{5}\left|\nabla v^{h}\right|^{2}+h^{5}+h^{5}|\nabla V|+h^{5}\left|\nabla^{2} V\right|+h^{5}\left|\vec{r}^{h}\right|\right. \\
& \left.\quad+h^{5}|\nabla V|^{2}\left|\nabla^{2} V\right|+h^{5}|\nabla V|\left|\nabla^{2} V\right|^{2}\right) \mathrm{d} x \leq C \epsilon_{0} .
\end{aligned}
$$

We therefore conclude that:

$$
\begin{equation*}
\limsup _{h \rightarrow 0}\left|\mathcal{E}_{1}^{h}\right| \leq C \epsilon_{0} \tag{4.11}
\end{equation*}
$$

Step 4. Consider now the error due to integrating on the residual subdomain:

$$
\mathcal{E}_{2}^{h}=\frac{1}{h^{4}} \int_{\Omega^{1} \backslash \Omega_{h}} W\left(\nabla u^{h} A^{-1}\left(x^{\prime}, h x_{3}\right)\right) \mathrm{d} x \leq \frac{C}{h^{4}} \int_{\Omega^{1} \backslash \Omega_{h}}\left|2 \operatorname{sym}\left(Q_{0}^{t} Z_{1}^{h}\right)+\left(Z_{1}^{h}\right)^{t} Z_{1}^{h}\right|^{2}+\left|Z_{2}^{h}\right|^{2} \mathrm{~d} x .
$$

Observe that, since the matrix field $\left[\nabla v^{h} \mid \vec{p}^{h}\right]$ is Lipschitz, we have:

$$
\left|\operatorname{sym}\left(Q_{0}^{t}\left[\nabla v^{h} \mid \vec{p}^{h}\right]\right)\left(x^{\prime}\right)\right| \leq C\left\|\nabla v^{h}\right\|_{W^{1, \infty}} \operatorname{dist}\left(x^{\prime},\left\{v^{h}=V\right\}\right) \leq \frac{C \epsilon_{0}}{h} \operatorname{dist}\left(x^{\prime},\left\{v^{h}=V\right\}\right) \rightarrow 0 \quad \text { in } L^{\infty}(\omega)
$$

The last inequality above follows by a standard argument by contradiction. If there were a sequence $x^{\prime h} \in \omega$ such that $\operatorname{dist}\left(x^{\prime h},\left\{v^{h}=V\right\}\right) \geq c h$, this would imply that: $\left|\left\{x^{\prime} ; v^{h}\left(x^{\prime}\right) \neq V\left(x^{\prime}\right)\right\}\right| \geq\left|\Omega \cap B\left(x^{h}, c h\right)\right| \geq c h^{2}$, contradicting (4.6). Consequently, by (4.5), (4.8), (4.6):

$$
\begin{aligned}
\left|\mathcal{E}_{2}^{h}\right| \leq & \frac{C}{h^{4}} \\
& \int_{\Omega^{1} \backslash \Omega_{h}} h^{2}\left|\operatorname{sym}\left(Q_{0}^{t}\left[\nabla v^{h} \mid \vec{p}^{h}\right]\right)\right| \mathrm{d} x \\
& \quad+\frac{C}{h^{4}} \int_{\Omega^{1} \backslash \Omega_{h}} h^{4}\left|\nabla w^{h}\right|^{2}+h^{4}\left|\nabla v^{h}\right|^{4}+h^{4}\left|\nabla^{2} v^{h}\right|^{2}+h^{4}\left|\vec{r}^{h}\right|^{2}+h^{4}+h^{6}\left|\nabla v^{h}\right|^{4} \mathrm{~d} x \\
\leq & \frac{C}{h^{4}} o\left(h^{2}\right)\left|\Omega^{1} \backslash \Omega_{h}\right|+\frac{C}{h^{4}} \sqrt{h}\left\|\nabla w^{h}\right\|_{L^{\infty}} h^{7 / 2}\left|\Omega^{1} \backslash \mathcal{U}^{h}\right|^{1 / 2}\left\|\nabla w^{h}\right\|_{L^{2}} \\
& +C\left|\Omega^{1} \backslash \mathcal{U}^{h}\right|\left\|\nabla v^{h}\right\|_{L^{8}}^{4}+C h\left\|\nabla^{2} v^{h}\right\|_{L^{\infty}} \frac{1}{h}\left\|\nabla^{2} v^{h}\right\|_{L^{2}}\left|\Omega^{1} \backslash \mathcal{U}^{h}\right|^{1 / 2}+\frac{1}{h}\left(\sqrt{h}\left\|\vec{r}^{h}\right\|_{L^{\infty}}\right)^{2}\left|\Omega^{1} \backslash \mathcal{U}^{h}\right| \\
& +\left(h\left\|\nabla^{2} v^{h}\right\|_{\left.L^{\infty}\right)^{2}\left\|\nabla v^{h}\right\|_{L^{4}}^{2}\left|\Omega^{1} \backslash \mathcal{U}^{h}\right|^{1 / 2} \quad \rightarrow 0 \quad \text { as } h \rightarrow 0 .}\right.
\end{aligned}
$$

Thus:

$$
\limsup _{h \rightarrow 0} \frac{1}{h^{4}} E^{h}\left(u^{h}\right) \leq \limsup _{h \rightarrow 0} \frac{1}{h^{4}} \int_{\Omega_{h}} \frac{1}{2} \mathcal{Q}_{3}\left(A^{-1}\left(\operatorname{sym}\left(Q_{0}^{t} Z_{1}^{h}\right)+\frac{1}{2}\left(Z_{1}^{h}\right)^{t} Z_{1}^{h}\right) A^{-1}\right) \mathrm{d} x+C \epsilon_{0} .
$$

Now on $\Omega_{h}$ we have:

$$
\begin{aligned}
2 \operatorname{sym}( & \left.Q_{0}^{t} Z_{1}^{h}\right)+\left(Z_{1}^{h}\right)^{t} Z_{1}^{h} \\
= & 2 h^{2}\left(\operatorname{sym}\left(Q_{0}^{t}\left[\nabla w^{h} \mid \vec{q}^{h}\right]\right)+\frac{x_{3}^{2}}{2} \operatorname{sym}\left(Q_{0}^{t}\left[\nabla \vec{d}_{0} \mid \vec{k}_{0}\right]\right)+x_{3} \operatorname{sym}\left(Q_{0}^{t}\left[\nabla \vec{p} \mid \vec{r}^{h}\right]\right)\right) \\
& +h^{2}\left([\nabla V \mid \vec{p}]^{t}[\nabla V \mid \vec{p}]+x_{3}^{2}\left[\nabla \vec{b}_{0} \mid \vec{d}_{0}\right]^{t}\left[\nabla \vec{b}_{0} \mid \vec{d}_{0}\right]+2 x_{3} \operatorname{sym}\left([\nabla V \mid \vec{p}]^{t}\left[\nabla \vec{b}_{0} \mid \vec{d}_{0}\right]\right)\right)+\mathcal{E}^{h},
\end{aligned}
$$

where the present error $\mathcal{E}^{h}$ is estimated by:

$$
\begin{align*}
&\left|\mathcal{E}^{h}\right| \leq C\left(h^{3}|\nabla V|\left|\nabla w^{h}\right|+h^{3}|\nabla V|+h^{3}|\nabla V||\nabla \vec{p}|+h^{3}|\nabla V|\left|\vec{r}^{h}\right|\right. \\
&+h^{4}\left|\nabla w^{h}\right|^{2}+h^{3}\left|\nabla w^{h}\right|+h^{4}\left|\nabla w^{h}\right|\left|\nabla \vec{p}+h^{4}\right| \nabla w^{h}| | \vec{r}^{h} \mid+h^{3} \\
&\left.+h^{3}|\nabla \vec{p}|+h^{3}\left|\vec{r}^{h}\right|+h^{4}+h^{4}|\nabla \vec{p}|+h^{4}\left|\vec{r}^{h}\right|+h^{4}|\nabla \vec{p}|^{2}+h^{4}\left|\vec{r}^{h}\right|^{2}\right)  \tag{4.12}\\
& \leq C h^{2}\left(o(1) \sqrt{h}|\nabla V|+\epsilon_{0}^{2}\left|\nabla^{2} V\right|+o(1) \sqrt{h}+o(1) \epsilon_{0} \sqrt{h}\right) .
\end{align*}
$$

Consequently:

$$
\begin{aligned}
& \limsup \frac{1}{h \rightarrow 0} E^{h} E^{h}\left(u^{h}\right) \\
& \leq \limsup _{h \rightarrow 0} \frac{1}{2} \int_{\Omega_{h}} \mathcal{Q}_{3}\left(A ^ { - 1 } \left(\operatorname{sym}\left(Q_{0}^{t}\left[\nabla w^{h} \mid \vec{q}^{h}\right]\right)+\frac{1}{2} x_{3}^{2} \operatorname{sym}\left(Q_{0}^{t}\left[\nabla \vec{d}_{0} \mid \vec{k}_{0}\right]\right)+x_{3} \operatorname{sym}\left(Q_{0}^{t}\left[\nabla \vec{p} \mid \vec{r}^{h}\right]\right)\right.\right. \\
& \left.\left.\quad+\frac{1}{2}[\nabla V \mid \vec{p}]^{t}[\nabla V \mid \vec{p}]+\frac{1}{2} x_{3}^{2}\left[\nabla \vec{b}_{0} \mid \vec{d}_{0}\right]^{t}\left[\nabla \vec{b}_{0} \mid \vec{d}_{0}\right]+x_{3} \operatorname{sym}\left([\nabla V \mid \vec{p}]^{t}\left[\nabla \vec{b}_{0} \mid \vec{d}_{0}\right]\right)\right) A^{-1}\right) \mathrm{d} x+C \epsilon_{0} \\
& =\underset{h \rightarrow 0}{\limsup } \frac{1}{2} \int_{\Omega_{h}} \mathcal{Q}_{3}\left(A ^ { - 1 } \left(\operatorname{sym}\left(Q_{0}^{t}\left[\nabla w^{h} \mid \vec{q}^{h}\right]\right)+\frac{1}{2}[\nabla V \mid \vec{p}]^{t}[\nabla V \mid \vec{p}]\right.\right. \\
& \left.\left.\quad+\frac{1}{2} x_{3}^{2} \operatorname{sym}\left(Q_{0}^{t}\left[\nabla \vec{d}_{0} \mid \vec{k}_{0}\right]\right)+\frac{1}{2} x_{3}^{2}\left[\nabla \vec{b}_{0} \mid \vec{d}_{0}\right]^{t}\left[\nabla \vec{b}_{0} \mid \vec{d}_{0}\right]\right) A^{-1}\right) \\
& \quad+\mathcal{Q}_{3}\left(A^{-1}\left(x_{3} \operatorname{sym}\left(Q_{0}^{t}\left[\nabla \vec{p} \mid \vec{r}^{h}\right]\right)+x_{3} \operatorname{sym}\left([\nabla V \mid \vec{p}]^{t}\left[\nabla \vec{b}_{0} \mid \vec{d}_{0}\right]\right)\right) A^{-1}\right) \mathrm{d} x+C \epsilon_{0} .
\end{aligned}
$$

Denoting:

$$
I_{1}\left(x^{\prime}\right)=\operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla w^{h}\right)+\frac{1}{2}\left(\nabla v^{h}\right)^{t} \nabla v^{h}, \quad I_{2}\left(x^{\prime}\right)=\frac{1}{2} \operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla \vec{d}_{0}\right)+\frac{1}{2}\left(\nabla \vec{b}_{0}\right)^{t} \nabla \vec{b}_{0},
$$

we have:

$$
\begin{aligned}
\mathcal{Q}_{3} & \left(A^{-1}\left(I_{1}^{*}\left(x^{\prime}\right)+\operatorname{sym}\left(c\left(x^{\prime}, I_{1}\left(x^{\prime}\right)\right) \otimes e_{3}\right)+x_{3}^{2} I_{2}^{*}\left(x^{\prime}\right)+x_{3}^{2} \operatorname{sym}\left(c\left(x^{\prime}, I_{2}\left(x^{\prime}\right)\right) \otimes e_{3}\right)\right) A^{-1}\right) \\
& =\mathcal{Q}_{3}\left(A^{-1}\left(\left(I_{1}\left(x^{\prime}\right)+x_{3}^{2} I_{2}\left(x^{\prime}\right)\right)^{*}+\operatorname{sym}\left(c\left(x^{\prime}, I_{1}\left(x^{\prime}\right)+x_{3}^{2} I_{2}\left(x^{\prime}\right)\right) \otimes e_{3}\right)\right) A^{-1}\right) \\
& =\mathcal{Q}_{2, A}\left(I_{1}\left(x^{\prime}\right)+x_{3}^{2} I_{2}\left(x^{\prime}\right)\right),
\end{aligned}
$$

where we have used the definition and linearity of the minimizing map $c$. Recalling the definitions of the curvature forms $I\left(x^{\prime}\right), I I\left(x^{\prime}\right)$ and $I I I\left(x^{\prime}\right)$ in (3.29), observe that $I_{2}\left(x^{\prime}\right)=I I\left(x^{\prime}\right)$ and that $I_{1}$ converges to $I$ in $L^{2}$ by (4.4). Hence:

$$
\begin{aligned}
\limsup _{h \rightarrow 0} \frac{1}{h^{4}} E^{h}\left(u^{h}\right) & \leq \frac{1}{2} \int_{\Omega^{1}} \mathcal{Q}_{2, A}\left(I\left(x^{\prime}\right)+x_{3}^{2} I I\left(x^{\prime}\right)\right) \mathrm{d} x+\frac{1}{2} \int_{\Omega^{1}} \mathcal{Q}_{2, A}\left(x_{3} I I I\left(x^{\prime}\right)\right) \mathrm{d} x+C \epsilon_{0} \\
& =\mathcal{I}_{4}(V, \mathrm{e})+C \epsilon_{0} .
\end{aligned}
$$

Since $\epsilon_{0}>0$ was arbitrary, the proof is completed by a diagonal argument.

## 5. Discussion of the von Kármán-like functional (3.5)

Theorems 3.1 and 4.1 imply, as usual in this setting, convergence of almost-minimizers:
Corollary 5.1. If $u^{h} \in W^{1,2}\left(\Omega^{h}, \mathbb{R}^{3}\right)$ is a minimizing sequence to $h^{-4} E^{h}$, that is:

$$
\lim _{h \rightarrow 0}\left(\frac{1}{h^{4}} E^{h}\left(u^{h}\right)-\inf \frac{1}{h^{4}} E^{h}\right)=0
$$

then the appropriate renormalizations $y^{h}=\left(\bar{R}^{h}\right)^{t} u^{h}\left(x^{\prime}, h x_{3}\right)-c^{h} \in W^{1,2}\left(\Omega^{1}, \mathbb{R}^{3}\right)$ obey the convergence statements of Theorem 3.1 (i), (ii), (iii). The convergence of $h^{-1} \operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla V^{h}\right)$ to e in (iii) is strong in $L^{2}(\omega)$. Moreover, any limit $(V, \mathrm{e})$ minimizes the functional $\mathcal{I}_{4}$ on the product of spaces in (1.14) and (1.13).

Proof. The proof is standard. The only possibly nontrivial part is the strong convergence of the scaled tangential strains in (iii), which can be deduced as in Theorem 2.5 in [42].

Let us now compare the functional (3.5) with the von-Kármán energy of thin shells that has been derived in [42]. We will see that the arguments and the stretching and bending terms in both energies are related via the parametrization $y_{0}$ of the surface $y_{0}(\omega)$ in (1.8).

Recall that when $S$ is a smooth 2 d surface in $\mathbb{R}^{3}$, the $\Gamma$-limit of the scaled elastic energies $h^{-4}\left(\frac{1}{h} \int_{S^{h}} W\left(\nabla u^{h}\right)\right)$ on thin shells $S^{h}$ with mid-surface $S$, is:

$$
\begin{equation*}
\tilde{\mathcal{I}}_{4, S}(\tilde{V}, \tilde{\mathrm{e}})=\frac{1}{2} \int_{S} \mathcal{Q}_{2}\left(\tilde{\mathrm{e}}-\frac{1}{2}\left(\tilde{A}^{2}\right)_{t a n}\right) \mathrm{d} y+\frac{1}{24} \int_{S} \mathcal{Q}_{2}\left((\nabla(\tilde{A} \vec{N})-\tilde{A} \Pi)_{t a n}\right) \mathrm{d} y . \tag{5.1}
\end{equation*}
$$

Above, $\Pi$ stands for the shape operator of $S$ and $\vec{N}$ is the unit normal vector to $S$. The subscript tan means taking the restriction of a quadratic form (or an operator) to the tangent space $T_{y} S$. The arguments of $\tilde{\mathcal{I}}_{4, S}$ are:
(i) First order infinitesimal isometries $\tilde{V}$ on $S$. These are vector fields $\tilde{V} \in W^{2,2}\left(S, \mathbb{R}^{3}\right)$ with skew symmetric covariant derivative, so that one may define:

$$
\begin{equation*}
\tilde{A} \in W^{1,2}(S, \operatorname{so}(3)) \quad \text { with } \quad \tilde{A}(y) \tau=\partial_{\tau} \tilde{V}(y) \quad \forall y \in S \quad \forall \tau \in T_{y} S ; \tag{5.2}
\end{equation*}
$$

(ii) Finite strains ẽ on $S$. These are tensor fields $\tilde{\mathrm{e}} \in L^{2}\left(S, \mathbb{R}_{s y m}^{2 \times 2}\right)$ such that:

$$
\begin{equation*}
\tilde{\mathrm{e}}=L^{2}-\lim _{h \rightarrow 0} \operatorname{sym}\left(\nabla \tilde{w}_{h}\right)_{t a n} \quad \text { for some } \quad \tilde{w}_{h} \in W^{1,2}\left(S, \mathbb{R}^{3}\right) \tag{5.3}
\end{equation*}
$$

In the present setting, denote $S=y_{0}(\omega)$ and observe that the $1-1$ correspondence between $\tilde{V}$ in (5.2) and $V$ in (3.3) is given by the change of variables $V=\tilde{V} \circ y_{0}$. The skew symmetric tensor field $\tilde{A}$ on $T_{y} S$ is then uniquely given by:

$$
\begin{equation*}
\tilde{A}\left(y_{0}\left(x^{\prime}\right)\right) \partial_{e} y_{0}=\partial_{e} V\left(x^{\prime}\right) \quad \text { and } \quad \tilde{A} \vec{b}_{0}=\vec{p} \quad \forall e \in \mathbb{R}^{2} \tag{5.4}
\end{equation*}
$$

and the extended strains in (5.3) are related to (4.1) by:

$$
\left\langle\tilde{\mathrm{e}}\left(y_{0}\left(x^{\prime}\right)\right) \partial_{e} y_{0}, \partial_{e} y_{0}\right\rangle=\left\langle\mathrm{e}\left(x^{\prime}\right) e, e\right\rangle \quad \forall e \in \mathbb{R}^{2}
$$

Recall that the first term in the functional (5.1) measures the difference of order $h^{2}$, between the (Euclidean) metric on $S$ and the metric of the deformed surface. Indeed, the amount of stretching of $S$ in the direction $\tau \in T_{y} S$, induced by the deformation $u_{h}=i d+h \tilde{V}+h^{2} \tilde{w}$, has the expansion:

$$
\left|\partial_{\tau} u_{h}\right|^{2}-|\tau|^{2}=h^{2}\left(2\left\langle\partial_{\tau} \tilde{w}, \tau\right\rangle+\left|\partial_{\tau} \tilde{V}\right|^{2}\right)+\mathcal{O}\left(h^{3}\right)=2 h^{2}\left(\langle(\operatorname{sym} \nabla \tilde{w}) \tau, \tau\rangle-\frac{1}{2}\left\langle\tilde{A}^{2} \tau, \tau\right\rangle\right)+\mathcal{O}\left(h^{3}\right) .
$$

The leading order quantity on the right hand side above coincides with:

$$
\langle(\operatorname{sym} \nabla w) e, e\rangle+\frac{1}{2}\left\langle\partial_{e} V, \partial_{e} V\right\rangle=\left\langle\left(\operatorname{sym} \nabla w+\frac{1}{2}(\nabla V)^{t} \nabla V\right) e, e\right\rangle,
$$

where we write $\tau=\partial_{e} y_{0}$, for any $e \in \mathbb{R}^{2}$. This is precisely the argument of the first term in $\mathcal{I}_{4}(V, \mathrm{e})$, modulo the correction $\left(\nabla \vec{b}_{0}\right)^{t} \nabla \vec{b}_{0}$ (equal to the third fundamental form on $S$ in case $\vec{b}_{0}=\vec{N}$ ), due to the incompatibility of the ambient Euclidean metric of $S^{h}$ with the given prestrain $G$ on $\Omega^{h}$.

The second term in (5.1) measures the difference of order $h$, between the shape operator $\Pi$ on $S$ and the shape operator $\Pi^{h}$ on the deformed surface $(i d+h \tilde{V})(S)$ whose unit normal we denote by $\vec{N}^{h}$. The amount of bending of $S$, in the direction $\tau \in T_{y} S$, induced by the deformation $u_{h}=i d+h \tilde{V}$ can be estimated by [42]:

$$
\begin{aligned}
(\operatorname{Id}+h \tilde{A})^{-1} \Pi^{h} & (\operatorname{Id}+h \tilde{A}) \tau-\Pi \tau=(\operatorname{Id}+h \tilde{A})^{-1}\left(\partial_{\tau} \vec{N}^{h}+\mathcal{O}\left(h^{2}\right)\right) \tau-\Pi \tau \\
& =(\operatorname{Id}+h \tilde{A})^{-1}\left((\operatorname{Id}+h \tilde{A}) \Pi \tau+h\left(\partial_{\tau} A\right) \vec{N}+\mathcal{O}\left(h^{2}\right)\right)-\Pi \tau \\
& =(\operatorname{Id}-h \tilde{A}) h\left(\partial_{\tau} \tilde{A}\right) \vec{N}+\mathcal{O}\left(h^{2}\right) \\
& =h\left(\partial_{\tau} \tilde{A}\right) \vec{N}+\mathcal{O}\left(h^{2}\right)=h(\nabla(\tilde{A} \vec{N})-\tilde{A} \Pi)+\mathcal{O}\left(h^{2}\right)
\end{aligned}
$$

The leading order term in this expansion coincides with $\left(\nabla y_{0}\right)^{t} \nabla \vec{p}+(\nabla V)^{t} \nabla \vec{b}_{0}$ when $\vec{b}_{0}=\vec{N}$, as:

$$
\begin{aligned}
\left\langle\left(\partial_{\tau} \tilde{A}\right) \vec{b}_{0}, \tau\right\rangle & =\left\langle\left(\partial_{e}\left(\tilde{A} \vec{b}_{0}\right), \partial_{e} y_{0}\right\rangle-\left\langle\left(\tilde{A} \partial_{e} \vec{b}_{0}, \partial_{e} y_{0}\right\rangle=\left\langle\left(\partial_{e} \vec{p}, \partial_{e} y_{0}\right\rangle+\left\langle\left(\partial_{e} \vec{b}_{0}, \tilde{A} \partial_{e} y_{0}\right\rangle\right.\right.\right.\right. \\
& =\left\langle\left(\nabla y_{0}\right)^{t} \nabla \vec{p} e, e\right\rangle-\left\langle(\nabla V)^{t} \nabla \vec{b}_{0} e, e\right\rangle,
\end{aligned}
$$

in view of (5.4), where again $\tau=\partial_{e} y_{0} \in T_{y_{0}\left(x^{\prime}\right)} S$, for any $e \in \mathbb{R}^{2}$. This is the argument in the second term in $\mathcal{I}_{4}(V, \mathrm{e})$. In the section 6 we identify geometric significance of the last term in (3.5).

## 6. Scaling optimality and examples

In this section, we prove the following result, asserting in particular that under conditions (1.7) and $\operatorname{Riem}(G) \not \equiv 0$, the scaling $h^{4}$ is optimal, i.e.:

$$
\exists c, C>0 \quad c h^{4} \leq \inf _{u^{h} \in W^{1,2}\left(\Omega^{h}, \mathbb{R}^{3}\right)} E^{h}\left(u^{h}\right) \leq C h^{4} \quad \text { as } h \rightarrow 0 .
$$

In addition, we complete the study of two examples already mentioned in [5], where computations illustrate the content of Theorem 6.1.

Theorem 6.1. Assume (1.7), together with:

$$
\begin{equation*}
\operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla \vec{d}_{0}\right)+\left(\nabla \vec{b}_{0}\right)^{t} \nabla \vec{b}_{0}=0 \tag{6.1}
\end{equation*}
$$

where $y_{0}, \vec{b}_{0}$ and $\vec{d}_{0}$ are defined in (1.8), (1.9), (1.11). Then the metric $G$ is flat, i.e. $\operatorname{Riem}(G) \equiv 0$ in $\Omega^{h}$. Equivalently: $\min E^{h}=0$ for all $h$.

Observe that when $\vec{b}_{0}=\vec{N}$, then by (1.11) there must be $\vec{d}_{0}=0$, and hence condition (6.1) means that $\vec{N}$ is constant. This is consistent with our previous observation that when $G e_{3}=e_{3}$, then already condition (1.8) is enough to conclude immersability of $G$ in $\mathbb{R}^{3}$. Equivalently, $G_{2 \times 2}$ is immersible in $\mathbb{R}^{2}$, so that indeed $y_{0}(\omega)$ must be planar in this case.

Towards a proof of Theorem 6.1, recall that $\operatorname{Riem}(G)$ is the covariant Riemann curvature tensor, whose components $R_{i k l m}$ and their relation to the contravariant-covariant curvatures in $R_{s t q}^{p}$ are:

$$
\begin{aligned}
& R_{i k l m}=\frac{1}{2}\left(\partial_{k l} G_{i m}+\partial_{i m} G_{k l}-\partial_{k m} G_{i l}-\partial_{i l} G_{k m}\right)+G_{n p}\left(\Gamma_{k l}^{n} \Gamma_{i m}^{p}-\Gamma_{k m}^{n} \Gamma_{i l}^{p}\right) \\
& R_{i k l m}=G_{i s} R_{k l m}^{s},
\end{aligned}
$$

where we used the Einstein summation convention and the Christoffel symbols:

$$
\begin{equation*}
\Gamma_{k l}^{n}=\frac{1}{2} G^{n s}\left(\partial_{k} G_{s l}+\partial_{l} G_{s k}-\partial_{s} G_{k l}\right) . \tag{6.2}
\end{equation*}
$$

In view of the symmetries in $\operatorname{Riem}(G)$ of a 3-dimensional metric $G$, its flatness is equivalent to the vanishing of the following curvatures:
$R_{1212}, R_{1213}, R_{1223}, R_{1313}, R_{1323}, R_{2323}$.
The proof of Theorem 6.1 is a consequence of the following observation.
Theorem 6.2. Assume (1.7) and let $y_{0}, \vec{b}_{0}$ and $\vec{d}_{0}$ be defined as in (1.8), (1.11). Then:

$$
\operatorname{sym}\left(\left(\nabla y_{0}\right)^{t} \nabla \vec{d}_{0}\right)+\left(\nabla \vec{b}_{0}\right)^{t} \nabla \vec{b}_{0}=\left[\begin{array}{ll}
R_{1313} & R_{1323}  \tag{6.3}\\
R_{1323} & R_{2323}
\end{array}\right] .
$$

Proof. Step 1. We have:

$$
\begin{aligned}
R_{1313} & =-\frac{1}{2} \partial_{11} G_{33}+G_{n p}\left(\Gamma_{13}^{n} \Gamma_{13}^{p}-\Gamma_{11}^{n} \Gamma_{33}^{p}\right), \\
R_{2323} & =-\frac{1}{2} \partial_{22} G_{33}+G_{n p}\left(\Gamma_{23}^{n} \Gamma_{23}^{p}-\Gamma_{22}^{n} \Gamma_{33}^{p}\right), \\
R_{1323} & =-\frac{1}{2} \partial_{12} G_{33}+G_{n p}\left(\Gamma_{13}^{n} \Gamma_{23}^{p}-\Gamma_{12}^{n} \Gamma_{33}^{p}\right) .
\end{aligned}
$$

Besides, in view of (1.11):

$$
\begin{aligned}
i, j=1,2 \quad \frac{1}{2}\left(\left\langle\partial_{i} y_{0}, \partial_{j} \vec{d}_{0}\right\rangle+\left\langle\partial_{j} y_{0}, \partial_{i} \vec{d}_{0}\right\rangle\right) & =\frac{1}{2}\left(\partial_{j}\left\langle\partial_{i} y_{0}, \vec{d}_{0}\right\rangle+\partial_{i}\left\langle\partial_{j} y_{0}, \vec{d}_{0}\right\rangle\right)-\left\langle\partial_{i j} y_{0}, \vec{d}_{0}\right\rangle \\
& =-\frac{1}{2} \partial_{i j} G_{33}-\left\langle\partial_{i j} y_{0}, \vec{d}_{0}\right\rangle
\end{aligned}
$$

since: $\partial_{j}\left\langle\partial_{i} y_{0}, \vec{d}_{0}\right\rangle+\partial_{i}\left\langle\partial_{j} y_{0}, \vec{d}_{0}\right\rangle=-\partial_{i j}\left|\vec{b}_{0}\right|^{2}=-\partial_{i j} G_{33}$. Consequently, (6.3) will follow provided that:

$$
\begin{equation*}
\forall i, j=1,2 \quad\left\langle\partial_{i j} y_{0}, \vec{d}_{0}\right\rangle=G_{n p} \Gamma_{i j}^{n} \Gamma_{33}^{p} \quad \text { and } \quad\left\langle\partial_{i} \vec{b}_{0}, \partial_{j} \vec{b}_{0}\right\rangle=G_{n p} \Gamma_{i 3}^{n} \Gamma_{j 3}^{p} . \tag{6.4}
\end{equation*}
$$

Step 2. Before proving (6.4) we gather some useful formulas. Note that $\partial_{i} G=2 \operatorname{sym}\left(\left(\partial_{i} Q\right)^{t} Q\right)$ for $i=1,2$. Therefore, by direct inspection:

$$
\begin{equation*}
\forall i, j, k=1,2 \quad\left\langle\partial_{i j} y_{0}, \partial_{k} y_{0}\right\rangle=\frac{1}{2}\left(\partial_{i} G_{k j}+\partial_{j} G_{k i}-\partial_{k} G_{i j}\right) . \tag{6.5}
\end{equation*}
$$

Also, recall that condition (1.8) is equivalent to (see [5], proof of Theorem 5.3, formula (5.8)):

$$
\begin{equation*}
\forall i, j=1,2 \quad\left\langle\partial_{i j} y_{0}, \vec{b}_{0}\right\rangle=\frac{1}{2}\left(\partial_{i} G_{j 3}+\partial_{j} G_{i 3}\right) . \tag{6.6}
\end{equation*}
$$

Therefore, for all $i, j=1,2$ :

$$
\begin{align*}
& \left\langle\partial_{j} y_{0}, \partial_{i} \vec{b}_{0}\right\rangle=\partial_{i}\left\langle\partial_{j} y_{0}, \vec{b}_{0}\right\rangle-\left\langle\partial_{i j} y_{0}, \vec{b}_{0}\right\rangle=\frac{1}{2}\left(\partial_{i} G_{j 3}-\partial_{j} G_{i 3}\right),  \tag{6.7}\\
& \left\langle\partial_{i} \vec{b}_{0}, \vec{b}_{0}\right\rangle=\frac{1}{2} \partial_{i} G_{33} .
\end{align*}
$$

We now express $\partial_{i j} y_{0}, \partial_{i} \vec{b}_{0}$ and $\vec{d}_{0}$ in the basis $\left\{\partial_{1} y_{0}, \partial_{2} y_{0}, \vec{b}_{0}\right\}$, writing:

$$
\begin{equation*}
\partial_{i j} y_{0}=\alpha_{i j}^{1} \partial_{1} y_{0}+\alpha_{i j}^{2} \partial_{2} y_{0}+\alpha_{i j}^{3} \vec{b}_{0}, \quad \partial_{i} \vec{b}_{0}=\beta_{i}^{1} \partial_{1} y_{0}+\beta_{i}^{2} \partial_{2} y_{0}+\beta_{i}^{3} \vec{b}_{0}, \quad \vec{d}_{0}=\gamma^{1} \partial_{1} y_{0}+\gamma^{2} \partial_{2} y_{0}+\gamma^{3} \vec{b}_{0} \tag{6.8}
\end{equation*}
$$

By (6.5), (6.6), (6.7) and (1.11), it follows that:

$$
\begin{aligned}
G\left(\alpha_{i j}^{1}, \alpha_{i j}^{2}, \alpha_{i j}^{3}\right)^{t} & =G Q_{0}^{-1} \partial_{i j} y_{0}=Q_{0}^{t} \partial_{i j} y_{0} \\
& =\frac{1}{2}\left(\partial_{i} G_{1 j}+\partial_{j} G_{1 i}-\partial_{1} G_{i j}, \partial_{i} G_{2 j}+\partial_{j} G_{2 i}-\partial_{2} G_{i j}, \partial_{i} G_{3 j}+\partial_{j} G_{3 i}\right), \\
G\left(\beta_{i}^{1}, \beta_{i}^{2}, \beta_{i}^{3}\right)^{t} & =G Q_{0}^{-1} \partial_{i} \vec{b}_{0}=Q_{0}^{t} \partial \vec{b}_{0}=Q_{0}^{t} \partial_{i} \vec{b}_{0}=\frac{1}{2}\left(\partial_{i} G_{13}-\partial_{1} G_{i 3}, \partial_{i} G_{23}-\partial_{2} G_{i 3}, \partial_{i} G_{33}\right)^{t}, \\
G\left(\gamma^{1}, \gamma^{2}, \gamma^{3}\right)^{t} & =G Q_{0}^{-1} \vec{d}_{0}=Q_{0}^{t} \vec{d}_{0}=-\frac{1}{2}\left(\partial_{1} G_{33}, \partial_{2} G_{33}, 0\right)^{t} .
\end{aligned}
$$

In view of (6.2) we then obtain, for all $i, j=1,2$ :

$$
\left(\alpha_{i j}^{1}, \alpha_{i j}^{2}, \alpha_{i j}^{3}\right)=\left(\Gamma_{i j}^{1}, \Gamma_{i j}^{2}, \Gamma_{i j}^{3}\right), \quad\left(\beta_{i}^{1}, \beta_{i}^{2}, \beta_{i}^{3}\right)=\left(\Gamma_{i 3}^{1}, \Gamma_{i 3}^{2}, \Gamma_{i 3}^{3}\right), \quad\left(\gamma^{1}, \gamma^{2}, \gamma^{3}\right)^{t}=\left(\Gamma_{33}^{1}, \Gamma_{33}^{2}, \Gamma_{33}^{3}\right),
$$

so that (6.8) becomes:

$$
\begin{equation*}
\partial_{i j} y_{0}=\Gamma_{i j}^{1} \partial_{1} y_{0}+\Gamma_{i j}^{2} \partial_{2} y_{0}+\Gamma_{i j}^{3} \vec{b}_{0}, \quad \partial_{i} \vec{b}_{0}=\Gamma_{i 3}^{1} \partial_{1} y_{0}+\Gamma_{i 3}^{2} \partial_{2} y_{0}+\Gamma_{i 3}^{3} \vec{b}_{0}, \quad \vec{d}_{0}=\Gamma_{33}^{1} \partial_{1} y_{0}+\Gamma_{33}^{2} \partial_{2} y_{0}+\Gamma_{33}^{3} \vec{b}_{0} . \tag{6.9}
\end{equation*}
$$

Step 3. We now prove (6.4). Recalling $Q_{0}^{T} Q_{0}=G$, the scalar products of expressions in (6.9) are:

$$
\begin{aligned}
& \left\langle\partial_{i j} y_{0}, \vec{d}_{0}\right\rangle=\left\langle\Gamma_{i j}^{n} \partial_{n} y_{0}+\Gamma_{i j}^{3} \vec{b}_{0}, \Gamma_{33}^{p} \partial_{p} y_{0}+\Gamma_{33}^{3} \vec{b}_{0}\right\rangle=G_{n p} \Gamma_{i j}^{n} \Gamma_{33}^{p}, \\
& \left\langle\partial_{i} \vec{b}_{0}, \partial_{j} \vec{b}_{0}\right\rangle=\left\langle\Gamma_{i 3}^{n} \partial_{n} y_{0}+\Gamma_{i 3}^{3} \vec{b}_{0}, \Gamma_{j 3}^{p} \partial_{p} y_{0}+\Gamma_{j 3}^{3} \vec{b}_{0}\right\rangle=G_{n p} \Gamma_{i 3}^{n} \Gamma_{j 3}^{p},
\end{aligned}
$$

exactly as claimed in (6.4). This ends the proof of Theorem 6.2 and also of Theorem 6.1.
We now compute the energy $\mathcal{I}_{4}(V, \mathrm{e})$ in two particular cases:

$$
G\left(x^{\prime}, x_{3}\right)=\operatorname{diag}\left(1,1, \lambda\left(x^{\prime}\right)\right) \quad \text { and } \quad G\left(x^{\prime}, x_{3}\right)=\lambda\left(x^{\prime}\right) \operatorname{Id}_{3}
$$

corresponding to the prestrain with differential shrinking factor only in the normal direction (in the first case), and to the isotropic prestrain (in the second case).

Let $\vec{p}$ be as in the definition (3.6). Writing: $\vec{p}=\alpha^{1} \partial_{1} y_{0}+\alpha^{2} \partial_{2} y_{0}+\alpha^{3} \vec{b}_{0}$, we obtain:

$$
G\left(\alpha^{1}, \alpha^{2}, \alpha^{3}\right)^{t}=-\left(\left\langle\partial_{1} V, \vec{b}_{0}\right\rangle,\left\langle\partial_{2} V, \vec{b}_{0}\right\rangle, 0\right)^{t} .
$$

Consequently:

$$
\begin{equation*}
\vec{p}=-G^{1 i}\left\langle\partial_{i} V, \vec{b}_{0}\right\rangle \partial_{1} y_{0}-G^{2 i}\left\langle\partial_{i} V, \vec{b}_{0}\right\rangle \partial_{2} y_{0}-G^{3 i}\left\langle\partial_{i} V, \vec{b}_{0}\right\rangle \vec{b}_{0} \tag{6.10}
\end{equation*}
$$

Lemma 6.3. Let $\lambda: \bar{\omega} \rightarrow \mathbb{R}$ be smooth and strictly positive. Consider the metric of the form: $G\left(x^{\prime}, x_{3}\right)=$ $\operatorname{diag}\left(1,1, \lambda\left(x^{\prime}\right)\right)$. Then:
(i) $G$ is immersible in $\mathbb{R}^{3}$ if and only if:

$$
M_{\lambda}=\nabla^{2} \lambda-\frac{1}{2 \lambda} \nabla \lambda \otimes \nabla \lambda \equiv 0 \quad \text { in } \omega,
$$

while the condition $M_{\lambda} \not \equiv 0$ is equivalent to: $c h^{4} \leq \inf E^{h} \leq C h^{4}$.
(ii) The $\Gamma$-limit energy functional $\mathcal{I}_{4}$ in (3.5) becomes:

$$
\begin{aligned}
& \forall w \in W^{1,2}\left(\omega, \mathbb{R}^{2}\right) \quad \forall v \in W^{2,2}(\omega, \mathbb{R}) \\
& \mathcal{I}_{4}(v, w)= \frac{1}{2} \int_{\omega} \mathcal{Q}_{2}\left(\operatorname{sym} \nabla w+\frac{1}{2} \nabla v \otimes \nabla v+\frac{1}{96 \lambda} \nabla \lambda \otimes \nabla \lambda\right) \mathrm{d} x^{\prime} \\
&+\frac{1}{24} \int_{\omega} \mathcal{Q}_{2}\left(\sqrt{\lambda} \nabla^{2} v\right)+\frac{1}{5760} \int_{\omega} \mathcal{Q}_{2}\left(M_{\lambda}\right) \mathrm{d} x^{\prime},
\end{aligned}
$$

where $\mathcal{Q}_{2}$ is independent of $x^{\prime}$ and it is defined by $\mathcal{Q}_{2, \text { Id }}$ in (3.4).
Proof. Part (i) of the assertion has been shown in [5]. For (ii), note first that:

$$
y_{0}\left(x^{\prime}\right)=x^{\prime} \quad \text { and } \quad Q_{0}=A=\operatorname{diag}(1,1, \sqrt{\lambda}) .
$$

Consequently, directly from (3.4) we see that $\mathcal{Q}_{2, A}=\mathcal{Q}_{2, \text { Id }}$, which we denote simply by $\mathcal{Q}_{2}$.
Further, in view of (4.1), every admissible limiting strain $\mathrm{e} \in \mathcal{S}$ has the form $\mathrm{e}=\operatorname{sym} \nabla w$ for some $w \in$ $W^{1,2}\left(\omega, \mathbb{R}^{2}\right)$. Also, without loss of generality, every admissible limiting displacement $V$ is of the form: $V=(0,0, v)$ for some $v \in W^{2,2}(\omega, \mathbb{R})$. We now compute, using (1.10), (6.9), (6.10):

$$
\vec{b}_{0}=\sqrt{\lambda} e_{3}, \quad \vec{d}_{0}=-\frac{1}{2}\left(\partial_{1} \lambda, \partial_{2} \lambda, 0\right), \quad \vec{p}=-\sqrt{\lambda}\left(\partial_{1} v, \partial_{2} v, 0\right)
$$

Therefore:

$$
\begin{aligned}
& \left(\nabla \vec{b}_{0}\right)^{t} \nabla \vec{b}_{0}=\frac{1}{4 \lambda} \nabla \lambda \otimes \nabla \lambda, \quad\left(\nabla y_{0}\right)^{t} \nabla \vec{d}_{0}=-\frac{1}{2} \nabla^{2} \lambda \\
& \left(\nabla y_{0}\right)^{t} \nabla \vec{p}=-\frac{1}{2 \sqrt{\lambda}} \nabla v \otimes \nabla \lambda-\sqrt{\lambda} \nabla^{2} v, \quad(\nabla V)^{t} \nabla \vec{b}_{0}=\frac{1}{2 \sqrt{\lambda}} \nabla v \otimes \nabla \lambda
\end{aligned}
$$

This ends the proof of Lemma 6.3 in view of (3.5).
Lemma 6.4. Let $\lambda: \bar{\omega} \rightarrow \mathbb{R}$ be smooth and strictly positive. Consider the metric $G\left(x^{\prime}, x_{3}\right)=\lambda\left(x^{\prime}\right) \operatorname{Id}_{3}$. Denote $f=$ $\frac{1}{2} \log \lambda$. Then:
(i) Condition (1.8) is equivalent to $\Delta f=0$, which is also equivalent to the immersability of the metric $G_{2 \times 2}$ in $\mathbb{R}^{2}$.
(ii) Under condition (1.8), condition (6.1) can be directly seen as equivalent to $\operatorname{Ric}(G)=0$ and therefore to the immersability of $G$.
(iii) The $\Gamma$-limit energy functional in (3.5) has the following form:

$$
\begin{aligned}
\mathcal{I}_{4}(V, \mathrm{e}) & =\frac{1}{2} \int_{\omega} e^{-2 f} \mathcal{Q}_{2}\left(\mathrm{e}+\frac{1}{2}(\nabla V)^{t} \nabla V+\frac{1}{24} e^{2 f} \nabla f \otimes \nabla f\right) \mathrm{d} x^{\prime} \\
& +\frac{1}{24} \int_{\omega} \mathcal{Q}_{2}\left(2 \nabla V_{3} \otimes \nabla f-\nabla^{2} V_{3}-\left\langle\nabla V_{3}, \nabla f\right\rangle \mathrm{Id}_{2}\right) \mathrm{d} x^{\prime}+\frac{1}{1440} \int_{\omega} \mathcal{Q}_{2}\left(e^{f} \operatorname{Ric}(G)_{2 \times 2}\right) \mathrm{d} x^{\prime}
\end{aligned}
$$

where $\mathcal{Q}_{2}$ is as in Lemma 6.3, and where $\operatorname{Ric}(G)_{2 \times 2}$ denotes the tangential part of the Ricci curvature tensor of $G$, i.e.:

$$
\operatorname{Ric}(G)_{2 \times 2}=\left[\begin{array}{ll}
R_{11} & R_{12} \\
R_{12} & R_{22}
\end{array}\right]
$$

Proof. The part (i) has been deduced in [5], together with the expression:

$$
\begin{equation*}
\operatorname{Ric}(G)=-\left(\nabla^{2} f-\nabla f \otimes \nabla f\right)^{*}-\left(\Delta f+|\nabla f|^{2}\right) \operatorname{Id}_{3} \tag{6.11}
\end{equation*}
$$

We now consider the case when (1.8) holds. By $(i)$ the metric $G_{2 \times 2}$ is immersible in $\mathbb{R}^{2}$ and in particular $\vec{N}=e_{3}$. Writing $V=\left(V_{1}, V_{2}, V_{3}\right)$, from (1.10), (6.9) and (6.10) we obtain:

$$
\begin{aligned}
& \vec{b}_{0}=\sqrt{\lambda} e_{3}, \quad \vec{d}_{0}=-\left(\partial_{1} f \partial_{1} y_{0}+\partial_{2} f \partial_{2} y_{0}\right), \quad \vec{p}=-\frac{1}{\sqrt{\lambda}}\left(\partial_{1} V_{3} \partial_{1} y_{0}+\partial_{2} V_{3} \partial_{2} y_{0}\right) . \\
& \left(\nabla \vec{b}_{0}\right)^{t} \nabla \vec{b}_{0}=e^{2 f} \nabla f \otimes \nabla f, \quad(\nabla V)^{t} \nabla \vec{b}_{0}=e^{f} \nabla V_{3} \otimes \nabla f
\end{aligned}
$$

Further, observe that: $\partial_{i} \vec{d}_{0}=-\left(\partial_{1 i} f \partial_{1} y_{0}+\partial_{2 i} f \partial_{2} y_{0}+\partial_{1} f \partial_{1 i} y_{0}+\partial_{2} f \partial_{2 i} y_{0}\right)$, and so:

$$
\frac{1}{\lambda}\left\langle\partial_{1} y_{0}, \partial_{1} \vec{d}_{0}\right\rangle=-\frac{1}{\lambda}\left(\lambda \partial_{11} f+\frac{1}{2} \partial_{1} \lambda \partial_{1} f+\frac{1}{2} \partial_{2} \lambda \partial_{2} f\right)=-\left(\partial_{11} f+|\nabla f|^{2}\right)
$$

In the same manner, we arrive at:

$$
\frac{1}{\lambda}\left\langle\partial_{2} y_{0}, \partial_{2} \vec{d}_{0}\right\rangle=-\left(\partial_{22} f+|\nabla f|^{2}\right), \quad \frac{1}{\lambda}\left\langle\partial_{2} y_{0}, \partial_{1} \vec{d}_{0}\right\rangle=-\partial_{12} f, \quad \frac{1}{\lambda}\left\langle\partial_{1} y_{0}, \partial_{2} \vec{d}_{0}\right\rangle=-\partial_{21} f
$$

Consequently, $\left(\nabla y_{0}\right)^{t} \nabla \vec{d}_{0}$ is already a symmetric matrix, and:

$$
\left(\nabla y_{0}\right)^{t} \nabla \vec{d}_{0}=-e^{2 f}\left(\nabla^{2} f+|\nabla f|^{2} \mathrm{Id}_{2}\right)
$$

In particular, under condition $\Delta f=0$, the formula (6.11) yields:

$$
\operatorname{sym}\left(\nabla y_{0}\right)^{t} \nabla \vec{d}_{0}+\left(\nabla \vec{b}_{0}\right)^{t} \nabla \vec{b}_{0}=e^{2 f} \operatorname{Ric}(G)_{2 \times 2}
$$

which we see to be equivalent with $\nabla f=0$ and hence with $\operatorname{Ric}(G)=0$. This establishes (ii).

We now compute the remaining quantities appearing in the expression of $\mathcal{I}_{4}$. Firstly:

$$
\nabla \vec{p}=\frac{1}{2 \lambda^{3 / 2}} \nabla y_{0}\left(\nabla V_{3} \otimes \nabla \lambda\right)-\frac{1}{\sqrt{\lambda}} \nabla y_{0} \nabla^{2} V_{3}-\frac{1}{\sqrt{\lambda}}\left(\partial_{1} V_{3}\left(\partial_{11} y_{0}, \partial_{12} y_{0}\right)+\partial_{2} V_{3}\left(\partial_{12} y_{0}, \partial_{22} y_{0}\right)\right) .
$$

Using the relations between $\left\langle\partial_{i j} y_{0}, \partial_{k} y_{0}\right\rangle$ and $\partial_{l} G$ in (6.5), we obtain:

$$
\left(\nabla y_{0}\right)^{t} \nabla \vec{p}=\frac{1}{2 \lambda^{3 / 2}} G_{2 \times 2} \nabla V_{3} \otimes \nabla \lambda-\frac{1}{\sqrt{\lambda}} G_{2 \times 2} \nabla^{2} V_{3}-\frac{1}{2 \sqrt{\lambda}}\left[\begin{array}{c|c}
\left\langle\nabla V_{3}, \nabla \lambda\right\rangle & \left\langle\nabla V_{3}, \nabla \lambda^{\perp}\right\rangle \\
\hline-\left\langle\nabla V_{3}, \nabla \lambda^{\perp}\right\rangle \mid\left\langle\nabla V_{3}, \nabla \lambda\right\rangle
\end{array},\right.
$$

and therefore:

$$
\operatorname{sym}\left(\nabla y_{0}\right)^{t} \nabla \vec{p}=\sqrt{\lambda} \operatorname{sym}\left(\nabla V_{3} \otimes \nabla f\right)-\sqrt{\lambda} \nabla^{2} V_{3}-\sqrt{\lambda}\left\langle\nabla V_{3}, \nabla \lambda\right\rangle \operatorname{Id}_{2} .
$$

In a similar manner, it follows that:

$$
\operatorname{sym}\left(\nabla y_{0}\right)^{t} \nabla \vec{d}_{0}=-\lambda\left(\nabla^{2} f+|\nabla f|^{2} \mathrm{Id}_{2}\right) .
$$

Since $\mathcal{Q}_{2, A}\left(x^{\prime}\right)=\lambda^{-1} \mathcal{Q}_{2}$, the formula in (3.5) becomes:

$$
\begin{align*}
\mathcal{I}_{4}(V, \mathrm{e})= & \frac{1}{2} \int_{\omega} e^{-2 f} \mathcal{Q}_{2}\left(\mathrm{e}+\frac{1}{2}(\nabla V)^{t} \nabla V+\frac{1}{24} e^{2 f} \nabla f \otimes \nabla f\right) \mathrm{d} x^{\prime} \\
& +\frac{1}{24} \int_{\omega} e^{-2 f} \mathcal{Q}_{2}\left(2 e^{f} \nabla V_{3} \otimes \nabla f-e^{f} \nabla^{2} V_{3}-e^{f}\left\langle\nabla V_{3}, \nabla f\right\rangle \mathrm{Id}_{2}\right) \mathrm{d} x^{\prime}  \tag{6.12}\\
& +\frac{1}{1440} \int_{\omega} e^{-2 f} \mathcal{Q}_{2}\left(e^{2 f} \operatorname{Ric}(G)_{2 \times 2}\right) \mathrm{d} x^{\prime},
\end{align*}
$$

which implies the result.

## Conflict of interest statement

There is no conflict of interest.

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## Appendix A. A proof of Corollary 2.4

Let us show that for any sequence of deformations $u^{h}$ such that $\lim _{h \rightarrow 0} h^{-2} E^{h}\left(u^{h}\right)=0$, there exist matrix fields $R^{h} \in W^{1,2}(\omega, \mathrm{SO}(3))$ such that:

$$
\begin{equation*}
\frac{1}{h} \int_{\Omega^{h}}\left|\nabla u^{h}(x)-R^{h}\left(x^{\prime}\right)\left(Q_{0}\left(x^{\prime}\right)+x_{3} B_{0}\left(x^{\prime}\right)\right)\right|^{2} \mathrm{~d} x \leq C\left(E^{h}\left(u^{h}\right)+h^{4}\right) \tag{A.1}
\end{equation*}
$$

and:

$$
\begin{equation*}
\int_{\omega}\left|\nabla R^{h}\left(x^{\prime}\right)\right|^{2} \mathrm{~d} x^{\prime} \leq \frac{C}{h^{2}}\left(E^{h}\left(u^{h}\right)+h^{4}\right) . \tag{A.2}
\end{equation*}
$$

This will prove Corollary 2.4. For $\mathcal{U}^{h}$ as in Lemma 2.3, we let:

$$
E^{h}\left(u^{h}, \mathcal{U}^{h}\right)=\frac{1}{h} \int_{\mathcal{U}^{h}} W\left(\nabla u^{h} A^{-1}\right) \mathrm{d} x
$$

and we obtain:

$$
\begin{equation*}
\frac{1}{h} \int_{\mathcal{U}^{h}}\left|\nabla u^{h}(x)-\bar{R}^{h}\left(Q_{0}\left(x^{\prime}\right)+x_{3} B_{0}\left(x^{\prime}\right)\right)\right|^{2} \mathrm{~d} x \leq C\left(E^{h}\left(u^{h}, \mathcal{U}^{h}\right) \mathrm{d} x+h^{3}\left|\mathcal{U}^{h}\right|\right) \tag{A.3}
\end{equation*}
$$

$\underline{\text { Step 1. For every } x^{\prime} \in \omega \text {, denote } D_{x^{\prime}, \delta}=B\left(x^{\prime}, \delta\right) \cap \omega \text { and } B_{x^{\prime}, \delta, h}=D_{x^{\prime}, \delta} \times(-h / 2, h / 2) \text {. For short, we write } B_{x^{\prime}, 2 h}=}$ $\overline{B_{x^{\prime}, 2 h, h}}$ and $B_{x^{\prime}, h}=B_{x^{\prime}, h, h}$. Apply (A.3) to the set $\mathcal{U}^{h}=B_{x^{\prime}, 2 h}$ to get a rotation $R_{x^{\prime}, 2 h} \in \mathrm{SO}(3)$ such that, with a universal constant $C$ :

$$
\begin{equation*}
\frac{1}{h} \int_{B_{x^{\prime}, 2 h}}\left|\nabla u^{h}(z)-R_{x^{\prime}, 2 h}\left(Q_{0}\left(z^{\prime}\right)+z_{3} B_{0}\left(z^{\prime}\right)\right)\right|^{2} \mathrm{~d} z \leq C\left(E^{h}\left(u^{h}, B_{x^{\prime}, 2 h}\right)+h^{3}\left|B_{x^{\prime}, 2 h}\right|\right) \tag{A.4}
\end{equation*}
$$

Consider a family of mollifiers $\eta_{x^{\prime}} \in C^{\infty}(\omega, \mathbb{R})$, parametrized by $x^{\prime} \in \omega$ :

$$
\int_{\omega} \eta_{x^{\prime}}=\frac{1}{h}, \quad\left\|\eta_{x^{\prime}}\right\|_{L^{\infty}(\omega)} \leq \frac{C}{h^{3}}, \quad\left\|\nabla_{x^{\prime}} \eta_{x^{\prime}}\right\|_{L^{\infty}(\omega)} \leq \frac{C}{h^{4}} \quad \text { and }\left(\operatorname{supp} \eta_{x^{\prime}}\right) \cap \omega \subset D_{x^{\prime}, h} .
$$

Define $\tilde{R}^{h} \in W^{1,2}\left(\omega, \mathbb{R}^{3 \times 3}\right)$ as:

$$
\begin{equation*}
\tilde{R}^{h}\left(x^{\prime}\right)=\int_{\Omega^{h}} \eta_{x^{\prime}}\left(z^{\prime}\right) \nabla u^{h}(z)\left(Q_{0}\left(z^{\prime}\right)+z_{3} B_{0}\left(z^{\prime}\right)\right)^{-1} \mathrm{~d} z \tag{A.5}
\end{equation*}
$$

We then have:

$$
\begin{align*}
& \frac{1}{h} \int_{B_{x^{\prime}, h}}\left|\nabla u^{h}(z)-\tilde{R}^{h}\left(z^{\prime}\right)\left(Q_{0}\left(z^{\prime}\right)+z_{3} B_{0}\left(z^{\prime}\right)\right)\right|^{2} \mathrm{~d} z \\
\leq & \frac{C}{h} \int_{B_{x^{\prime}, 2 h}}\left|\nabla u^{h}(z)-R_{x^{\prime}, 2 h}\left(Q_{0}\left(z^{\prime}\right)+z_{3} B_{0}\left(z^{\prime}\right)\right)\right|^{2} \mathrm{~d} z+\frac{C}{h} \int_{B_{x^{\prime}, h}}\left|\tilde{R}^{h}\left(z^{\prime}\right)-R_{x^{\prime}, 2 h}\right|^{2}\left|Q_{0}\left(z^{\prime}\right)+z_{3} B_{0}\left(z^{\prime}\right)\right|^{2} \mathrm{~d} z \\
\leq & C\left(E^{h}\left(u^{h}, B_{x^{\prime}, 2 h}\right)+h^{3}\left|B_{x^{\prime}, 2 h}\right|\right)+\frac{C}{h} \int_{B_{x^{\prime}, h}}\left|\tilde{R}^{h}\left(z^{\prime}\right)-R_{x^{\prime}, 2 h}\right|^{2} \mathrm{~d} z \tag{A.6}
\end{align*}
$$

where we have used (A.4) and $\left\|Q_{0}\left(z^{\prime}\right)+z_{3} B_{0}\left(z^{\prime}\right)\right\|_{L^{\infty}} \leq C$. Now, for every $z^{\prime} \in B_{x^{\prime}, h}$ we have:

$$
\begin{align*}
\left|\tilde{R}^{h}\left(z^{\prime}\right)-R_{x^{\prime}, 2 h}\right|^{2} & =\left|\int_{\Omega^{h}} \eta_{z^{\prime}}\left(y^{\prime}\right) \nabla u^{h}(y)\left(Q_{0}\left(y^{\prime}\right)+y_{3} B_{0}\left(y^{\prime}\right)\right)^{-1} \mathrm{~d} y-R_{x^{\prime}, 2 h}\right|^{2} \\
& =\left|\int_{\Omega^{h}} \eta_{z^{\prime}}\left(y^{\prime}\right)\left(\nabla u^{h}(y)-R_{x^{\prime}, 2 h}\left(Q_{0}\left(y^{\prime}\right)+y_{3} B_{0}\left(y^{\prime}\right)\right)\right)\left(Q_{0}\left(z^{\prime}\right)+y_{3} B_{0}\left(z^{\prime}\right)\right)^{-1} \mathrm{~d} y\right|^{2} \\
& \leq C\left(\int_{B_{z^{\prime}, h}} \eta_{z^{\prime}}\left(y^{\prime}\right)^{2} \mathrm{~d} y\right)\left(\int_{B_{z^{\prime}, h}}\left|\nabla u^{h}(y)-R_{x^{\prime}, 2 h}\left(Q_{0}\left(y^{\prime}\right)+y_{3} B_{0}\left(y^{\prime}\right)\right)\right|^{2} \mathrm{~d} y\right) \\
& \leq \frac{C}{h^{2}} \int_{B_{x^{\prime}, 2 h}}\left|\nabla u^{h}(y)-R_{x^{\prime}, 2 h}\left(Q_{0}\left(y^{\prime}\right)+y_{3} B_{0}\left(y^{\prime}\right)\right)\right|^{2} \mathrm{~d} y \leq \frac{C}{h^{2}}\left(E^{h}\left(u^{h}, B_{x^{\prime}, 2 h}\right)+h^{3}\left|B_{x^{\prime}, 2 h}\right|\right) . \tag{A.7}
\end{align*}
$$

In a similar way, in view of $\int_{\Omega^{h}} \nabla_{z^{\prime}} \eta_{z^{\prime}}\left(y^{\prime}\right) \mathrm{d} y=0$, it follows that:

$$
\begin{aligned}
\left|\nabla \tilde{R}^{h}\left(z^{\prime}\right)\right|^{2} & =\left(\int_{\Omega^{h}} \nabla_{z^{\prime}} \eta_{z^{\prime}}\left(y^{\prime}\right) \nabla u^{h}(y)\left(Q_{0}\left(y^{\prime}\right)+y_{3} B_{0}\left(y^{\prime}\right)\right)^{-1} \mathrm{~d} y\right)^{2} \\
& =\left(\int_{B_{x^{\prime}, 2 h}} \nabla_{z^{\prime}} \eta_{z^{\prime}}\left(y^{\prime}\right)\left(\nabla u^{h}(y)\left(Q_{0}\left(y^{\prime}\right)+y_{3} B_{0}\left(y^{\prime}\right)\right)^{-1}-R_{x^{\prime}, 2 h}\right) \mathrm{d} y\right)^{2} \\
& \leq C \int_{\Omega^{h}}\left|\nabla_{z^{\prime}} \eta_{z^{\prime}}\left(y^{\prime}\right)\right|^{2} \mathrm{~d} y \int_{B_{x^{\prime}, 2 h}}\left|\nabla u^{h}(y)-R_{x^{\prime}, 2 h}\left(Q_{0}\left(y^{\prime}\right)+y_{3} B_{0}\left(y^{\prime}\right)\right)\right|^{2} \mathrm{~d} y \\
& \leq \frac{C}{h^{4}}\left(E^{h}\left(u^{h}, B_{x^{\prime}, 2 h}\right)+h^{3}\left|B_{x^{\prime}, 2 h}\right|\right) .
\end{aligned}
$$

From (A.7) we obtain:

$$
\begin{aligned}
\int_{B_{x^{\prime}, h}}\left|\tilde{R}^{h}\left(z^{\prime}\right)-R_{x^{\prime}, 2 h}\right|^{2} \mathrm{~d} z & \leq \frac{C}{h^{2}} \int_{B_{x^{\prime}, h}}\left(E^{h}\left(u^{h}, B_{x^{\prime}, 2 h}\right)+h^{4}\left|B_{x^{\prime}, 2 h}\right|\right) \mathrm{d} z \\
& \leq \operatorname{Ch}\left(E^{h}\left(u^{h}, B_{x^{\prime}, 2 h}\right)+h^{3}\left|B_{x^{\prime}, 2 h}\right|\right)
\end{aligned}
$$

and therefore by (A.6) we further see that:

$$
\begin{equation*}
\frac{1}{h} \int_{B_{x^{\prime}, h}}\left|\nabla u^{h}(z)-\tilde{R}^{h}\left(z^{\prime}\right)\left(Q_{0}\left(z^{\prime}\right)+z_{3} B_{0}\left(z^{\prime}\right)\right)\right|^{2} \mathrm{~d} z \leq C\left(E^{h}\left(u^{h}, B_{x^{\prime}, 2 h}\right)+h^{3}\left|B_{x^{\prime}, 2 h}\right|\right) \tag{A.8}
\end{equation*}
$$

Step 2. Covering $\Omega^{h}$ by a finite family of sets $\left\{B_{x^{\prime}, h}\right\}$, such that the intersection number of the doubled covering $\overline{\left\{B_{x^{\prime}, 2 h}\right\}}$ is independent of $h$, applying (A.8) and summing over the covering, it follows that:

$$
\frac{1}{h} \int_{\Omega^{h}}\left|\nabla u^{h}(z)-\tilde{R}^{h}\left(z^{\prime}\right)\left(Q_{0}\left(z^{\prime}\right)+z_{3} B_{0}\left(z^{\prime}\right)\right)\right|^{2} \mathrm{~d} z \leq C\left(E^{h}\left(u^{h}\right)+h^{4}\right) .
$$

In a similar fashion we obtain:

$$
\int_{D_{x^{\prime}, h}}\left|\nabla \tilde{R}^{h}\left(z^{\prime}\right)\right|^{2} \mathrm{~d} z \leq \frac{C}{h^{4}} \int_{D_{x^{\prime}, h}}\left(E^{h}\left(u^{h}, B_{x^{\prime}, 2 h}\right)+h^{3}\left|B_{x^{\prime}, 2 h}\right|\right) \mathrm{d} z \leq \frac{C}{h^{2}}\left(E^{h}\left(u^{h}, B_{x^{\prime}, 2 h}\right)+h^{3}\left|B_{x^{\prime}, 2 h}\right|\right),
$$

and by the same covering argument:

$$
\int_{\Omega^{h}}\left|\nabla \tilde{R}^{h}\left(z^{\prime}\right)\right|^{2} \mathrm{~d} z \leq \frac{C}{h^{2}}\left(E^{h}\left(u^{h}\right)+h^{4}\right) .
$$

Step 3. Note that, in the above two estimates, we can replace $\tilde{R}^{h}$ by $R^{h}=\mathbb{P}_{\mathrm{SO}(3)} \tilde{R}^{h} \in W^{1,2}(\omega, \mathrm{SO}(3))$. Firstly, the projection in question is well defined in view of (A.7), since:

$$
\operatorname{dist}^{2}\left(\tilde{R}^{h}, \mathrm{SO}(3)\right) \leq\left|\tilde{R}^{h}-R_{x^{\prime}, 2 h}\right| \leq \frac{C}{h^{2}}\left(E^{h}\left(u^{h}\right)+h^{4}\right),
$$

which is small because of the hypothesis $\alpha<2$. Moreover:

$$
\begin{aligned}
& \frac{1}{h} \int_{B_{x^{\prime}, h}}\left|\nabla u^{h}(z)-R^{h}\left(z^{\prime}\right)\left(Q_{0}\left(z^{\prime}\right)+z_{3} B_{0}\left(z^{\prime}\right)\right)\right|^{2} \mathrm{~d} z \\
& \quad \leq \frac{C}{h} \int_{B_{x^{\prime}, h}}\left|\nabla u^{h}(z)-\tilde{R}^{h}\left(z^{\prime}\right)\left(Q_{0}\left(z^{\prime}\right)+z_{3} B_{0}\left(z^{\prime}\right)\right)\right|^{2} \mathrm{~d} z+\frac{C}{h} \int_{B_{x^{\prime}, h}}\left|\tilde{R}^{h}\left(z^{\prime}\right)-R^{h}\left(z^{\prime}\right)\right|^{2}\left|Q_{0}\left(z^{\prime}\right)+z_{3} B_{0}\left(z^{\prime}\right)\right|^{2} \mathrm{~d} z \\
& \quad \leq C\left(E^{h}\left(u^{h}, B_{x^{\prime}, 2 h}\right)+h^{3}\left|B_{x^{\prime}, 2 h}\right|\right)
\end{aligned}
$$

because of (A.8) and (A.7). Finally, the previous covering argument clearly implies (A.1), and $\int_{\omega}\left|\nabla R^{h}\right|^{2} \mathrm{~d} z \leq$ $C \int_{\omega}\left|\nabla \tilde{R}^{h}\right|^{2} \mathrm{~d} z$ yields (A.2).

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[^1]:    ${ }^{1}$ In most intermediary results of this paper, the simple connectedness of $\omega$ is not needed. However, it is so in order that the vanishing of the Riemann tensor of a metric $G$ implies existence of an equidimensional isometric immersion of $G$.

[^2]:    2 We call e an extended strain because it is a limit of the usual strains.

